Chapter 5

Third Order Nonlinear Delay Difference Equation-II

5. Third Order Nonlinear Delay Difference Equation-II

5.1 Introduction

In this chapter, we continue our study on the oscillation of more general third order nonlinear delay difference equation of the form

$$\Delta \left(a_n (\Delta (b_n(\Delta x_n)^{\alpha}))^{\beta} \right) + q_n \boldsymbol{f}(x_{n-\tau}) = 0, n \in \mathrm{N}_0, \qquad (5.1.1)$$

subject to the following conditions:

(C_1) $\{a_n\}$, $\{b_n\}$ and $\{q_n\}$ are positive real sequences;

(C_2) α and β are ratio of odd positive integers;

(C_3) τ is a nonnegative integer;

(C_4) $\boldsymbol{f} : \mathrm{R} \to \mathrm{R}$ is continuous functions such that $uf(u) > 0$ for $u = 0$ and $-\boldsymbol{f}(-uv) \geq \boldsymbol{f}(uv) \geq \boldsymbol{f}(u)\boldsymbol{f}(v)$ for $uv > 0$.

By a solution of equation (5.1.1), we mean a real sequence $\{x_n\}$ and satisfying equation (5.1.1) for all $n \in \mathrm{N}_0$. We consider only those solution $\{x_n\}$ of equation (5.1.1) which satisfy $\sup\{|x_n| : n \geq N\} > 0$ for all $n \geq N \in \mathrm{N}_0$.

In [34, 62], the authors studied the third difference equation (5.1.1) when $\alpha = \beta = 1$ and established some oscillation and asymptotic behavior of solution, and in [60], the authors considered the oscillatory and asymptotic behavior of solutions of the equation (5.1.1) when $\beta = 1$. Motivated by this observation, in this chapter we obtain some sufficient conditions for the oscillation of all solutions of equation (5.1.1).

In Section 5.2, we establish some sufficient conditions for the oscillation of all solutions of equation (5.1.1) and in Section 5.3, we present some examples to illustrate the main results. The results obtained in this chapter complement and generalize the results established in [34, 60, 62].

5.2 Oscillation Theorems

In this section, we establish some new oscillation theorem for equation (5.1.1). Throughout this chapter we use the following notations without further mention:

$$\delta_{n,n_0} = \sum_{s=n_0}^{n-1} b_s^{-1/\alpha},$$

$$\delta_n = \sum_{s=n_0}^{n} a_s^{-1/\beta},$$

and

$$\bar{\delta}_n = \sum_{s=n_0}^{n} b_s^{-1/\beta}.$$

We begin with the following lemma.

Lemma 5.2.1. *Assume that for all sufficiently large $N_1 \in \mathbb{N}_0$, there is a $N > N_1$ such that $n - \tau > N_1$ for $n \geq N$ and*

(H_1) *either*

$$\sum_{s=n_0}^{\infty} \frac{1}{a_s^{1/\beta}} = \infty \tag{5.2.1}$$

or

$$\sum_{n=N}^{\infty} a_n^{-1/\beta} \left(\sum_{s=N}^{n} q_s \boldsymbol{f}(\bar{\delta}_{s-\tau}^{1/\alpha}) \boldsymbol{f}(\delta_{s-\tau,N}) \right)^{1/\beta} = \infty; \tag{5.2.2}$$

(H_2) *either*

$$\sum_{s=n_0}^{\infty} \frac{1}{b_s^{1/\alpha}} = \infty \tag{5.2.3}$$

or

$$\sum_{n=n_0}^{\infty} b_n^{-1/\alpha} \left[\sum_{s=n_0}^{n} a_s^{-1/\beta} \left(\sum_{t=s_0}^{s} q_t \boldsymbol{f}(\bar{\delta}_{s-\tau}) \right)^{1/\beta} \right]^{1/\alpha} = \infty \tag{5.2.4}$$

hold. Let $\{x_n\}$ be an eventually positive solution of equation (5.1.1). *Then one of the following two cases holds:*

(i) $\Delta x_n > 0,\ \ \Delta(b_n(\Delta x_n)^{\alpha}) > 0$ *for all* $n \geq N$;

(ii) $\Delta x_n < 0,\ \ \Delta(b_n(\Delta x_n)^{\alpha}) > 0$ *for all* $n \geq N.$

Proof. Let $\{x_n\}$ be a positive solution, from the equation (5.1.1), we have

$$\Delta\left(a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}\right) = -q_n f(x_{n-\tau}) < 0 \ \text{ for } \ n \geq n_1.$$

Consequently $(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}$ is strictly decreasing and then Δx_n and $\Delta(b_n(\Delta x_n)^{\alpha})$ are eventually of one sign. We claim that $\Delta(b_n(\Delta x_n)^{\alpha}) > 0$. If not, then we have two cases:

Case(i). There exists $n_2 \geq n_1$, sufficiently large, such that

$$\Delta x_n > 0, \ \text{ and } \ \Delta(b_n(\Delta x_n)^{\alpha}) < 0 \ \text{ for } \ n \geq n_2.$$

Case(ii). There exists $n_2 \geq n_1$, sufficiently large, such that

$$\Delta x_n < 0, \ \text{ and } \ \Delta(b_n(\Delta x_n)^{\alpha}) < 0 \ \text{ for } \ n \geq n_2.$$

For the Case(i), we have $b_n(\Delta x_n)^{\alpha}$ is strictly decreasing and there exists a negative constant M such that

$$a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta} < M \quad \text{for all} \quad n \geq n_2$$

or

$$\Delta(b_n(\Delta x_n)^{\alpha}) < \frac{M^{1/\beta}}{a_n^{1/\beta}}.$$

Summing from n_2 to $n-1$, we get

$$b_n(\Delta x_n)^{\alpha} \leq b_{n_2}(\Delta x_{n_2})^{\alpha} + M^{1/\beta}\sum_{s=n_2}^{n-1}\frac{1}{a_s^{1/\beta}}.$$

Letting $n \to \infty$ and using (5.2.1) then $b_n(\Delta x_n)^\alpha \to -\infty$, which contradicts that $\Delta x_n > 0$. Hence (5.2.2) is satisfied, we have

$$\begin{aligned} x_n - x_{n_3} &= \sum_{s=n_3}^{n-1} \Delta x_s \\ &= \sum_{s=n_3}^{n-1} b_s^{-1/\alpha}((\Delta x_s)^\alpha)^{1/\alpha} b_s^{1/\alpha} \\ &\geq (b_n(\Delta x_n)^\alpha)^{1/\alpha} \sum_{s=n_3}^{n-1} b_s^{-1/\alpha} \quad \text{for all} \quad n \geq n_3, \end{aligned}$$

and hence

$$\begin{aligned} x_n &\geq (b_n(\Delta x_n)^\alpha)^{1/\alpha} \sum_{s=n_3}^{n-1} b_s^{-1/\alpha} \quad \text{for all} \quad n \geq n_3 \\ &\geq (b_n(\Delta x_n)^\alpha)^{1/\alpha} \delta_{n,n_3} \quad \text{for all} \quad n \geq n_3. \end{aligned}$$

There exists a $n_4 \geq n_3$ with $n - \tau \geq n_3$ for all $n \geq n_4$ such that

$$x_{n-\tau} \geq (b_{n-\tau}(\Delta x_{n-\tau})^\alpha)^{1/\alpha} \delta_{n-\tau,n_3} \quad \text{for all} \quad n \geq n_4.$$

From equation (5.1.1)

$$0 \geq \Delta(a_n(\Delta y_n)^\beta) + q_n \boldsymbol{f}(y_{n-\tau}^{1/\alpha}) \boldsymbol{f}(\delta_{n-\tau,n_3}) \quad \text{for all} \quad n \geq n_4 \tag{5.2.5}$$

where $y_n = b_n(\Delta x_n)^\alpha$. It is clear that $y_n > 0$ and $\Delta y_n < 0$. It follows that

$$\Delta(a_n(\Delta y_n)^\beta) \leq 0 \quad \text{for all} \quad n \geq n_4.$$

Summing from $n-1$ to n_4, we get

$$a_n(\Delta y_n)^\beta - a_{n_4}(\Delta y_{n_4})^\beta \leq 0$$

or

$$-a_n(\Delta y_n)^\beta \geq -a_{n_4}(\Delta y_{n_4})^\beta$$

or

$$\Delta y_n \geq \frac{-a_{n_4}^{1/\beta}(\Delta y_{n_4})}{a_n^{1/\beta}} \quad \text{for all} \quad n \geq n_4.$$

Summing the last inequality from n to ∞, we obtain

$$-[y_\infty - y_n] \geq -\sum_{s=n}^{\infty} \frac{a_{s_4}^{1/\beta}(\Delta y_{s_4})}{a_s^{1/\beta}}$$

or

$$y_n \geq -a_{n_4}^{1/\beta}(\Delta y_{n_4}) \sum_{s=n}^{\infty} a_s^{-1/\beta}$$

hence

$$y_n \geq -a_{n_4}^{1/\beta}(\Delta y_{n_4})\delta_n$$

or

$$y_n \geq k_1\delta_n \quad \text{for all} \quad n \geq n_5$$

where $k_1 = -a_{n_4}^{1/\beta}(\Delta y_{n_4}) > 0$. There exists a $n_5 \geq n_4$ with $n - \tau \geq n_4$ for all $n \geq n_5$ such that

$$y_{n-\tau} \geq k_1\delta_{n-\tau} \quad \text{for all} \quad n \geq n_5.$$

Summing (5.2.5) from n_5 to $n-1$ and using the above inequality, we get

$$\sum_{s=n_5}^{n-1} q_s \boldsymbol{f}(y_{s-\tau}^{1/\alpha})\boldsymbol{f}(\delta_{s-\tau,n_3}) \leq a_{n_5}\Delta(y_{n_5})^{\beta} - a_n\Delta(y_n)^{\beta}$$

or

$$\sum_{s=n_5}^{n-1} q_s \boldsymbol{f}(k_1^{1/\alpha}\delta_{s-\tau}^{1/\alpha})\boldsymbol{f}(\delta_{s-\tau,n_3}) \leq -a_n\Delta(y_n)^{\beta}.$$

Now using the condition (C_4) we have

$$\sum_{s=n_5}^{n-1} q_s \boldsymbol{f}(k_1^{1/\alpha})\boldsymbol{f}(\delta_{s-\tau}^{1/\alpha})\boldsymbol{f}(\delta_{s-\tau,n_3}) \leq -a_n\Delta(y_n)^{\beta}$$

or

$$\left[\frac{\boldsymbol{f}(k_1^{1/\alpha})}{a_n}\sum_{s=n_5}^{n-1} q_s \boldsymbol{f}(\delta_{s-\tau}^{1/\alpha})\boldsymbol{f}(\delta_{s-\tau,n_3})\right]^{1/\beta} \leq -\Delta(y_n).$$

Summing the above inequality from n_5 to ∞, we get

$$(\boldsymbol{f}(k_1^{1/\alpha}))^{1/\beta} \sum_{s=n_5}^{\infty} \frac{1}{a_s^{1/\beta}}\left[\sum_{t=s_5}^{s-1} q_t \boldsymbol{f}(\delta_{t-\tau}^{1/\alpha})\boldsymbol{f}(\delta_{t-\tau,t_3})\right]^{1/\beta} \leq y_{n_5} < \infty$$

which contradicts the condition (5.2.2).

For the Case(ii), we have

$$b_n(\Delta x_n)^\alpha \le b_{n_2}(\Delta x_{n_2})^\alpha = c < 0$$

or

$$\Delta x_n \le \frac{c^{1/\alpha}}{b_n^{1/\alpha}}.$$

Summing the last inequality from n_2 to $n-1$, we get

$$x_n \le x_{n_2} + c^{1/\alpha} \sum_{s=n_2}^{n-1} b_s^{-1/\alpha}.$$

Letting $n \to \infty$, then (5.2.3) yields $x_n \to -\infty$. This contradicts that $x_n > 0$. Otherwise, if (5.2.4) is satisfied. One can choose $n_3 \ge n_2$ with $n - \tau \ge n_2$ for all $n \ge n_3$ such that

$$x_n \ge -(b_n(\Delta x_n)^\alpha)^{1/\alpha} \sum_{s=n_3}^{n-1} b_s^{-1/\alpha}$$

or

$$x_{n-\tau} \ge -(b_{n-\tau}(\Delta x_{n-\tau})^\alpha)^{1/\alpha} \sum_{s=n_3}^{n-\tau-1} b_s^{-1/\alpha}$$

hence

$$x_{n-\tau} \ge k_2 \bar{\delta}_{n-\tau} \quad \text{for} \quad n \ge n_3$$

where $k_2 = -(b_{n-\tau}(\Delta x_{n-\tau})^\alpha)^{1/\alpha}$. Then equation (5.1.1) and (C_4) yield.

$$\begin{aligned}
\Delta \left[a_n (\Delta (b_n(\Delta x_n)^\alpha))^\beta \right] &= -q_n f(x_{n-\tau}) \\
&\le -q_n f(k_2 \bar{\delta}_{n-\tau}) \\
&\le -q_n f(k_2) f(\bar{\delta}_{n-\tau})
\end{aligned}$$

or

$$\Delta \left[a_n (\Delta (b_n(\Delta x_n)^\alpha))^\beta \right] \le q_n L f(\bar{\delta}_{n-\tau}),$$

where $L = -f(k_2)$, summing the above inequality from n_3 to $n-1$, we get

$$a_n(\Delta\,(b_n(\Delta x_n)^{\alpha}))^{\beta} \le L\sum_{s=n_3}^{n-1} q_s f(\bar{\delta}_{s-\tau})$$

or

$$\Delta\,(b_n(\Delta x_n)^{\alpha}) \le \frac{L^{1/\beta}}{a_n^{1/\beta}}\left(\sum_{s=n_3}^{n-1} q_s f(\bar{\delta}_{s-\tau})\right)^{1/\beta}.$$

Summing the last inequality from n_3 to $n-1$, we have

$$b_n(\Delta x_n)^{\alpha} \le L^{1/\beta}\sum_{s=n_3}^{n-1} a_s^{-1/\beta}\left(\sum_{t=n_3}^{s-1} q_t f(\bar{\delta}_{t-\tau})\right)^{1/\beta}$$

or

$$\Delta x_n \le \frac{L^{1/\alpha\beta}}{b_n^{1/\alpha}}\left[\sum_{s=n_3}^{n-1} a_s^{-1/\beta}\left(\sum_{t=n_3}^{s-1} q_t f(\bar{\delta}_{t-\tau})\right)^{1/\beta}\right]^{1/\alpha}.$$

Again summing the last inequality from n_3 to $n-1$, we have

$$x_n \le L^{1/\alpha\beta}\sum_{s=n_3}^{n-1} b_s^{-1/\alpha}\left[\sum_{t=n_3}^{s-1} a_t^{-1/\beta}\left(\sum_{j=n_3}^{t-1} q_j f(\bar{\delta}_{j-\tau})\right)^{1/\beta}\right]^{1/\alpha}.$$

From condition (5.2.4) we get $x_n \to -\infty$ as $n \to \infty$ which contradicts that x_n is a positive solution of equation (5.1.1). Then we have $\Delta(b_n(\Delta x_n)^{\alpha}) > 0$ for $n \ge n_1$ and of one sign thus either $\Delta x_n > 0$ or $\Delta x_n < 0$. The proof is now complete. □

Lemma 5.2.2. *Assume that* (H_1) *and* (H_2) *hold. Let* $\{x_n\}$ *be an eventually positive solution of the equation* (5.1.1) *for all* $n \in \mathbb{N}_0$ *and suppose that Case (ii) of Lemma 5.2.1 holds. If*

$$\sum_{n=n_0}^{\infty} b_n^{-1/\alpha}\left(\sum_{s=n}^{\infty} a_s^{-1/\beta}\Big(\sum_{t=s}^{\infty} q_t\Big)^{1/\beta}\right)^{1/\alpha} = \infty \qquad (5.2.6)$$

then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a positive solution of the equation (5.1.1) then there exists $A \ge 0$ such that $\lim_{n\to\infty} x_n = A$. Assume $A > 0$, then we have $x_{n-\tau} \ge A$ for $n \ge n_2 \ge n_1$. Summing the equation (5.1.1) from n to ∞, we have

$$a_n(\Delta\,(b_n(\Delta x_n)^{\alpha}))^{\beta} \ge \sum_{s=n}^{\infty} q_s f(x_{s-\tau}) \ge f(x_{n-\tau})\sum_{s=n}^{\infty} q_s$$

or

$$\Delta\left(b_n(\Delta x_n)^{\alpha}\right) \ge \left(\frac{f(A)}{a_n}\right)^{1/\beta}\left(\sum_{s=n}^{\infty} q_s\right)^{1/\beta}.$$

Summing the last inequality from n to ∞, we have

$$-b_n(\Delta x_n)^{\alpha} \ge (f(A))^{1/\beta}\sum_{s=n}^{\infty} a_s^{-1/\beta}\left(\sum_{t=s}^{\infty} q_t\right)^{1/\beta}$$

or

$$-\Delta x_n \ge \frac{(f(A))^{1/\alpha\beta}}{b_n^{1/\alpha}}\left(\sum_{s=n}^{\infty} a_s^{-1/\beta}\left(\sum_{t=s}^{\infty} q_t\right)^{1/\beta}\right)^{1/\alpha}.$$

Again summing the last inequality from n_2 to ∞, we get

$$x_{n_2} \ge (f(A))^{1/\alpha\beta}\sum_{n=n_2}^{\infty} b_n^{-1/\alpha}\left(\sum_{s=n}^{\infty} a_s^{-1/\beta}\left(\sum_{t=s}^{\infty} q_t\right)^{1/\beta}\right)^{1/\alpha}.$$

This contradicts to the condition (5.2.6). The proof is complete. □

Theorem 5.2.1. *Let* (H_1) *and* (H_2) *hold and there exists an integer* σ *such that*

$$\sigma > \tau. \tag{5.2.7}$$

If both first order delay difference equations

$$\Delta y_n + q_n f(y_{n-\tau}^{1/\alpha\beta})\, f\left(\sum_{s=n_2}^{n-\tau-1} b_s^{-1/\alpha}\left(\sum_{t=n_2}^{s} a_t^{-1/\beta}\right)^{1/\alpha}\right) = 0, \tag{5.2.8}$$

and

$$\Delta x_n + (f(x_{n+2\sigma-\tau}))^{1/\alpha\beta} b_n^{-1/\alpha}\left(\sum_{s=n}^{n+\sigma} a_s^{-1/\beta}\left(\sum_{t=s}^{s+\sigma} q_t\right)^{1/\beta}\right)^{1/\alpha} = 0 \tag{5.2.9}$$

are oscillatory, then equation (5.1.1) *is oscillatory.*

Proof. Assume that (5.1.1) has a nonoscillatory solution. Without loss of generality, there is a $n_1 \ge n_0 \in \mathbb{N}_0$ sufficiently large such that $x_n > 0$ and $x_{n-\tau} > 0$ for all $n \ge n_1$. From the equation (5.1.1), we have

$$\Delta\left(a_n(\Delta(b_n(\Delta x_n)^{\alpha}))^{\beta}\right) = -q_n f(x_{n-\tau}) < 0 \quad \text{for all} \quad n \ge n_1.$$

Thus $a_n\Delta\,(b_n(\Delta x_n)^{\alpha})$ is strictly decreasing then $\Delta\,(b_n(\Delta x_n)^{\alpha})$ and Δx_n are eventually of one sign. Then from Lemma 5.2.1, we have the following cases for sufficiently large $n_2 \geq n_1$

(i) $\Delta x_n > 0,\ \ \Delta(b_n(\Delta x_n)^{\alpha}) > 0,$

(ii) $\Delta x_n < 0,\ \ \Delta(b_n(\Delta x_n)^{\alpha}) > 0.$

Case(i). Let $a_n(\Delta\ (b_n(\Delta x_n)^{\alpha})^{\beta} = y_n$, then we have

$$\Delta\,(b_n\,(\Delta x_n)^{\alpha}) = \frac{y_n^{1/\beta}}{a_n^{1/\beta}}.$$

Summing the last inequality from n_2 to $n-1$, we have

$$b_n(\Delta x_n)^{\alpha} = b_{n_2}(\Delta x_{n_2})^{\alpha} + \sum_{s=n_2}^{n-1} a_s^{-1/\beta} y_s^{1/\beta} \geq y_n^{1/\beta} \sum_{s=n_2}^{n-1} a_s^{-1/\beta}$$

or

$$\Delta x_n \geq y_n^{1/\alpha\beta}\frac{1}{b_n^{1/\alpha}}\left(\sum_{s=n_2}^{n-1} a_s^{-1/\beta}\right)^{1/\alpha}.$$

Summing the last inequality from n_2 to $n-1$, we get

$$\begin{aligned} x_n &\geq x_{n_2} + \sum_{s=n_2}^{n-1} y_s^{1/\alpha\beta} b_s^{-1/\alpha}\left(\sum_{t=n_2}^{s-1} a_t^{-1/\beta}\right)^{1/\alpha} \\ &\geq y_n^{1/\alpha\beta}\sum_{s=n_2}^{n-1} b_s^{-1/\alpha}\left(\sum_{t=n_2}^{s-1} a_t^{-1/\beta}\right)^{1/\alpha}. \end{aligned}$$

There exists $n_3 \geq n_2$ such that $n-\tau \geq n_2$ for all $n \geq n_3$. Then

$$x_{n-\tau} \geq y_{n-\tau}^{1/\alpha\beta}\sum_{s=n_2}^{n-\tau-1} b_s^{-1/\alpha}\left(\sum_{t=n_2}^{s-1} a_t^{-1/\beta}\right)^{1/\alpha} \quad \text{for all} \quad n \geq n_3.$$

This and the equation (5.1.1), (C_4) yield for all $n \geq n_3$,

$$-\Delta y_n = q_n \boldsymbol{f}(x_{n-\tau}) \geq q_n \boldsymbol{f}(y_{n-\tau}^{1/\alpha\beta})\boldsymbol{f}\left(\sum_{s=n_2}^{n-\tau-1} b_s^{1/\alpha}\left(\sum_{t=n_2}^{s-1} a_t^{-1/\beta}\right)^{1/\alpha}\right).$$

Summing the last inequality from n to ∞, we get

$$y_n \geq \sum_{s=n}^{\infty} q_s f(y_{s-\tau}^{1/\alpha\beta}) f\left(\sum_{t=n_2}^{s-\tau-1} b_t^{1/\alpha} \left(\sum_{j=n_2}^{t-1} a_j^{-1/\beta}\right)^{1/\alpha}\right).$$

The function y_n is strictly decreasing, and by Theorem 6.19.3 of [1] there exists a positive solution of equation (5.2.8) which tends to zero, this contradicts that equation (5.2.8) is oscillatory.

Case(ii). Summing the equation (5.1.1) from n to $n + \sigma$, we have

$$a_n (\Delta (b_n(\Delta x_n)^{\alpha}))^{\beta} \geq \sum_{s=n}^{n+\sigma} q_s f(x_{s-\tau})$$

or

$$\Delta (b_n(\Delta x_n)^{\alpha}) \geq \left(\frac{f(x_{n+\sigma-\tau})}{a_n}\right)^{1/\beta} \left(\sum_{s=n}^{n+\sigma} q_s\right)^{1/\beta}.$$

Summing the above inequality from n to $n + \sigma$, we obtain

$$-b_n(\Delta x_n)^{\alpha} \geq \sum_{s=n}^{n+\sigma} \left(\frac{f(x_{s+\sigma-\tau})}{a_s}\right)^{1/\beta} \left(\sum_{t=s}^{s+\sigma} q_t\right)^{1/\beta}$$

or

$$-(\Delta x_n)^{\alpha} \geq \frac{(f(x_{n+2\sigma-\tau}))^{1/\beta}}{b_n} \sum_{s=n}^{n+\sigma} a_s^{-1/\beta} \left(\sum_{t=s}^{s+\sigma} q_t\right)^{1/\beta}$$

or

$$-\Delta x_n \geq \frac{(f(x_{n+2\sigma-\tau}))^{1/\alpha\beta}}{b_n^{1/\alpha}} \left(\sum_{s=n}^{n+\sigma} a_s^{-1/\beta} \left(\sum_{t=s}^{s+\sigma} q_t\right)^{1/\beta}\right)^{1/\alpha}.$$

Summing the last inequality from n to ∞, we get

$$x_n \geq (f(x_{n+2\sigma-\tau}))^{1/\alpha\beta} \sum_{s=n}^{\infty} b_s^{-1/\alpha} \left(\sum_{t=s}^{s+\sigma} a_t^{-1/\beta} \left(\sum_{j=t}^{t+\sigma} q_j\right)^{1/\beta}\right)^{1/\alpha}.$$

Since by Lemmas 5.2.1 and 5.2.2, there exists a positive solution of equation (5.2.9) which tends to zero, this contradicts that equation (5.2.9) is oscillatory. The proof is complete. □

Theorem 5.2.2. *Assume that the first order delay difference equation* (5.2.8) *is oscillatory, condition* (5.2.6), (H_1) *and* (H_2) *hold. Then every solution* $\{x_n\}$ *of equation* (5.1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Proof. The proof follows from Theorem 5.2.1 of Case(i) and Lemma 5.2.2 and hence the details are omitted. □

5.3 Examples

In this section, we present some examples to illustrate the main results.

Example 5.3.1. *Consider the difference equation*

$$\Delta\left(n\Delta\left(\frac{1}{n^2}(\Delta x_n)^{1/3}\right)^3\right) + \frac{1}{n}x_{n-2} = 0, \quad n \ge 1. \tag{5.3.1}$$

Here $f(u) = u$, $q_n = \frac{1}{n}$, $a_n = n$, $b_n = \frac{1}{n^2}$, $\tau = 2$, $\alpha = \frac{1}{3}$, *and* $\beta = 3$. *Further* $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \infty$, $\sum_{n=1}^{\infty} n^6 = \infty$. *It is easy to see that condition* (5.2.6) *holds. Further the equation* (5.2.8) *reduces to*

$$\Delta y_n + \frac{1}{n}\sum_{s=1}^{n-3} n^6\left(\sum_{t=1}^{s}\frac{1}{t^{1/3}}\right)^3 y_{n-2} = 0. \tag{5.3.2}$$

Then by Theorem 7.5.1 of [36], the equation (5.3.2) *is oscillatory, provided that*

$$\liminf_{n\to\infty} \sum_{s=n-2}^{n-1}\frac{1}{s}\left(\sum_{t=1}^{s-1} t^6\left(\sum_{j=1}^{t}\frac{1}{j^{1/3}}\right)^3\right) > \left(\frac{2}{3}\right)^3,$$

and according to Theorem 5.2.2 every nonoscillatory solution of equation (5.3.1) *tends to zero as* $n \to \infty$.

Example 5.3.2. *Consider the difference equation*

$$\Delta\left(n\Delta\left(\frac{1}{n^3}(\Delta x_n)^3\right)^{1/3}\right) + \frac{1}{n^2}x_{n-2} = 0, \quad n \ge 1. \tag{5.3.3}$$

Here $f(u) = u$, $q_n = \frac{1}{n^2}$, $a_n = n$, $b_n = \frac{1}{n^3}$, $\tau = 2$, $\alpha = 3$, *and* $\beta = \frac{1}{3}$. *Further* $\sum_{n=1}^{\infty} \frac{1}{n^3} = \infty$, $\sum_{n=1}^{\infty} n^9 = \infty$. *It is easy to see that condition* (5.2.6) *holds. Further* *it is easy to see that all conditions of Corollary 5.2.2 are satisfied. Hence every nonoscillatory solution of equation* (5.3.3) *tends to zero as* $n \to \infty$.

We conclude this chapter with the following remark.

Remark 5.3.1. *It would be interesting to extend the results of this chapter to the equation* (5.1.1) *when* $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\beta}} < \infty$ *and/or* $\sum_{n=n_0}^{\infty} \frac{1}{b_n^{1/\alpha}} < \infty$.