Chapter 1

Introduction

1.Introduction

1.1 Difference Equations

The dynamical systems in which time independent variable having only a discrete set of values often lead to mathematical models of discrete event dynamical systems which are also called difference equations. In the past few decades the study of difference equations has already drawn a great deal of attention, not only among mathematicians, but from various other disciplines as well. This comes out, in large part, from the use of these equations in the formulation and analysis of discrete time systems, the numerical integration of differential equations by difference schemes, and several other fields. For the basic theory of difference equations and its applications to various fields, one can refer the monographs by Agarwal [1], and Agarwal et al. [3, 4].

Before giving a formal definition to difference equation, let us introduce the following notations which are used in this thesis:

$$\mathrm{N}_0 = \{0,\ 1,\ 2,\ .\ .\ .\},$$

and

$$\mathrm{N}(n_0) = \{n_0,\ n_0 + 1,\ n_0 + 2,\ .\ .\ .\},\quad n_0 \in \mathrm{N}_0.$$

A difference equation of order m is defined as a functional relation of the form

$$F(n, x_n, x_{n+1},\ ..., x_{n+m}) = 0,\quad n \in \mathrm{N}_0 \tag{1.1.1}$$

or

$$\Delta^m x_n + f(n, x_n, \Delta x_n, \cdots, \Delta^{m-1} x_n) = 0,\quad n \in \mathrm{N}_0 \tag{1.1.2}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and in general $\Delta^i x_n = \Delta(\Delta^{i-1} x_n)$, $i = 2, 3, \cdots, m$. A function x_n defined on N_0 is said to be a solution of the difference equation (1.1.1) or (1.1.2) if the values of x_n reduce the equation (1.1.1) or (1.1.2) to an identity over N_0.

As in the case of differential equations, the difference equations can also be classified as ordinary, delay, advanced and neutral type difference equations.

A difference equation of the form

$$\Delta^m x_n + f(n, x_n, x_{n+1}, \cdots, x_{n+m}) = 0, \quad n \in N_0, \tag{1.1.3}$$

is called an *ordinary difference equation*.

An equation of the form

$$\Delta^m x_n + f(n, x_{n-A}) = 0, \quad n \in N_0, \tag{1.1.4}$$

is called a *delay difference equation* if A is a positive integer and *advanced type* if A is a negative integer.

An equation of the form

$$\Delta^m (x_n + p_n x_{n-k}) + f(n, x_{n-A}) = 0, \quad n \in N_0, \tag{1.1.5}$$

where k and A are integers, is called a difference equation of *neutral type*.

1.2 Motivation

Most of the difference equations appearing in real world problem are nonlinear in nature, and it is a well known fact that these equations cannot be solvable to get a closed form solutions. In the absence of closed form solutions, a rewarding alternative is to study the qualitative behavior of solutions of these equations.

The origin of the large part of the qualitative theory of difference equations may be traced back to the fundamental work of Poincare at the end of the nineteenth century, see [1] and the references cited therein. In the qualitative theory of difference equations oscillatory behavior of solutions play an important role. A non trivial solution of a difference equation is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise. These types of solutions occur in many physical phenomena such as, vibrating mechanical systems, electrical circuits and in population dynamics.

The literature on the oscillation and nonoscillation of solutions of nonlinear delay and neutral type difference equations have grown to such an extent that it is impossible to mention all the authors who had contributed on this subject. For example, Agarwal et al. [1–10], Aktas et al. [11], Arul et al. [12–15], Bolat et al. [17, 18], Budincevic [19], Candan et al. [20], Chatzarakis et al. [21, 22], Chen et al. [23], S.S.Chang et al. [24, 25], Dosla et al. [26], Elabbasy et al. [27], Elayadi [28], Grace et al. [29–33], Graef et al. [34, 35], Gyori et al. [36], Han et al. [37], Hartman [38], Hooker and Patula [40], Jiang et al. [41], Jinfa [42], Karpuz et al. [43], Kelley et al. [44], Koplatadze et al. [45, 46], Ladas et al [47], Lalli et al. [48], H.J.Li et al. [49], W.T.Li et al. [50–52], Lourdu Marian et al. [53], Migda [54], Ocalan et al. [55], Parhi et al. [56], Popenda et al. [57], Saker et al. [58–60], Sandra Pinelas et al. [61], Schmeidel [62], Smith [63], Sun et al. [64], Tang et al. [65], Thandapani et al. [66–79], J.Wang et al. [80], D.M. Wang et al. [81], X.Wang et al. [82], Wei [83], Yildiz et al. [84], Zafer et al. [85, 86], B.G.Zhang et al. [87, 88], C.Zhang [89], G.Zhang [90, 91], and Zjou et al. [92] have done extensive work on this topic.

This motivated us to study the oscillation and nonoscillation of solutions of nonlinear delay and neutral type difference equations. The study of delay and neutral type difference equations is both an interesting and useful area of research since such type of equations arise in population dynamics when maturation and gestation are included, in cobweb models in economics when demand depends on current price but supply depends on the price at an earlier time and in electric networks containing lossless transmission lines.

Keeping in view of the importance of the subject, and in the light of the above trend, we obtained some significant results on the following topics:

1. **Existence of nonoscillatory solutions of first order nonlinear neutral difference equations;**

2. **Oscillation of second order neutral delay difference equation;**

3. **Oscillation of third order nonlinear difference equation;**

4. **Oscillation criteria for third order nonlinear delay difference equation;**

5. **Oscillatory behavior of even order quasilinear delay difference equation.**

1.3 Plan of the Thesis

This thesis consists of six chapters including this introductory chapter.

In Chapter 2, we discuss the existence of nonoscillatory solutions of first order nonlinear neutral difference equations of the form

$$\Delta((x(n) - p(n)x(n-\tau))^{\alpha}) + Q(n)G(x(n-\sigma)) = 0, \qquad (1.3.1)$$

$$\Delta((x(n) - p(n)x(n-\tau))^{\alpha}) + \sum_{s=c}^{d} Q(n,s)G(x(n-s)) = 0, \qquad (1.3.2)$$

and

$$\Delta\left[x(n) - \sum_{s=a}^{b} p(n,s)x(n-s)\right]^{\alpha} + \sum_{s=c}^{d} Q(n,s)G(x(n-s)) = 0, \qquad (1.3.3)$$

for $n \in \mathbb{N}_0$, where σ, τ, a, b, c, and d are positive integers with $a < b$ and $c < d$, $\{p(n)\}$, $\{Q(n)\}$ and $\{Q(n,s)\}$ are nonnegative real sequences, $G(x)$ is a positive continuous real valued function with $xG(x) > 0$ for $x = 0$, and α is a ratio of odd positive integers. Section 2.1 presents necessary introduction and motivation. In Section 2.2, we establish some sufficient conditions for the existence of nonoscillatory solutions of equations (1.3.1)-(1.3.3), and in Section 2.3 we present some examples to illustrate the main results. The results established in this chapter generalize and complement to those given in [68, 70].

In Chapter 3, we are concerned with the oscillatory behavior of second order neutral delay difference equation of the form

$$\Delta\left(a_n\Delta(x_n + p_n x_{\tau(n)})\right) + q_n x_{\sigma(n)} = 0, \quad n \in \mathbb{N}_0, \tag{1.3.4}$$

where $\{q_n\}$ and $\{p_n\}$ are nonnegative real sequences with $0 \le p_n \le p_0 < \infty$, $\{a_n\}$ is positive real sequence with $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$, and $\{\tau(n)\}$ and $\{\sigma(n)\}$ are sequences of integers such that $\lim_{n\to\infty} \tau(n) = \infty$ and $\lim_{n\to\infty} \sigma(n) = \infty$. Section 3.1, presents necessary introduction and motivation. In Section 3.2, we establish some sufficient conditions for the oscillation of all solutions of equation (1.3.4) and some examples are given in Section 3.3 to illustrate the main results. The results obtained in this chapter complement and extend the results established in [1, 10, 12, 16, 49, 58, 67, 69].

In Chapter 4, we study the oscillatory behavior of third order nonlinear delay difference equation of the form

$$\Delta\left(a_n(\Delta^2 x_n)^{\alpha}\right) + q_n f(x_{\sigma(n)}) = 0, n \in \mathbb{N}_0, \tag{1.3.5}$$

where $\{q_n\}$ and $\{a_n\}$ are nonnegative real sequences and $\sum_{s=n_0}^{\infty} \frac{1}{a_s^{1/\alpha}} = \infty$, α is a ratio of odd positive integers, $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing function such that $uf(u) > 0$ for $u = 0$ and $-f(-uv) \ge f(uv) \ge f(u)f(v)$ for $uv > 0$ and $\{\sigma(n)\}$ is a sequence of positive integer such that $\sigma(n) \le n$ and $\lim_{n\to\infty} \sigma(n) = \infty$. In Section 5.1, necessary introduction and motivation are provided and in Section 4.2, we establish some sufficient conditions for the oscillation of all solutions of equation (1.3.5). Examples are given in Section 4.3 to illustrate the main results. The results presented in this chapter generalize and improve those obtained in [3, 34, 60, 62, 73].

Chapter 5, deals with the oscillation of third order nonlinear delay difference equation of the form

$$\Delta\left(a_n\left(\Delta\left(b_n(\Delta x_n)^{\alpha}\right)\right)^{\beta}\right) + q_n f(x_{n-\tau}) = 0, n \in N_0, \qquad (1.3.6)$$

where $\{a_n\}$, $\{b_n\}$ and $\{q_n\}$ are positive real sequences, α and β are ratio of odd positive integers, τ is a nonnegative integer and $f : R \to R$ is continuous function such that $uf(u) > 0$ for $u = 0$ and $-f(-uv) \geq f(uv) \geq f(u)f(v)$ for $uv > 0$. In Section 5.1, necessary introduction and motivation are provided, In Section 5.2, we establish some sufficient conditions for the oscillation of all solutions of equation (1.3.6) and some examples are given in Section 5.3 to illustrate the main results. The results obtained in this chapter complement and generalize the results established in [34, 60, 62].

Finally, in Chapter 6, we discuss the oscillatory behavior of even order quasilinear delay difference equation of the form

$$\Delta\left(a_n(\Delta^{m-1}x_n)^{\alpha}\right) + q_n x^{\beta}_{\tau_n} = 0, n \in N_0 \qquad (1.3.7)$$

where α and β are the ratio of odd positive integers, $\{q_n\}$ and $\{a_n\}$ are positive real sequences with $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\alpha}} < \infty$ and $\Delta a_n \geq 0$ for all $n \in N(n_0)$ and $\{\tau_n\}$ is a nondecreasing sequence of integers and $\tau_n \leq n$ with $\tau_n \to \infty$ as $n \to \infty$. Section 6.1 provides necessary introduction and motivation. In Section 6.2, we establish some sufficient conditions for the oscillation of all solutions of equation (1.3.7) and in Section 6.3, we present some examples to illustrate the main results. The results established in this chapter improve and complement to those given in [17, 18, 54, 84].

Thus, we have obtained some new results, improve, generalize, and extended some of the existing results on the oscillatory and asymptotic behavior of delay and neutral difference equations. Further, examples are presented in the text to illustrate the importance of the results.