Chapter 2

First Order Neutral Difference Equation

2. First Order Neutral Difference Equation

2.1 Introduction

In this chapter, we discuss the existence of nonoscillatory solutions of first order nonlinear neutral difference equations of the form

$$\Delta\left((x(n) - p(n)x(n-\tau))^{\alpha}\right) + Q(n)G(x(n-\sigma)) = 0, \quad n \in N_0, \tag{2.1.1}$$

$$\Delta\left((x(n) - p(n)x(n-\tau))^{\alpha}\right) + \sum_{s=c}^{d} Q(n,s)G(x(n-s)) = 0, \quad n \in N_0, \tag{2.1.2}$$

and

$$\Delta\left[x(n) - \sum_{s=a}^{b} p(n,s)x(n-s)\right]^{\alpha} + \sum_{s=c}^{d} Q(n,s)G(x(n-s)) = 0 \tag{2.1.3}$$

for $n \in N_0$, subject to the following conditions:

(C_1) α is a ratio of odd positive integers;

(C_2) σ, τ, a, b, c, and d are positive integers with $a < b$ and $c < d$;

(C_3) $\{p(n)\}$, $\{Q(n)\}$ and $\{Q(n,s)\}$ are nonnegative real sequences;

(C_4) $G(x)$ is a positive continuous real valued function with $xG(x) > 0$ for $x = 0$.

Let $\theta = max\{\tau, \sigma\}$. By a solution of equations (2.1.1)-(2.1.3), we mean a real sequence $\{x(n)\}$ defined and satisfying equations (2.1.1)-(2.1.3) for all $n \in N_0$.

In recent years, there has been much research concerning the oscillation of first order neutral delay difference equations, see for example [1, 3, 4, 13, 35, 55, 56, 66, 83] and the references cited therein. In [2, 3, 15, 23, 24, 53, 68, 71], the authors investigated the existence of nonoscillatory solutions of first order difference equations.

Following this trend, in this chapter we obtain some new sufficient conditions for the existence of nonoscillatory solutions of equations (2.1.1)-(2.1.3).

In Section 2.2, we establish some sufficient conditions for the existence of nonoscillatory solutions of equations (2.1.1)-(2.1.3) and in Section 2.3, we present some examples to illustrate the main results. The results established in this chapter generalize and complement to those given in [68, 70].

2.2 Nonoscillation Theorems

In this section, we present some sufficient conditions for existence of bounded nonoscillatory solutions of equations (2.1.1) - (2.1.3). We begin with the following lemma.

Lemma 2.2.1. (Knaster-Tarski Fixed Point Theorem)
Let B be a partially ordered Banach space with ordering $\leq$. Let M be a subset of B with the following properties: the infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $T : M \to M$ be an increasing mapping, that is, $x \leq y$ implies $Tx \leq Ty$. Then T has a fixed point in M.

The proof of Lemma 2.2.1 can be found in [4].

Theorem 2.2.1. *Assume that $0 \leq p(n) \leq p < 1$, G is nondecreasing and*

$$\sum_{n=n_0}^{\infty} Q(n) < \infty, \tag{2.2.1}$$

then equation (2.1.1) has a bounded nonoscillatory solution.

Proof. Let B be the set of all bounded real valued sequences with the supremum norm,

$$||x|| = \sup_{x_n \in B} |x_n| < \infty.$$

Then clearly B is a Banach space. We can define a partial ordering as follows: for given $x_1, x_2 \in B$, $x_1 \leq x_2$ means that $x_1(n) \leq x_2(n)$ for $n \geq n_0 \in \mathbb{N}_0$. Define

$$S = \{x \in B : C_1 \leq x(n) \leq C_2,\ n \geq n_0\},$$

where C_1 and C_2 are positive constants such that

$$C_1 \le \beta < (1-p)C_2.$$

If $\tilde{}$ $x_1 \in S$ and $x_1 = \inf S$. In addition, if $\emptyset \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \le x(n) \le \mu,\ C_1 \le \lambda,\ \mu \le C_2;\ n \ge n_0\}.$$

Let $\tilde{}$ $x_2 \in S$ and $x_2 = \sup S^*$.

From the condition (2.2.1) there exists $n_1 > n_0$ with

$$n_1 \ge n_0 + max\{\tau,\ \sigma\} \tag{2.2.2}$$

sufficiently large that

$$\sum_{s=n}^{\infty} Q(s) \le \frac{[(1-p)C_2]^{\alpha} - \beta^{\alpha}}{G(C_2)},\ n \ge n_1. \tag{2.2.3}$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} p(n)x(n-\tau) + [\beta^{\alpha} + \sum_{s=n}^{\infty} Q(s)G(x(s-\sigma))]^{1/\alpha}, & n \ge n_1 \\ (Tx_1)(n), & n_0 \le n \le n_1. \end{cases}$$

For $n \ge n_1$ and $x \in S$, by making use of (2.2.3), we obtain

$$\begin{aligned}(Tx)(n) &\le pC_2 + \left[\beta^{\alpha} + G(C_2)\sum_{s=n}^{\infty} Q(s)\right]^{1/\alpha} \\ &\le pC_2 + \left[\beta^{\alpha} + G(C_2)\frac{[(1-p)C_2]^{\alpha} - \beta^{\alpha}}{G(C_2)}\right]^{1/\alpha} \\ &\le pC_2 + [[(1-p)C_2]^{\alpha}]^{1/\alpha} \\ &\le C_2,\end{aligned}$$

and

$$(Tx)(n) \ge \beta \ge C_1.$$

Hence $Tx \in S$ for every $x \in S$. Let $x_1,\ x_2 \in S$ with $x_1 \le x_2$. Since G is nondecreasing, $Tx_1 \le Tx_2$, that is, T is an increasing mapping. Then by the

Knaster-Tarski fixed point theorem, there exists a positive $x \in S$ such that $Tx = x$. Thus $\{x(n)\}$ is a bounded nonoscillatory solution of equation (2.1.1), which complete the proof. □

Theorem 2.2.2. *Assume that* $1 < p \leq p(n) \leq p_0 < \infty$, *G is nonincreasing and* (2.2.1) *holds, then equation* (2.1.1) *has a bounded nonoscillatory solution.*

Proof. Let B be a Banach space as defined in Theorem 2.2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B$, $x_1 \leq x_2$ means that $x_1(n) \leq x_2(n)$ for $n \geq n_0 \in N_0$. Define

$$S = \{x \in B : C_3 \leq x(n) \leq C_4,\ n \geq n_0\},$$

where C_3 and C_4 are positive constants such that

$$(p_0 - 1)C_3 < \beta \leq (p - 1)C_4.$$

If $\tilde{}$ $x_1 \in S$ and $x_1 = \inf S$. In addition, if $\emptyset \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \leq x(n) \leq \mu,\ C_3 \leq \lambda,\ \mu \leq C_4;\ n \geq n_0\}.$$

Let $\tilde{}$ $x_2 \in S$ and $x_2 = \sup S^*$.

From the condition (2.2.1) there exists $n_1 > n_0$ with

$$n_1 + \tau \geq n_0 + \sigma \qquad (2.2.4)$$

sufficiently large that

$$\sum_{s=n}^{\infty} Q(s) \leq \frac{\beta^\alpha - [(p_0 - 1)C_3]^\alpha}{G(C_3)},\ n \geq n_1. \qquad (2.2.5)$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} \frac{1}{p(n+\tau)} + [x(n+\tau) + (\beta^\alpha - \sum_{s=n+\tau}^{\infty} Q(s)G(x(s-\sigma)))^{1/\alpha}], & n \geq n_1 \\ (Tx_1)(n), & n_0 \leq n \leq n_1. \end{cases}$$

For $n \geq n_1$ and $x \in S$ by using (2.2.5), we obtain

$$\begin{aligned}(Tx)(n) &\leq \frac{1}{p}\left[C_4 + (\beta^{\alpha})^{1/\alpha}\right] \\ &\leq \frac{1}{p}[C_4 + \beta] \\ &\leq \frac{1}{p}[C_4 + (1-p)C_4] \\ &\leq C_4,\end{aligned}$$

and

$$\begin{aligned}(Tx)(n) &\geq \frac{1}{p(n+\tau)}\left\{C_3 + \left[\beta^{\alpha} - G(C_3)\sum_{s=n+\tau}^{\infty} Q(s)\right]^{1/\alpha}\right\} \\ &\geq \frac{1}{p(n+\tau)}\left[C_3 + \left(\beta^{\alpha} - G(C_3)\frac{\beta^{\alpha} - [(p_0-1)C_3]^{\alpha}}{G(C_3)}\right)^{1/\alpha}\right] \\ &\geq \frac{1}{p(n+\tau)}\left[C_3 + (\beta^{\alpha} - \beta^{\alpha} + [(p_0-1)C_3]^{\alpha})^{1/\alpha}\right] \\ &\geq \frac{1}{p(n+\tau)}[C_3 + p_0C_3 - C_3] \\ (Tx)(n) &\geq C_3.\end{aligned}$$

Thus, $Tx \in S$ for every $x \in S$. Let $x_1, x_2 \in S$ with $x_1 \leq x_2$ and since G is nonincreasing, $Tx_1 \leq Tx_2$, that is, T is an increasing mapping. Then by the Knaster-Tarski fixed point theorem, there exists a positive $x \in S$ such that $Tx = x$. Thus $\{x(n)\}$ is a bounded nonoscillatory solution of equation (2.1.1), which complete the proof. □

Theorem 2.2.3. *Assume that* $0 \leq p(n) \leq p < 1$, *G is nondecreasing and*

$$\sum_{n=n_0}^{\infty}\sum_{s=c}^{d} Q(n,s) < \infty, \tag{2.2.6}$$

then equation (2.1.2) *has a bounded nonoscillatory solution.*

Proof. Let B be a Banach space as defined in Theorem 2.2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B$, $x_1 \leq x_2$ means that $x_1(n) \leq x_2(n)$

for $n \geq n_0 \in \mathbb{N}_0$. Define

$$S = \{x \in B : C_5 \leq x(n) \leq C_6,\ n \geq n_0\},$$

where C_5 and C_6 are positive constants such that

$$C_5 \leq \beta < (p-1)C_6.$$

If $\tilde{}$ $x_1 \in S$ and $x_1 = \inf S$. In addition, if $\emptyset \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \leq x(n) \leq \mu,\ C_5 \leq \lambda,\ \mu \leq C_6;\ n \geq n_0\}.$$

Let $\tilde{}$ $x_2 \in S$ and $x_2 = \sup S^*$.

From the condition (2.2.6) there exists $n_1 > n_0$ with

$$n_1 \leq n_0 + max\{\tau,\ d\}$$

sufficiently large that

$$\sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i) \leq \frac{[(1-p)C_6]^{\alpha} - \beta^{\alpha}}{G(C_6)},\ n \geq n_1.$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} p(n)x(n-\tau) + \left[\beta^{\alpha} + \sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i)G(x(s-i))\right]^{1/\alpha}, & n \geq n_1 \\ Tx_1(n), & n_0 \leq n \leq n_1. \end{cases}$$

The remaining part of the proof is similar to that of Theorem 2.2.1, and hence the details are omitted. □

Theorem 2.2.4. *Assume that* $1 < p \leq p(n) \leq p_0 < \infty$, *G is nonincreasing and* (2.2.6) *holds, then equation* (2.1.2) *has a bounded nonoscillatory solution.*

Proof. Let B be a Banach space as defined in Theorem 2.2.1. We can define a partial ordering as follows: for given $x_1,\ x_2 \in B,\ x_1 \leq x_2$ means that $x_1(n) \leq x_2(n)$ for $n \geq n_0 \in \mathbb{N}_0$. Define

$$S = \{x \in B : C_7 \leq x(n) \leq C_8,\ n \geq n_0\},$$

where C_7 and C_8 are positive constants such that

$$(p_0 - 1)C_7 < \beta \le (p - 1)C_8.$$

If $\tilde{}$ $x_1 \in S$ and $x_1 = \inf S$. In addition, if $\emptyset \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \le x(n) \le \mu,\ C_7 \le \lambda,\ \mu \le C_8;\ n \ge n_0\}.$$

Let $\tilde{}$ $x_2 \in S$ and $x_2 = \sup S^*$.

From the condition (2.2.6) there exists $n_1 > n_0$ with

$$n_1 + \tau \ge n_0 + d$$

sufficiently large that

$$\sum_{s=n+\tau}^{\infty} \sum_{i=c}^{d} Q(s, i) \le \frac{\beta^\alpha - [(p_0 - 1)C_7]^\alpha}{G(C_7)},\ n \ge n_1.$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} \frac{1}{p(n+\tau)} + [x(n + \tau) + \left(\beta^\alpha - \sum_{s=n+\tau}^{\infty} \sum_{i=c}^{d} Q(s, i)G(x(s - i))\right)^{\frac{1}{\alpha}}], & n \ge n_1 \\ (Tx_1)(n), & n_0 \le n \le n_1. \end{cases}$$

The remaining part of the proof is similar to that of Theorem 2.2.2, and hence the details are omitted. □

Theorem 2.2.5. *Assume that* $0 \le \sum_{s=a}^{b} p(n, s) \le p < 1$, *G is nondecreasing and* (2.2.6) *holds, then equation* (2.1.3) *has a bounded nonoscillatory solution.*

Proof. Let B be a Banach space as defined in Theorem 2.2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B$, $x_1 \le x_2$ means that $x_1(n) \le x_2(n)$ for $n \ge n_0 \in \mathbb{N}_0$. Define

$$S = \{x \in B : C_9 \le x(n) \le C_{10},\ n \ge n_0\},$$

where C_9 and C_{10} are positive constants such that

$$C_9 \le \beta < (1-p)C_{10}.$$

If $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\emptyset \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \le x(n) \le \mu, \; C_9 \le \lambda, \; \mu \le C_{10}; \; n \ge n_0\}.$$

Let $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$.

From the condition (2.2.6) there exists $n_1 > n_0$ with

$$n_1 \ge n_0 + max\{b, d\}$$

sufficiently large that

$$\sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i) \le \frac{[(1-p)C_{10}]^\alpha - \beta^\alpha}{G(C_{10})}, \quad n \ge n_1.$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} -\sum_{s=a}^{b} p(n,s)x(n-s) \\ \quad + \left(\beta^\alpha + \sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i)G(x(s-i))\right)^{1/\alpha}, & n \ge n_1 \\ (Tx_1)(n), & n_0 \le n \le n_1. \end{cases}$$

The remaining part of the proof is similar to that of Theorem 2.2.1, and hence the details are omitted. □

2.3 Examples

In this section, we present some examples to illustrate the main results.

Example 2.3.1. *Consider the difference equation*

$$\Delta\left(x(n) - \frac{1}{4}x(n-1)\right)^3 + \frac{7}{2^{2n+8}}x(n-2) = 0, \quad n \ge 0. \qquad (2.3.1)$$

Here $\alpha = 3$, $p(n) = \frac{1}{4}$, $Q(n) = \frac{7}{2^{2n+8}}$, *and* $\tau = 1$, $\sigma = 2$. *By taking* $G(x) = x$, *we see that* $\sum_{n=1}^{\infty} Q(n) < \infty$. *Further it is easy to verify that all other conditions of Theorem 2.2.1 are satisfied. Therefore the equation* (2.3.1) *has a bounded nonoscillatory solution. In fact,* $\{x(n)\} = \{2^{-n}\}$ *is one such solution of equation* (2.3.1).

Example 2.3.2. *Consider the difference equations*

$$\Delta\left(x(n) - \frac{1}{2}x(n-3)\right)^{3} + \sum_{s=1}^{2} \frac{1}{n+s}x(n-s) = 0, \quad n \geq 0. \tag{2.3.2}$$

Here $\alpha = 3$, $p(n) = \frac{1}{2}$, $Q(n, s) = \frac{1}{n+s}$, $\tau = 3$ *and* $c = 1$, $d = 2$. *By taking* $G(x) = x$, *we see that all conditions of Theorem 2.2.3 are satisfied and hance every solution of equation* (2.3.2) *has a bounded nonoscillatory.*

Example 2.3.3. *Consider the difference equations*

$$\Delta\left(x(n) - \sum_{s=1}^{2} \frac{1}{2(n+s-1)}x(n-s)\right) + \sum_{s=2}^{3} \frac{1}{(n+s)^2}x(n-s) = 0, \quad n \geq 0. \tag{2.3.3}$$

Here $\alpha = 1$, $p(n) = \frac{1}{2(n+s-1)}$, $Q(n, s) = \frac{1}{(n+s)^2}$, $a = 1$, $b = 2$ *and* $c = 2$, $d = 3$. *By taking* $G(x) = x$, *we see that all conditions of Theorem 2.2.5 are satisfied and hance every solution of equation* (2.3.3) *has a bounded nonoscillatory.*

We conclude this chapter with the following remark.

Remark 2.3.1. *It would be interesting to obtain conditions for the oscillation of all solutions of equations* (2.1.1)-(2.1.3).