Chapter 4
The Harper system: Fractal attractors, Fractal spectra, and Gaps

As shown in the previous Chapter, the transformation of the Harper equation into a dynamical map simplifies the problem to some extent and provides a new method to study the overall spectrum. This Chapter is concerned with details regarding the Harper Equation. One focus is on what can be learned about the spectrum and eigenstates of the Harper equation through a study of the dynamics of the Harper map. As discussed in Chapter 3 at the duality point, \( \epsilon = 1 \), the eigenspectrum is given exactly by the zeros of the Lyapunov exponent, \( \lambda(E) \). We show here that topological properties of the orbits of the mapping provide a unique labeling of the gaps in the spectrum. The gap indices also determine the scaling behavior of the gaps as a function of the coupling constant [103]. The behavior of the gaps as a function of coupling constant is important for a number of problems, and we show how the gaps can be uniquely labeled by mapping them on to a Cayley tree. The gap labels are winding numbers, which are topological invariants for orbits of the iterative mapping. These determine the scaling behavior of the gaps at, above, and by duality, below, the critical coupling, \( \epsilon = \epsilon_c \equiv 1 \). We discuss the general scheme for labeling the gaps for particular choices of \( \omega \) such as the golden mean or silver mean ratios, as well as for other algebraic and transcendental irrational ratios.
Figure 4.1: Phase diagram of the Harper equation for $\omega = (\sqrt{5} - 1)/2$.

In this Chapter the variation of Lyapunov exponent with the system parameter and its relation to the formation of SNAs via symmetry breaking is described in Section 4.2. Different bifurcation routes for the formation of SNAs are also discussed. Gaps in the eigenvalue spectrum and their labeling and scaling properties are discussed in detail. The SNAs of the Harper map are themselves of interest. Regions (in parameter space) where they occur have an interesting hierarchical geometry, being organized in fractal “tongue-like” zones [103]. Apart from discussing the phase diagram for this system as a function of $E$ and $\epsilon$ for fixed irrational $\omega$, I also describe the SNAs in the Harper system and characterize them via correlation, Fourier spectra and phase sensitivity.

### 4.1 Phase diagram

The overall dynamical behavior in the Harper map can be summarized in a phase diagram, shown in Fig. (4.1) for the case of $\omega = \frac{\sqrt{5} - 1}{2}$, namely the golden mean ratio and Fig. (4.2a) for the silver mean ($\sqrt{2} - 1$) case.

The Harper equation is invariant under the transformation $\psi_n \rightarrow (-1)^n \psi_n$, $\epsilon \rightarrow -\epsilon$, $E \rightarrow -E$, so that it suffices to consider the case $E, \epsilon$ nonnegative.
We compute the Lyapunov exponent and determine the dynamical state in the different regions demarcated in the figures. Below $\epsilon_c$, the dynamics is either on invariant curves (I) or on tori (Q), while above $\epsilon_c$, there are regions of SNA (S) and tori (Q). The SNA regions exist only at and above $\epsilon_c$. As was first argued by Ketoja and Satija [72], in the strong-coupling limit of large $\epsilon$, the dynamics is guaranteed to be on SNAs; this argument can be extended down to $\epsilon_c$. Above $\epsilon_c$, therefore, the dynamics is largely on SNAs. The gap regions extend above $\epsilon_c$. Therefore, by continuity, the SNA regions emanate from the eigenvalues at $\epsilon_c$. These have a tongue-like shape, and since each eigenvalue is associated with one such region, these are hierarchically organized in the same manner as are the energy eigenvalues: SNAs occur in fractal tongues.

Shown in Fig. (4.2b) is a (schematic) detail in the neighborhood of one (and therefore of every) tongue: the region of 3-frequency quasiperiodic orbits or tori for $\epsilon < 1$ are mirrored by SNA regions for $\epsilon > 1$. The major differences between the cases of different $\omega$ arise from the manner in which the regions of tori (Q) are arranged.

4.2 Symmetry Breaking: SNAs in Harper Map

The variation of the Lyapunov exponent with $\epsilon$ is simple if $E$ is an eigenvalue. Below $\epsilon = 1$, the exponent is zero and the state is extended, while above it, $\lambda \sim -\ln \epsilon$ and the state is localized; see Fig. (4.3) for the case of $E = 0$. Of particular note is the discontinuity (in slope) at $\epsilon_c$: the localization transition can therefore be viewed as a bifurcation.

We now turn to the description of the dynamical attractors in the Harper map. Strange nonchaotic dynamics in the Harper system is known to be created through a number of different routes or scenarios. Unlike other examples of quasiperiodically forced systems wherein SNAs are formed by mechanisms that have parallels with routes to chaos in similar unforced systems, here the routes to SNA can be quite distinctive. As has been described earlier, at band center, namely for $E = 0$, orbits of the Harper map below $\epsilon_c$ lie on invariant curves that consist of two branches (these are denoted C for central and N for noncentral in Fig. (4.4)).

These branches originate in the fact that the mapping for $\epsilon = 0$ has a
Figure 4.2: Phase diagram for Harper equation for $\omega = (\sqrt{2} - 1)$ the line corresponding to $\epsilon = 1$ separates the localized states from the extended one.
Figure 4.3: Lyapunov exponent versus $\epsilon$ for the eigenvalue $E = 0$. The extended to localized transition occurs at $\epsilon = \epsilon_c = 1$. All the states corresponding to $\lambda = 0$ are extended while the states corresponding to $\lambda > 0$ are localized for the eigenvalue $E = 0$.

Figure 4.4: Invariant curve for the Harper map corresponding to the band center extended state, the central and noncentral branches being indicated by C and N respectively.
period-2 fixed point and no period-1 fixed point. Different initial points
on the \((x, \phi)\) plane generate different invariant curves, the two branches of
which are widely separated at \(\epsilon = 0\), but as \(\epsilon \rightarrow 1\), these branches begin to
approach each other. For a particular curve, the branches collide at a dense
set of points, at \(\epsilon_c\), and indeed, all the different curves merge at \(\epsilon_c\) to give a
single attractor which has a dense set of singularities (Fig. 4.5)).

At the merging point, the Lyapunov exponent remains zero. The distance
between the two branches of a given invariant curve decreases as a power in
\(\epsilon - \epsilon_c\), as does the distance between any pair of invariant curves (Fig. 4.6). As
previously noted, the homoclinic collision route to SNA is accompanied by
a symmetry-breaking. Below \(\epsilon_c\) when the Lyapunov exponent is zero, there
is a quasiperiodic symmetry in the stretch exponents and this symmetry is
lost above \(\epsilon_c\), once the attractor has negative Lyapunov exponent.

In order to understand this, it is convenient to study the local expansion
and contraction rates by calculating the return-map for the stretch expo-
nents, Shown in Fig. (4.7a,b) for Harper map for \(E = 0\) and \(\epsilon = 0.6\). There
is a reflection symmetry evident, namely \((x, y \rightarrow -y, -x)\) Fig. (4.7a). Due
to this symmetry, the positive and the negative terms in stretch exponent
are exactly cancel each other and gives a zero Lyapunov exponent.

This form of symmetry is maintained for \(0 < \epsilon < 1\), above which this
symmetry is broken Fig. (4.7). This may be due to the fact that the negative
stretch exponents exceed the positive ones and giving the negative value of
Lyapunov exponent \(\lambda\). Therefore there is no balance between the positive
and negative component of stretch exponent and leads to symmetry breaking
Fig. (4.7b). The attractor corresponding to this value of \(\epsilon\) has a dense set of
singularities and negative value of lyapunov exponent confirms the existence
of strange nonchaotic dynamics.

When the eigenvalue \(E\) differs from 0, say at the band-edge, the attractor
in the localized state is also a SNA which is born at \(\epsilon = 1\), with zero Lyapunov
exponent. While each eigenvalue at \(\epsilon_c\) is associated with a SNA, not all SNAs
arise from the above homoclinic collision route. Indeed, the more general
route to SNA in this system appears to be related to both the blowout
bifurcation as well as the so-called fractalization scenario for the formation
of SNAs, and this is discussed in the following subsection.
Figure 4.5: The fractal attractor corresponding to the critically localized state.

Figure 4.6: Scaling of the minimum vertical distance $d$ between a family of different invariant curves in the Harper map as a function of the parameter $(1 - \epsilon)$ for the eigenvalue $E = 0$. Different pairs of curves approach each other at different rates, but at $\epsilon_c$, all curves collide to form the critical SNA.
Figure 4.7: (a) First return map for the stretch exponent showing the quasiperiodic symmetry below $\epsilon_c$, and (b) the return map above $\epsilon_c$, when this symmetry is broken.
4.2.1 Other Bifurcations and Fractalization

Given the complexity of the phase diagram for the Harper system with the hierarchical organization of SNA ($S$) and torus ($Q,G$) regions, there are a number of bifurcations as parameters are varied. For fixed $\epsilon$ below $\epsilon_c$, there are bifurcations from 2–frequency to 3–frequency tori as $E$ is varied, the signature of this transition being the change of the Lyapunov exponent from a negative value to zero. There is therefore a discontinuity in the slope of the Lyapunov exponent versus $E$ curve. The 3–frequency tori ($Q$) densely cover the phase space, while the 2–frequency tori ($G$) are attractors of the dynamics (including the case when the Lyapunov exponent is exactly zero). This transition is also, therefore, accompanied by a breaking of the symmetry in the stretch exponents. On the other hand, by fixing $E$ and varying $\epsilon$, there are other bifurcations depending on the value of $E$. There can be transitions from 3–frequency tori to 2–frequency tori below $\epsilon_c$, and from 2–frequency tori to SNA above $\epsilon_c$. Shown in Fig. (4.8a) is the variation of the Lyapunov exponent with $\epsilon$ for $E = 1$; the bifurcations being signaled by a discontinuity in slope of the curve, with the Lyapunov exponent being strictly nonpositive throughout. SNAs are first formed at this energy at $\epsilon \approx 2.5200$, below which there is an attracting 2–frequency torus. This transition has some of the dynamical characteristics of the blowout bifurcation [11, 81, 109] both in the nature of the attractors before (Fig. (4.9a)) and after (Fig. (4.9b)) the transition as well as in the manner in which the Lyapunov exponent varies. The fluctuations in the local Lyapunov exponents, as measured in the variance of the distribution, are large on the SNAs and zero on the tori. This is shown in Fig. (4.8b). Note that the Cantor set structure of the gaps (whose widths decay only as a power in $\epsilon$) is reflected in the pattern of bifurcations along any typical line in the $(E,\epsilon)$ plane.

The $G \rightarrow S$ transition occurs along essentially all the boundaries of the SNA regions above $\epsilon_c$. When approached from below, the attractor of the dynamics can be seen to gradually develop from a smooth curve to one with wrinkles on all scales: this is reminiscent of the fractalization process [66]. The blowout nature of the transition, insofar as the attractor abruptly changes its volume with intermittent bursts, can be seen more clearly along the line $E = 3$. In the torus region $G$, the dynamical map here has a single invariant curve (Fig. (4.10a)), which develops kinks and wrinkles as $\epsilon$ is
Figure 4.8: The depressed blowout bifurcation along the line $E = 1$. (a) Variation of the Lyapunov exponent with $\varepsilon$ for $\omega$ the golden mean. (b) Fluctuations of the Lyapunov exponent (scaled by a factor of 1000).
Figure 4.9: The attractor (a) before (a two-frequency torus) and (b) after (a SNA) the bifurcation.

varied. The transition to SNA is at $\epsilon \approx 1.270(37567)$, when the attractor gets wrinkled enough and actually hits the boundary $x \to \infty$, Fig. (4.10b). This bifurcation can be clearly detected by measuring, say, the distance $\Delta = \min_{n} |1 - \tanh x_n|$, namely the closest point of the attractor from the boundary as a function of $\epsilon$. As can be seen in Fig. (4.10c), this quantity abruptly decreases at the transition.

4.3 Characterization of Harper SNAs

4.3.1 Correlations and Power spectra

Principal among the measures that can be used to distinguish SNAs from morphologically similar chaotic attractors or from (regular) attractors which are similar dynamically (namely they have similar values of the Lyapunov exponent) are quantities that examine correlations on the attractors [33, 116].

The autocorrelation function, which provides a quantitative measure of the extent to which dynamical properties are correlated, is defined for a
Figure 4.10: The bifurcations along the line $E = 3$, showing the attractor (a) before ($\epsilon = 1.270375\ldots$) and (b) after the $G \rightarrow S$ transition. (c) The distance from the attractor to the boundary as a function of $\epsilon$. 
Figure 4.11: For the $E = 0$ critical SNA for $\omega$ the golden mean (a) the correlation function, and (b) the power spectrum.
dynamical variable $x$ as

$$C(\tau) = \frac{\langle x_i x_{i+\tau} \rangle - \langle x_i \rangle \langle x_{i+\tau} \rangle}{\langle x_i^2 \rangle - \langle x_i \rangle^2},$$

(4.1)

where $i = 1, 2, \ldots$ is a discrete time index, $\tau = 0, 1, \ldots$ is the time shift, and $\langle \rangle$ denotes a time-average. If $x$ is periodic then $C(\tau)$ varies between -1 and 1, regaining its initial value $C(0)$ in a periodic fashion. If $x$ is chaotic, then $C(\tau)$ decays from its initial value of 1 to 0, around which latter value it fluctuates. On SNAs, the variables have correlation functions that do not recur exactly: $C(\tau)$ is also quasiperiodic and is nearly recurrent, oscillating between -1 and 1 (Fig. (4.11a)).

The Fourier transform of the correlation function also shows these differences as characteristic spectra. The Fourier transform of a discrete sequence $\{x_k\}$ is

$$T(\Omega, N) = \frac{1}{N} \sum_{k=1}^{N} x_k \exp(i2\pi k \Omega),$$

(4.2)

the power spectrum being

$$P(\omega) = \lim_{N \to \infty} \langle |T(\omega, N)|^2 \rangle.$$  

(4.3)

A chaotic signal typically has a continuous power spectrum, but for SNAs, the Fourier spectrum reflects the fact that the dynamics is neither chaotic nor regular, and the power spectrum consists of several peaks at frequencies corresponding to the quasiperiodicity in the dynamics; see Fig. (4.11b).

### 4.3.2 Phase Sensitivity Properties

Pikovsky and Feudel [116] introduced the phase-sensitivity exponent in order to characterize the strangeness of a attractor. This quantity measures the sensitivity of the dynamics with respect to change of phase of the external force. Given a map

$$x_{n+1} = f(x_n, \theta_n)$$

$$\theta_{n+1} = \theta_n + \omega \mod 1,$$

(4.4)

the maximal value of the derivative $\partial x/\partial \theta$ can provide a suitable tool to distinguish between strange and nonstrange geometry.
Figure 4.12: (a) Maxima of the partial sum $\gamma_N$ (cf. Eq. (4.6)) on the critical SNA at $\varepsilon_c$, $E=0$ in the Harper map, as a function of iteration length, $N$. (b) $\Gamma_N$, the minimum value of $\gamma_N$. 
Since the above derivative changes along the trajectory \((x_0, \theta_0), (x_1, \theta_1), (x_2, \theta_2), \ldots\), one can get a relation

\[
S_N = \frac{\partial x_{n+1}}{\partial \theta} = f(x_n, \theta_n) + f(x_n, \theta_n) \frac{\partial x_n}{\partial \theta},
\]

(4.5)

where \(S_N\) is the derivative with respect to external phase (we follow the notation of Pikovsky and Feudel [116]). The number of maxima in \(S_N\) up to time \(N\) is

\[
\gamma_N(x, \theta) = \max_{0 \leq n \leq N} |S_n|.
\]

(4.6)

The value of \(\gamma_N\) grows with \(N\), a consequence of the fact that the attractor is not smooth; see Fig. (4.12a). Also, as Pikovsky and Feudel [116] noted, the growth rate of the partial sum with time represent the strangeness of attractor, so it becomes necessary to calculate the minimum value of \(\gamma_N(x, \theta)\) with respect to randomly choose initial points,

\[
\Gamma_N = \min_{x, \theta} \max_{0 \leq n \leq N} |S_n|.
\]

(4.7)

From Fig. (4.12b) we see that \(\Gamma_N \sim N^\mu\), where \(\mu\) is the so-called phase sensitivity exponent. If the attractor is smooth (i.e. nonfractal) then the maximum derivative with respect to external phase is bounded, and this value saturates as a function of iteration. However, in the case of SNAs, the maximum derivative with respect to external phase increases indefinitely with iteration.

### 4.4 Gaps of the Harper spectrum

The eigenvalue spectrum of the Harper equation is singular-continuous [15] for irrational \(\omega\) at \(\epsilon_c\), namely it forms a Cantor set. The nature of the gaps in the spectrum has been of great interest in a number of different contexts in the past several years. In particular, how the gaps behave as a function of the potential strength, namely \(\epsilon\), has been a long-standing question [93, 150].

The eigenvalues can be simply deduced from the \(\lambda\) versus \(E\) curve at \(\epsilon_c\), as shown in Fig. (4.13a) for the case of \(\omega\) the golden mean ratio, or in
Figure 4.13: (a) Lyapunov exponent versus energy at \( \epsilon_c \) for \( \omega = (\sqrt{5} - 1)/2 \). Gap labels are indicated for the largest visible gaps. At every bifurcation, when \( \lambda = 0 \), the dynamics is on a SNA. (b) Lyapunov exponent versus energy at \( \epsilon_c \) for \( \omega = \sqrt{2} - 1 \). Gap labels are indicated, as in (a), for the largest visible gaps.
Fig. (4.13b) for $\omega$ the silver mean ratio. The tori regions of the Harper map correspond to gaps in the eigenvalue spectrum of the Harper equation. Orbits for such values of $E$ and $\epsilon$ are 1-d curves that wind across the cylinder an integral number of times. Within each gap, this integer index is constant and measures the number of times the orbit crosses the boundaries $x \to \pm \infty$. This essentially counts the number of changes of sign of the wavefunction (per unit length), and is related to a winding number for the orbit. These indices are indicated in Figs. (4.13a) and (4.13b) for the largest visible gaps.

### 4.4.1 Gap labeling Theorem

One of the most interesting features of the almost periodic lattice is that when a quasiperiodic modulation is applied, a series of gaps open in the energy spectrum of host lattice. Each gap is characterized by a unique topological gap index and finite width and depth. The integrated density of states (IDS) in each gap is found to be constant. Therefore IDS in particular gap is given by $N + M \omega$, where $N, M$ are integers and $\omega$ is an irrational number. These integers label the gap.

This is known as Gap labeling Theorem [14, 16] which relates the integrated density of states (IDS) to the rotation number for orbits [23, 64]. On a gap, the rotation number, which is constant, can be expressed as

$$\Omega = N \omega + M.$$  \hspace{1cm} (4.8)

For the case of the Harper system, further simplifications arise. Since the spectrum is symmetric about 0, for $E \geq 0$, the IDS increases from $1/2$ at the band center $E = 0$ to 1 at the band edge. On the gap labeled by the integer index $N$, the rotation number takes the value (cf. Eq. (4.8))

$$\Omega_N = \max\{N \omega, 1 - \{N \omega\}\},$$  \hspace{1cm} (4.9)

and the fact that the IDS must be a strictly monotonically increasing function of energy suffices in specifying the ordering of the gaps. One of the important features of the gap labeling theorem is that the gap can densely fill the energy spectrum. As result of this, energy band split into infinite number of gaps such that the total width of a gap found to be vanishes. Therefore the energy spectra found to be singular continuous. As shown in Fig. (4.14) for
Figure 4.14: The Integrated density of states (IDS) (scale on the right) and Lyapunov exponent ($\lambda$) (scale on the left) versus energy at $\epsilon_c$. The gap labels $k$ are indicated for the largest visible gaps. At every bifurcation, when $\lambda = 0$, the dynamics is on a SNA. On the gaps, the IDS takes the constant value $\Omega_k$ specified by Eq. (4.9).

Figure 4.15: The attractor for a value of $E$ corresponding to the gap $N = 8$. Note that the orbit has 8 branches that traverse the range $-\infty < x < \infty$. 
$\omega$ golden mean ratio, each gap can be labeled with a winding number or IDS given by Eq. (4.9). The attractor shown in Fig. (4.15) is an orbit in the gap labeled 8 (which lies between gaps 4 and 3) in the silver–mean Harper system.

For rational frequencies, the spectrum of the Harper equation consists of a finite number of bands. If $\omega = p/q$, $p, q$ integers, then there are $q$ or $q - 1$ bands depending on whether $q$ is odd or even, and therefore in the positive energy spectrum the number of gaps is $(q - 1)/2$ or $(q/2 - 1)$. For rational $\omega$ all orbits are periodic with period $q$. For energies in the gaps, the orbits form a 1-parameter family, indexed by initial phase, from which the winding number can be easily extracted.

As is well–known [73], an irrational number has a unique continued fraction representation. Thus an irrational frequency

$$\omega \equiv [a_1, a_2, a_3, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}}$$  \hspace{1cm} (4.10)

$a_i$'s integers, can be successively approximated by a unique series of rational approximant. By truncating the continued fraction representation up to the term $a_j$, one obtains the rational $p_j/q_j$; the sequence $p_j/q_j$, $j = 1, 2, \ldots$ converges, $\lim_{j \to \infty} p_j/q_j \to \omega$. For a given approximant $p_j/q_j$, the gap structure of the spectrum can be described in terms of the winding numbers. These run from 1 to $q_j/2 - 1$ or $(q_j - 1)/2$, as discussed in the preceding paragraph. The new gaps that appear for the next approximant, $p_{j+1}/q_{j+1}$, do not alter the inter se ordering of the gaps already present for lower order rational approximant. Thus the gap structure for irrational $\omega$ can be deduced by sequentially examining the structure of the gaps for the successive rational approximations to $\omega$. Each gap in the spectrum, or each region of tori in the map, can therefore be uniquely labeled by an integer index.

For algebraic irrationals which have periodic continued fraction representations, there are simplifications that permit a more compact description.
of the ordering scheme.

We now describe this for the case for the golden and silver mean ratios. The golden mean is the limiting ratio of successive Fibonacci numbers, \( F_k/F_{k+1} \), where \( F_k \) is given by the relation

\[
F_{k+1} = F_k + F_{k-1}, \quad k = 1, 2, \ldots, \text{with } F_0 = 0, F_1 = 1,
\]

the first few being 0, 1, 1, 2, 3, 5, 8, \ldots, while the silver mean is the limiting ratio of the successive integers of the family 0, 1, 2, 5, 12, 29, 70, \ldots, namely

\[
S_{k+1} = 2S_k + S_{k-1}, \quad k = 1, 2, \ldots, \text{with } S_0 = 0, S_1 = 1
\]

with \( \lim_{k \to \infty} S_k/S_{k+1} \to \sqrt{2} - 1 \).

4.4.2 Gap Ordering and the Cayley Tree

The ordering of the gaps can be best described by first constructing a graph for the particular irrational ratio in the following manner. The gap labels are unique and it is possible to specify their ordering as a function of energy. For the case of \( \omega \) the golden mean irrational, this is as follows. Consider a Cayley tree, arranged as shown in Fig. (4.16a), with each node (except the origin, labeled 0) having two successors. Nodes at the same horizontal level are at the same generation. The rightmost node at each generation is labeled by successive Fibonacci integers, \( F_{k+1} = F_{k-1} + F_k, F_0 = 1, F_1 = 2 \), while the leftmost are half the successive even Fibonacci integers. The non-Fibonacci numbers are then identified with nodes as follows: for given \( m \), the parent node \( i_m \) is the smallest available such that \( m + i_m \) is a Fibonacci number. The sub-tree rooted at node \( F_k \) contains a sequence \( F_jF_{j+k}, j = 1, \ldots \) which are placed alternately to the left and right; this suffices in determining the placement of all other integers within that subtree. Analogous result for the

1 Given an irrational \( \omega \), the reordering of the fractional parts of its integer multiples is a problem that has been encountered earlier in different contexts [112, 113, 135]. For any set of numbers \( y_j = \{j\omega\}, j = 0, 1, \ldots, m \), it is possible to construct an ordering function [112, 135] so that the set \( \{y\} \) is rearranged in ascending order. This procedure, which depends on the successive rational approximations to \( \omega \), gives the required permutation of the indices that reorders the gaps.

2 Subsidiary number-theoretic properties help in arranging the remaining integers. If \( m \) and \( m' \) are daughters of \( i_m \), then \( (m + i_m) \) and \( (m' + i_m) \) are consecutive Fibonacci numbers.
silver mean ratio can be obtained by consider a Cayley tree with each node (except the origin, labeled 0) having two successors. The rightmost node at each generation is labeled by successive $S_k$, while the leftmost are half the successive even $S_k$'s. The other integers are identified with the remaining nodes as follows: for given $m$, the mother node $i_m$ is the smallest available such that

- $m + i_m$ is one of the integers $S_k$. If $i_m$ already has two daughters, then the mother node is chosen such that
- $m + i_m$ is one of the integers $S_{k+1} - S_{k+1} = S_{k+2}$, namely 1,3,7,17,41,....

Complicated (but deducible [103]) rules determine which branch (left or right) the daughters are placed, but the above procedure results in an unique arrangement of all integers on the Cayley tree: see Figs. (4.16a) and (4.16b). The actual ordering of the gaps derives from the following additional consideration. Every pair of integers, $i_1$ and $i_2$, with $i_2 > i_1$, has two possibilities as to how they are relatively placed on this graph. Either

1. $i_1$ is an ancestor of $i_2$, i.e. there is a directed path connecting $i_2$ to $i_1$. If this path is to the left at node $i_1$, then $i_2 < i_1$. (If to the right, then $i_1 < i_2$.)
   or

2. $i_0$ is the most recent common ancestor of $i_1$ and $i_2$. If the path from $i_0$ to $i_1$ is on the left at $i_0$, then $i_1 < i_2$. (Similarly, if it is to the right, then $i_2 < i_1$.)

This gives a unique ordering of the integers with the relation $<$ being transitive (if $i < j$ and $j < m$ then $i < m$). Thus, for the golden mean, the ordering is

\[
\ldots < 4 < \ldots < 9 < \ldots < 1 < \ldots < 7 < \ldots < 2 < \ldots < 11 < \ldots < 21 < \ldots < 0.
\]

While for silver mean we get

\[
\ldots < 35 < \ldots < 11 < \ldots < 1 < \ldots < 4 < \ldots < 2 < \ldots < 7 < \ldots < 12 < \ldots < 0.
\]

74
Figure 4.16: (a) Ordering of the gaps for $\omega = (\sqrt{5} - 1)/2$, the golden mean ratio. Only part of the Cayley tree described in the text is shown for clarity. Each node has two daughters except for 0, which has only one. (b) The corresponding Cayley tree for the case of $\omega$ the silver mean, $\sqrt{2} - 1$. 
The gaps appear in precisely this order: if \( k < \ell \), then the gap with index \( k \) precedes the gap with index \( \ell \) in the positive energy spectrum of the Harper system: see Figs. (4.13a,b) where gap labels for the largest visible gaps are indicated. The labels have been determined by examination of orbits of the map as described.

4.4.3 Scaling of gap attributes

As can be seen in Figs. (4.13a) and (4.13b), each gap has two attributes: the width and the depth. The width is the spacing between two energy eigenvalues of the Harper equation, but the depth has no immediate significance, being just the minimum value that the Lyapunov exponent takes between two zeros. These are denoted \( w_m \) and \( d_m \) respectively, and both are functions of the coupling strength, \( \epsilon \). In general, both widths and depths decrease with order (i.e. with increasing gap index), and with increasing \( \epsilon \). However, and particularly in the case of the widths, the dependence on gap label is non-monotonic, in a manner described below first for golden mean ratio and then for silver mean ratio.

The depth has no obvious quantum-mechanical interpretation, \(-d_m\) merely being the minimum value that the Lyapunov exponent takes in the \( m \)th gap, and it decreases with order \( d_m > d_n \) if \( m < n \), scaling, at \( \epsilon_c \) as \( d_N \sim 1/N \) (see Fig. (4.17a)). The behavior of the gap widths is more complicated and depends on the details of the Cayley tree. These are nonmonotonic as a function of gap index, but come in families: gaps belonging to a given family scale as a power, \( w_N \sim 1/N^\theta \). The fastest decreasing are the Fibonacci gaps, \( 1,2,3,5,8,\ldots,F_k,\ldots (\theta \equiv \theta_r \approx 2.3) \), while the slowest is the family \( 1,4,17,\ldots,F_{1+3k}/2,\ldots (\theta \equiv \theta_l \approx 1.88) \): these are respectively the successive rightmost and leftmost nodes on the Cayley tree in Fig. (4.16a). Other families, which can be similarly defined on subtrees, also obey scaling, with exponents between \( \theta_l \) and \( \theta_r \). When the gaps are ordered by rank \( r \), then they scale as \( w_r \sim 1/r^2 \).

Above \( \epsilon_c \), the states are exponentially localized. For all localized states, irrespective of energy, the localization length or Lyapunov exponent is identical [7]. The gaps which dominate the spectrum at \( \epsilon_c \), persist for larger \( \epsilon \).
Figure 4.17: (a) Scaling of the gap widths, $w_N (\bullet)$, and depths $d_N (\circ)$ as a function of gap index, $N$, at $\epsilon = \epsilon_c$. For clarity, the depths have been multiplied by a factor of 10. The dashed line fitting the depths has slope -1. The dotted lines show the scaling of the two families of gaps; see the text for details. (b) Scaling of the gap widths, $w_N$ for the largest few gaps as a function of $\epsilon$ above $\epsilon_c$. The solid lines are the power-laws given in Eq. (4.13).
but decrease in width according to the (empirical) scalings (see Fig. (4.17b))

\[ w_N \sim \frac{1}{N^{\theta_e N - 1}} \]
\[ d_N \sim \frac{1}{N^{\theta_l N}} \]

where \( \theta \) is particular to the family to which the gap belongs. The dynamics of the Harper map corresponding to localized states is on SNAs [72], while that in the gaps continues to be on 1-dimensional attractors similar to those below \( \epsilon_c \). However, since the gaps decrease in width, most of the dynamics is now on SNAs. By continuity, therefore, the SNA regions must start at each eigenvalue at \( \epsilon_c \), and widen gradually since for large \( \epsilon \) the spectrum lies in the range \( 0 \leq E \leq 2 \epsilon \). The dynamics in the tongue–like regions is entirely on fractal attractors with a negative Lyapunov exponent, each of which starts at an eigenvalue at \( \epsilon_c \). The fractal (Cantor set) spectral structure is thus reflected in the fractal tongues as can be seen in the phase diagram for golden mean ratio as well as for silver mean ratio. However for silver mean ratio the scaling of depths found to be

\[ d_N \sim \frac{C}{N^{\theta_l N}} \]

as shown in Fig. (4.18). On the other hand, the widths must be considered in families labeled along a consistent branch of the Cayley tree. For example, for \( \omega = \sqrt{2} - 1 \) (see Fig. (4.16b)), the gaps 1,2,5,12,29,\ldots, the rightmost nodes at each generation form one family, while gaps 1,6,35,204,\ldots, the leftmost nodes, form another family, for each of which the widths scale, at \( \epsilon_c \), as

\[ w_N \sim \frac{1}{N^{\theta_r}} \]

the exponent \( \theta \) being particular to a given family of gaps. The two families of gaps enumerated above are the fastest and slowest decreasing, respectively, with exponents \( \theta_l \approx 2.05 \) and \( \theta_r \approx 2.31 \). For all other families, the exponents lie within the limits \( [\theta_l, \theta_r] \).

Wiegmann et al. [158] have studied the tight-binding Harper Hamiltonian for which they obtain the gap distribution, namely the number of gaps lying between \( D \) and \( D + dD \). They find that this quantity, \( \rho(D) \sim D^{-\gamma} \) where
γ is found to be approximately 3/2; similar exponents have been reported by Geisel et al. [42, 43] as well. If the gaps are rank-ordered, namely from largest downwards, disregarding the gap indices, then we observe that they decrease as

\[ w_r \sim \frac{1}{r^\mu} \]  

(4.17)

\[ \mu \approx 2, \] which is consistent with the power-laws obtained earlier [1, 42, 43, 158].