Chapter 4

Third Order Nonlinear Delay Difference Equation-I

4. Third Order Nonlinear Delay Difference Equation-I

4.1 Introduction

In this chapter, we are concerned with the oscillatory behavior of third order non-linear delay difference equation of the form

$$\Delta\left(a_n(\Delta^2 x_n)^{\alpha}\right) + q_n f(x_{\sigma(n)}) = 0, n \in \mathbb{N}_0, \quad (4.1.1)$$

subject to the following conditions:

(C_1) $\{q_n\}$ and $\{a_n\}$ are nonnegative real sequences and $\sum_{s=n_0}^{\infty} \frac{1}{a_s^{1/\alpha}} = \infty$;

(C_2) α is a ratio of odd positive integers;

(C_3) $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing functions such that $uf(u) > 0$ for $u = 0$ and $-f(-uv) \geq f(uv) \geq f(u)f(v)$ for $uv > 0$;

(C_4) $\{\sigma(n)\}$ is a sequence of positive integer such that $\sigma(n) \leq n$, and $\lim_{n\to\infty} \sigma(n) = \infty$.

By a solution of equation (4.1.1) we mean a real sequence $\{x_n\}$ which satisfies equation (4.1.1) for all $n \in \mathbb{N}_0$. We consider only those solution $\{x_n\}$ of equation (4.1.1) which satisfy $\sup\{|x_n| : n \geq N\} > 0$ for all $n \geq N \in \mathbb{N}_0$.

We say that a nontrivial solution $\{x_n\}$ of equation (4.1.1) is strongly decreasing if it satisfies

$$x_n \Delta x_n < 0, \quad (4.1.2)$$

and it said to be strongly increasing if

$$x_n \Delta x_n > 0. \quad (4.1.3)$$

In [63], the author considered the third order difference equation of the form

$$\Delta^3 x_n - q_n x_{n+2} = 0, \qquad (4.1.4)$$

and studied the oscillatory and asymptotic behavior of the solutions of equation (4.1.4) under the condition $q_n > 0$ for $n \geq n_0$.

In [57], the authors studied the difference equation of the form

$$\Delta^3 x_n + q_n x_n = 0, \qquad (4.1.5)$$

and established some sufficient conditions for the oscillatory solution of equation (4.1.5).

In [34], the authors studied the oscillatory and asymptotic behavior of solution of difference equation of the form

$$\Delta(a_n \Delta(b_n(\Delta x_n))) + q_n \boldsymbol{f}(x_{n-\sigma+1}) = 0, \quad n \geq n_0. \qquad (4.1.6)$$

In [58], the author considered the following difference equation

$$\Delta^3 x_n + p_n x_{n+1} = 0, \qquad (4.1.7)$$

and established oscillatory and asymptotic behavior of solutions of equation (4.1.7).

In [62], the author considered the difference equation of the form

$$\Delta(a_n \Delta(b_n \Delta x_n)) + q_n \boldsymbol{f}(x_{n+A}) = 0 \qquad (4.1.8)$$

where $A \in \{0, 1, 2, ...\}$ and derived oscillatory and asymptotic behavior of solution of equation (4.1.8).

In [60], the authors studied the oscillatory and asymptotic behavior of solutions of the equation

$$\Delta(a_n \Delta(b_n (\Delta x_n)^{\alpha})) + q_n \boldsymbol{f}(x_{n-\sigma}) = 0, \qquad (4.1.9)$$

under the conditions $\sum_{s=n_0}^{\infty} \frac{1}{a_s^{1/\alpha}} = \infty$ and $\sum_{s=n_0}^{\infty} \frac{1}{b_s} = \infty$.

Motivated by this observation, in this chapter we obtain some sufficient conditions for the oscillation of all solutions of equation (4.1.1).

In Section 4.2, we establish some sufficient conditions for the oscillation of all solutions of equation (4.1.1) and in Section 4.3, we present some examples to illustrate the main results. The results presented in this chapter generalize and improve those obtained in [3, 34, 60, 62, 73].

4.2 Oscillation Theorems

In this section, we establish some new oscillation theorem for equation (4.1.1). We begin with the following lemma.

Lemma 4.2.1. *Let $\{q_n\}$ be a positive real sequence, k be a positive integer and α be a ratio of odd positive integers. Then the difference inequality*

$$\Delta x_n + q_n x_{n-k}^{\alpha} \leq 0, \quad n \geq n_0 \in \mathbb{N}_0, \tag{4.2.1}$$

has an eventually positive solution if and only if the difference equation

$$\Delta x_n + q_n x_{n-k}^{\alpha} = 0, \quad n \geq n_0 \in \mathbb{N}_0, \tag{4.2.2}$$

has an eventually positive solution.

For the proof of Lemma 4.2.1, see [65].

Lemma 4.2.2. *Assume that $0 < \alpha < 1$ and k is a positive integer. Then every solution of equation* (4.2.2) *is oscillatory if and only if*

$$\sum_{n=n_0}^{\infty} q_n = \infty. \tag{4.2.3}$$

Proof. The proof of the lemma can be found in [65]. However for easy reference we present the proof of the lemma.

Sufficiency. Assume that (4.2.3) holds and suppose that $\{x_n\}$ is an eventually positive solution of (4.2.2). Then there exists an integer $n_1 \in N_0$ such that

$$x_{n-A} > 0 \text{ and } x_{n+1} - x_n \le 0 \text{ for } n \ge n_1. \tag{4.2.4}$$

Clearly (2.2.6), (4.2.3) and (4.2.4) imply that $\lim_{n\to\infty} x_n = 0$ and

$$x_n^{1-\alpha} - x_{n+1}^{1-\alpha} \ge (1-\alpha)x_n^{-\alpha}(x_n - x_{n+1}) \ge (1-\alpha)q_n, \; n \ge n_1.$$

It follows that

$$x_{n_1}^{1-\alpha} \ge (1-\alpha) \sum_{n=n_1}^{\infty} q_n,$$

which is a contradiction to (4.2.3).

Necessity. Assume that (4.2.2) holds and suppose that (4.2.3) is not true. Then there exists an integer $N > A$ such that $\sum_{s=N-A}^{\infty} q_s \le \frac{1}{2}$. Define a sequence $\{y_n\}$ as follows:

$$y_n = \frac{1}{2} + \sum_{i=n}^{\infty} q_i, \quad n \ge N - A.$$

Clearly, $\frac{1}{2} \le y_n \le 1$ for $n \ge N - A$ and

$$y_n \ge \frac{1}{2} + \sum_{i=n}^{\infty} q_i y_{i-A}^{\alpha}, \quad n \ge N.$$

From this, it is easy to see that the corresponding equation

$$x_n = \frac{1}{2} + \sum_{i=n}^{\infty} q_i x_{i-A}^{\alpha}, \quad n \ge N$$

has an eventually positive solution $\{x_n\}$. Clearly, $\{x_n\}$ is also an eventually positive solution of (4.2.2), which is a contradiction. This completes the proof. □

Lemma 4.2.3. *Let $\alpha > 1$. If there exists a $\lambda > \frac{1}{k}\log \alpha$ such that*

$$\liminf_{n\to\infty}[q_n exp(-e^{\lambda n})] > 0, \tag{4.2.5}$$

then every solution of equation (4.2.2) is oscillatory.

Proof. The proof of the lemma can be found in [65]. However for easy reference we present the proof of the lemma.

Let $\{x_n\}$ be a nonoscillatory solution of equation (4.2.2), which we can assume to be eventually positive solution. Then there exists an integer $N \geq n_0 \in N_0$ such that

$$x_{n-A} > 0 \text{ for } n \geq N.$$

By (4.2.5), there exist an integer $N_1 \in N_0$ and $\lambda_0 \in \left(\frac{1}{A}\log \alpha, \lambda\right)$ such that

$$q_n \geq e^{\lambda_0 n} exp\left(e^{\lambda_0 n}\right), \quad n \geq N_1. \tag{4.2.6}$$

Set

$$\varphi(n) = e^{\lambda_0 n}, \quad p_n = \varphi(n)e^{\varphi(n)} \text{ and } k = \alpha e^{-\lambda_0 A}.$$

Then $0 < k < 1$. Let $y_n = -\log x_n$ for $n \geq N_1 - A$. Then $y_n > 0$ for $n \geq N_1 - A$, and from (4.2.2), we have

$$1 - e^{y_n - y_{n+1}} = p_n e^{y_n - \alpha y_{n-fl}}, \quad n \geq N_1. \tag{4.2.7}$$

Consequently, we obtain

$$y_{n+1} - y_n \geq p_n e^{y_n - \alpha y_{n-fl}}, \quad n \geq N_1. \tag{4.2.8}$$

We consider the following two possible cases for $y_n - \alpha y_{n-A}$.

Case(i). Suppose $y_n - \alpha y_{n-A} \leq 0$ holds eventually. Choose an integer $N_2 \geq N_1$ such that

$$y_n - \alpha y_{n-A} \leq 0, \quad n \geq N_2.$$

It follows that

$$\frac{y_n}{\varphi(n)} \leq \frac{\alpha\varphi(n-A)}{\varphi(n)} \frac{y_{n-A}}{\varphi(n-A)} = k\frac{y_{n-A}}{\varphi(n-A)}, \quad n \geq N_2.$$

Set $z_n = \frac{y_n}{\varphi(n)}$, $n \geq N_2 - A$. Then

$$z_n \leq kz_{n-A}, \quad n \geq N_2. \tag{4.2.9}$$

Set $M = \max\{z_{N_2}, z_{N_2+1}, \ldots, z_{N_2+A-1}\}$. Then (4.2.9) implies that

$$z_n \le M\, k^{[(n-N_2)/A]}, \quad n \ge N_2.$$

Here $[x]$ denotes the greatest integer less than or equal to x. Hence

$$\lim_{n\to\infty} z_n = 0. \tag{4.2.10}$$

This shows that there exists an integer $N_3 > N_2$ such that

$$y_n < \frac{1}{1+\alpha}\varphi(n), \quad n \ge N_3. \tag{4.2.11}$$

From (4.2.8) and (4.2.11), we have

$$\begin{aligned}
\Delta y_n &\ge p_n e^{-(\alpha-1)y_{n-fl}} \\
&\ge p_n e^{-(\alpha-1)\varphi(n)/(1+\alpha)} \\
&\ge \lambda_0 e^{\lambda_0}\varphi(n) \\
&\ge \Delta\varphi(n).
\end{aligned}$$

Summing the last inequality from N_3 to n, we obtain

$$y_n > \varphi(n+1) - \varphi(N_3) + y_{N_3}, \quad \text{for} \quad n \ge N_3,$$

which is a contradiction to (4.2.11).

Case(ii). Now assume $y_n - \alpha y_{n-A} > 0$ holds. Then there exists a sequence $\{n_i\}$ of integers with $N_1 < n_1 < n_2 < \ldots$ such that

$$y_{n_i} - \alpha y_{n_i-A} > 0, \quad i = 1, 2, \ldots$$

which, together with (4.2.7) implies that

$$1 > q_{n_i} e^{y_{n_i} - \alpha y_{n_i-fl}} > q_{n_i} > 1, \quad i = 1, 2, \ldots.$$

This is a contradiction. The proof is complete. □

Lemma 4.2.4. *If* $\alpha = 1$ *and*

$$\liminf_{n\to\infty} \sum_{s=n-k}^{n-1} q_s > \left(\frac{k}{k+1}\right)^{k+1}, \tag{4.2.12}$$

then every solution of equation (4.2.2) *is oscillatory.*

Proof. The proof of the lemma can be found in [36]. However for easy reference we present the proof of the lemma.

Let $\{x_n\}$ be a nonoscillatory solution of equation (4.2.2), which we can assume to be positive eventually, and since $q_n \geq 0$ this solution $\{x_n\}$ is eventually decreasing. Therefore, using $x_n \leq x_{n-A}$ in (4.2.2), we obtain

$$q_n \leq 1 - \frac{x_{n+1}}{x_n}.$$

Now using arithmetic - geometric means inequality, we find

$$\begin{aligned} \frac{1}{A}\sum_{s=n-A}^{n-1} q_s &\leq 1 - \frac{1}{A}\sum_{s=n-A}^{n-1} \frac{x_{s+1}}{x_s} \\ &\leq 1 - \left(\frac{x_n}{x_{n-A}}\right)^{\frac{1}{fl}}. \end{aligned} \tag{4.2.13}$$

Setting $\beta = \frac{A^{fl}}{(A+1)^{fl+1}}$, from (4.2.13), we can choose a constant γ such that for n sufficiently large, $\beta < \gamma \leq \frac{1}{A}\sum_{s=n-A}^{n-1} q_s$. Therefore from (4.2.13) for all large n, $\left(\frac{x_n}{x_{n-fl}}\right)^{\frac{1}{fl}} \leq 1-\gamma$, which in particular implies that $0 < \gamma < 1$. Now since $\max_{0\leq\gamma\leq 1}\left((1-\gamma)\gamma^{\frac{1}{fl}}\right) = \beta^{\frac{1}{fl}}$, see [26], we have $1-\gamma \leq \beta^{\frac{1}{fl}}\gamma^{-\frac{1}{fl}}$ for $0 < \gamma < 1$, and hence it follows that

$$\left(\frac{x_n}{x_{n-A}}\right)^{\frac{1}{fl}} \leq \beta^{\frac{1}{fl}}\gamma^{-\frac{1}{fl}}$$

or

$$\frac{\gamma}{\beta} x_n \leq x_{n-A}. \tag{4.2.14}$$

Now using (4.2.14) instead of $x_n \leq x_{n-A}$ in (2.2.6) and repeating the arguments, we find $\left(\frac{\gamma}{\beta}\right)^2 x_n \leq x_{n-A}$ for all large n. Thus, by induction, for every $k \in N(1)$ there exists an integer n_k such that for all $n \in \mathrm{N}(n_k)$

$$\left(\frac{\gamma}{\beta}\right)^{k} x_n \le x_{n-A}. \tag{4.2.15}$$

Next, for sufficiently large n,

$$\sum_{s=n-A}^{n} q_s \ge \sum_{s=n-A}^{n-1} q_s \ge A\gamma = M.$$

Since $\gamma > \beta$, we can choose k such that

$$\left(\frac{\gamma}{\beta}\right)^{k} > \left(\frac{2}{M}\right)^{2}. \tag{4.2.16}$$

For this specific value of k, we consider n sufficiently large, say, n_1 so that for all $n \ge n_1$ all the above inequalities are satisfied. Then, for each $n \ge n_1 + A$ there exists an integer N with $n - A \le N \le n$ so that

$$\sum_{s=n-A}^{N} q_s \ge \frac{M}{2} \quad \text{and} \quad \sum_{s=N}^{n} q_s \ge \frac{M}{2}.$$

From (4.2.2) and the nonincreasing nature of $\{x_n\}$, we have

$$\begin{aligned} -x_{n-A} &\le \sum_{s=n-A}^{N} \Delta x_s = \sum_{s=n-A}^{N} (-q_s x_{s-A}) \\ &\le -\left(\sum_{s=n-A}^{N} q_s\right) x_{N-A} \le -\frac{M}{2} x_{N-A} \end{aligned}$$

and hence

$$\frac{M}{2} x_{N-A} \le x_{n-A}. \tag{4.2.17}$$

Similarly, we find

$$-x_N \le -\frac{M}{2} x_{n-A}$$

and so

$$\frac{M}{2} x_{n-A} \le x_N. \tag{4.2.18}$$

Combining (4.2.15), (4.2.17) and (4.2.18), we get

$$\left(\frac{\gamma}{\beta}\right)^{k} \le \frac{x_{N-A}}{x_N} \le \left(\frac{2}{M}\right)^{2}.$$

This contradicts (4.2.16). The proof is now complete. □

Lemma 4.2.5. *Let $\{x_n\}$ be a positive solution of equation* (5.1.1)*. Then $\Delta^2 x_n > 0$, eventually, and $\{x_n\}$ is either strongly increasing or strongly decreasing.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (5.1.1). We may assume that $x_n > 0$, eventually (if it is an eventually negative, the proof is similar). Then $\Delta(a_n(\Delta^2 x_n)^\alpha) < 0$, eventually. Thus, $a_n(\Delta^2 x_n)^\alpha$ is decreasing and of one sign and it follows from hypotheses (C_1) and (C_2) that there exists a $n_1 \geq n_0 \in \mathbb{N}_0$ such that $\Delta^2 x_n$ is of fixed sign for $n \geq n_1$. If we admit $\Delta^2 x_n < 0$ then there exists a constant $M > 0$ such that

$$a_n(\Delta^2 x_n)^\alpha \leq -M < 0, \quad n \geq n_1.$$

Summing from n_1 to $n - 1$, we obtain

$$\Delta x_n \leq \Delta x_{n_1} - M^{1/\alpha} \sum_{s=n_1}^{n-1} \frac{1}{a_s^{1/\alpha}}.$$

Letting $n \to \infty$ and using (C_1), we get $\Delta x_n \to \infty$. Thus, $\Delta x_n < 0$, which together with $\Delta^2 x_n < 0$ implies $x_n < 0$. This contradiction shows that $\Delta^2 x_n > 0$. Therefore Δx_n is increasing and thus either (4.1.2) or (4.1.3) holds, eventually. The proof is complete. □

Theorem 4.2.1. *If the first order delay difference equation*

$$\Delta y_n + q_n \boldsymbol{f}\left(\sum_{s=n_0}^{\sigma(n)-1} (\sigma(n) - s) a_s^{-1/\alpha}\right) \boldsymbol{f}(y_{\sigma(n)}^{1/\alpha}) = 0 \qquad (4.2.19)$$

is oscillatory, then every solution of equation (5.1.1) *is either oscillatory or strongly decreasing.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (5.1.1). We may assume that $x_n > 0$ for $n \geq n_0 \in \mathbb{N}_0$. From Lemma 4.2.5, we see that $\Delta^2 x_n > 0$ and $\{x_n\}$ is either strongly increasing or strongly decreasing.

Assume that $\{x_n\}$ is strongly increasing, that is $\Delta x_n > 0$, eventually. Using that $a_n(\Delta^2 x_n)^\alpha$ is decreasing, we are led to

$$\begin{aligned} \Delta x_n &\geq \sum_{s=n_1}^{n-1} \Delta^2 x_s = \sum_{s=n_1}^{n-1} a_s^{-1/\alpha} \left(a_s(\Delta^2 x_s)^\alpha\right)^{1/\alpha} \\ &\geq \left(a_n(\Delta^2 x_n)^\alpha\right)^{1/\alpha} \sum_{s=n_1}^{n-1} a_s^{-1/\alpha}. \end{aligned}$$

Summing the last inequality from n_2 to $n-1$, we have

$$x_n - x_{n_2} \geq \sum_{s=n_2}^{n-1} \left(a_s(\Delta^2 x_s)^\alpha\right)^{1/\alpha} \sum_{t=s_1}^{s-1} a_t^{-1/\alpha}$$

or

$$x_n \geq \left(a_n(\Delta^2 x_n)^\alpha\right)^{1/\alpha} \sum_{s=n_1}^{n-1} (n-s) a_s^{-1/\alpha}.$$

There exists a $n_3 \geq n_2$ such that for all $n \geq n_3$, one gets

$$x_{\sigma(n)} \geq (y_{\sigma(n)})^{1/\alpha} \sum_{s=n_3}^{\sigma(n)-1} (\sigma(n) - s) a_s^{-1/\alpha}, \qquad (4.2.20)$$

where $y_n = a_n(\Delta^2 x_n)^\alpha$. Combining (4.2.20) together with (5.1.1), we see that

$$\begin{aligned} -\Delta y_n &= q_n \boldsymbol{f}(x_{\sigma(n)}) \\ &\geq q_n \boldsymbol{f}\left(y_{\sigma(n)}^{1/\alpha} \sum_{s=n_3}^{\sigma(n)-1} (\sigma(n) - s) a_s^{-1/\alpha}\right) \\ &\geq q_n \boldsymbol{f}(y_{\sigma(n)}^{1/\alpha}) \boldsymbol{f}\left(\sum_{s=n_3}^{\sigma(n)-1} (\sigma(n) - s) a_s^{-1/\alpha}\right), \end{aligned}$$

where we have used (C_3). Thus $\{y_n\}$ is a positive and decreasing solution of the difference inequality

$$\Delta y_n + q_n \boldsymbol{f}(y_{\sigma(n)}^{1/\alpha}) \boldsymbol{f}\left(\sum_{s=n_3}^{\sigma(n)-1} (\sigma(n) - s) a_s^{-1/\alpha}\right) \leq 0.$$

Hence, by Theorem 6.19.3 of [1], there exists a positive solution of equation (4.2.19), which contradicts the oscillation of (4.2.19). Therefore $\{x_n\}$ is strongly decreasing. This completes the proof. □

Lemma 4.2.6. *Assume that $\{x_n\}$ is a positive decreasing solution of equation* (5.1.1)*. If*

$$\sum_{n=n_0}^{\infty}\sum_{s=n}^{\infty}\left(\frac{1}{a_s}\sum_{t=s}^{\infty}q_t\right)^{1/\alpha}=\infty \tag{4.2.21}$$

then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a positive solution of equation (5.1.1), and $\{x_n\}$ satisfies $x_n > 0$, $\Delta x_n < 0$, and $\Delta^2 x_n > 0$ for all $m \geq n_1 \in \mathrm{N}_0$. Then there exists $A \geq 0$ such that $\lim_{n\to\infty} x_n = A$. We shall prove that $A = 0$. Assume that $A > 0$, then we have $A < x_n < A + s$ for all $s > 0$ and $n \geq n_1 \geq n_0 \in \mathrm{N}_0$. From the equation (5.1.1), we have

$$\Delta\left(a_n(\Delta^2 x_n)^{\alpha}\right) = -q_n f(x_{\sigma(n)}) \leq -q_n L x^{\alpha}_{\sigma(n)}.$$

Summing the last inequality from n to ∞, we have

$$a_n(\Delta^2 x_n)^{\alpha} \geq L\sum_{s=n}^{\infty} q_s x^{\alpha}_{\sigma(s)}.$$

Using $x^{\alpha}_{\sigma(n)} \geq A^{\alpha}$, we get

$$\Delta^2 x_n \geq LA\left(\frac{1}{a_n}\sum_{s=n}^{\infty} q_s\right)^{\frac{1}{\alpha}}.$$

Summing the last inequality from n to ∞, we have

$$-\Delta x_n \geq LA\sum_{s=n}^{\infty}\left(\frac{1}{a_s}\sum_{t=s}^{\infty} q_t\right)^{\frac{1}{\alpha}}.$$

Summing the above inequality from n_1 to ∞, we obtain

$$x_{n_1} \geq LA\sum_{n=n_1}^{\infty}\sum_{s=n}^{\infty}\left(\frac{1}{a_s}\sum_{t=s}^{\infty} q_t\right)^{\frac{1}{\alpha}}.$$

This contradicts condition (4.2.21). Thus $A = 0$, this implies that $\lim_{n\to\infty} x_n = 0$. The proof is now complete. □

Combining Theorem 4.2.1 and Lemma 4.2.6 we get the following theorem.

Theorem 4.2.2. *Assume that* (4.2.21) *holds. If equation* (4.2.19) *is oscillatory, then every solution* $\{x_n\}$ *of equation* (5.1.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Now we eliminate strongly decreasing solutions of (5.1.1) to get its oscillation. We relax condition (4.2.21) and employ another one.

Theorem 4.2.3. *Assume that there exists a sequence* $\{\xi(n)\}$ *such that*

$$\Delta\xi(n) \geq 0, \quad \xi(n) > n \quad and \quad \eta(n) = \sigma(\xi(\xi(n))) < n. \tag{4.2.22}$$

If both the first order delay difference equation (4.2.19) *and*

$$\Delta z_n + \left(\sum_{s=n}^{\xi(n)} \frac{1}{a_s^{1/\alpha}}\left(\sum_{t=s}^{\xi(s)} q_t\right)^{1/\alpha}\right) f^{1/\alpha}(z_{\eta(n)}) = 0 \tag{4.2.23}$$

are oscillatory, then every solution of equation (5.1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (5.1.1). We may assume that $x_n > 0$. From Theorem 4.2.1, we see that $\{x_n\}$ is strongly decreasing.

Summing the equation (5.1.1) from n to $\xi(n)$ yields

$$a_n(\Delta^2 x_n)^\alpha \geq \sum_{s=n}^{\xi(n)} q_s f(x_{\sigma(s)}) \geq f(x_{\sigma(\xi(n))}) \sum_{s=n}^{\xi(n)} q_s.$$

Then

$$\Delta^2 x_n \geq \frac{f^{1/\alpha}(x_{\sigma(\xi(n))})}{a_n^{1/\alpha}} \left(\sum_{s=n}^{\xi(n)} q_s\right)^{1/\alpha}.$$

Summing the last inequality from n to $\xi(n)$, we get

$$\begin{aligned} -\Delta x_n &\geq \sum_{s=n}^{\xi(n)} \frac{f^{1/\alpha}(x_{\sigma(\xi(s))})}{a_s^{1/\alpha}} \left(\sum_{t=s}^{\xi(s)} q_t\right)^{1/\alpha} \\ &\geq f^{1/\alpha}(x_{\sigma(\xi(\xi(n)))}) \sum_{s=n}^{\xi(n)} \frac{1}{a_s^{1/\alpha}} \left(\sum_{t=s}^{\xi(s)} q_t\right)^{1/\alpha} \end{aligned}$$

or

$$-\Delta x_n \ge f^{1/\alpha}(x_{\eta(n)}) \sum_{s=n}^{\xi(n)} \frac{1}{a_s^{1/\alpha}} \left(\sum_{t=s}^{\xi(s)} q_t \right)^{1/\alpha} .$$

Summing the above inequality from n to ∞, we obtain

$$x_n \ge \sum_{s=n}^{\infty} f^{1/\alpha}(x_{\eta(s)}) \sum_{t=s}^{\xi(s)} \frac{1}{a_t^{1/\alpha}} \left(\sum_{j=t}^{\xi(t)} q_j \right)^{1/\alpha} . \tag{4.2.24}$$

Let us denote the right hand side of (4.2.24) by z_n. Then $z_n > 0$, and one can easily verify that $\{z_n\}$ is a solution of the difference inequality

$$\Delta z_n + \left(\sum_{s=n}^{\xi(n)} \frac{1}{a_s^{1/\alpha}} \left(\sum_{t=s}^{\xi(s)} q_t \right)^{1/\alpha} \right) f^{1/\alpha}(z_{\eta(n)}) \le 0.$$

Then Theorem 6.19.3 of [1] shows that the corresponding difference equation (4.2.23) also has a positive solution. This contradiction, completes the proof. □

For the special case of equation (5.1.1), when $f(u) = u^{\beta}$ we immediately have:

Corollary 4.2.1. *Assume that* β *is a quotient of odd positive integers. Let* $\sigma(n) > 0$. *Assume that there exists a sequence* $\xi(n)$ *such that* (4.2.22) *holds. If both equations*

$$\Delta y_n + A_n y_{\sigma(n)}^{\beta/\alpha} = 0, \tag{4.2.25}$$

and

$$\Delta z_n + B_n z_{\eta(n)}^{\beta/\alpha} = 0 \tag{4.2.26}$$

where

$$A_n = q_n \left(\sum_{s=n_0}^{\sigma(n)-1} (\sigma(n) - s) \right)^{\beta} ,$$

and

$$B_n = q_n \left(\sum_{s=n}^{\xi(n)} \frac{1}{a_s^{1/\alpha}} \left(\sum_{t=s}^{\xi(s)} q_t \right)^{1/\alpha} \right)^{\beta} ,$$

are oscillatory, then every solutions of equation

$$\Delta \left(a_n (\Delta^2 x_n)^{\alpha} \right) + q_n x_{\sigma(n)}^{\beta} = 0 \tag{4.2.27}$$

is oscillatory.

Corollary 4.2.2. *Let $\alpha = \beta$, $\sigma(n) = n - k$, $\xi(n) = n + m$, and $\eta(n) = n + 2m - k$ with $k > 2m$. If*

$$\lim_{n\to\infty} \inf \sum_{s=n-k}^{n-1} A_s > \left(\frac{k}{k+1}\right)^{k+1}, \tag{4.2.28}$$

and

$$\lim_{n\to\infty} \inf \sum_{s=n-k}^{n-1} B_s > \left(\frac{k}{k+1}\right)^{k+1}, \tag{4.2.29}$$

then every solution of equation (4.2.27) *is oscillatory.*

Proof. By applying (4.2.28) and (4.2.29) with Lemma 4.2.4, we can conclude that every solution of equation (4.2.27) is oscillatory. □

Corollary 4.2.3. *Let $\alpha > \beta$, $\sigma(n) = n - k$, $\xi(n) = n + m$, and $\eta(n) = n + 2m - k$ with $k > 2m$. If*

$$\sum_{n=n_0}^{\infty} A_n = \infty, \tag{4.2.30}$$

and

$$\sum_{n=n_0}^{\infty} B_n = \infty \tag{4.2.31}$$

then every solution of equation (4.2.27) *is oscillatory.*

Proof. The proof follows by applying Lemma 4.2.2 in Corollary 4.2.1 and hence the details are omitted. □

Corollary 4.2.4. *Let $\alpha < \beta$, $\sigma(n) = n - k$, $\xi(n) = n + m$, and $\eta(n) = n + 2m - k$ with $k > 2m$. If there exists a $\lambda > \frac{1}{k}\log\frac{\beta}{\alpha}$ such that*

$$\lim_{n\to\infty} \inf \left[A_n exp(-e^{\lambda n})\right] > 0, \tag{4.2.32}$$

and if there exists a $\lambda_1 > \frac{1}{k-2m}\log\frac{\beta}{\alpha}$ such that

$$\lim_{n\to\infty} \inf \left[B_n exp(-e^{\lambda_1 n})\right] > 0, \tag{4.2.33}$$

then every solution of equation (4.2.27) *is oscillatory.*

Proof. The proof follows by applying Lemma 4.2.3 in Corollary 4.2.1 and hence the details are omitted. □

4.3 Examples

In this section, we present some examples to illustrate the main results.

Example 4.3.1. *Consider the difference equation*

$$\Delta\left(n\Delta\left(\Delta^2 x_n\right)\right) + n x_{n-2} = 0, \quad n \geq 1. \tag{4.3.1}$$

Here $f(u) = u$, $q_n = n$, $a_n = n$, $\alpha = 1$, *and* $\sigma(n) = n - 2$. *It is easy to see that condition* (4.2.21) *holds. Further the equation* (4.2.19) *reduces to*

$$\Delta y_n + n \sum_{s=1}^{n-3} \frac{(n-2-s)}{s} y_{n-2} = 0. \tag{4.3.2}$$

Then by Lemma 4.2.4, the equation (4.3.2) *is oscillatory, provided that*

$$\liminf_{n\to\infty} \sum_{s=n-2}^{n-1} s \left(\sum_{t=1}^{s-3} \frac{(s-2-t)}{t} \right) > \left(\frac{2}{3}\right)^3,$$

and according to Theorem 4.2.2 every solution of equation (4.3.1) *is either oscillatory or tends to zero as* $n \to \infty$.

Example 4.3.2. *Consider the difference equation*

$$\Delta\left(n\Delta\left(\Delta^2 x_n\right)^3\right) + n x_{n-3}^3 = 0, \quad n \geq 1. \tag{4.3.3}$$

Here $f(u) = u^3$, $\alpha = \beta = 3$, $a_n = n$, $q_n = n$, *and* $\sigma(n) = n - 3$. *By taking* $\xi(n) = n + 1$, *we see that all conditions of Corollary 4.2.2 are satisfied and hence every solution of equation* (4.3.3) *is oscillatory.*

Example 4.3.3. *Consider the difference equation*

$$\Delta\left(n^3\Delta\left(\Delta^2 x_n\right)^3\right) + n x_{n-3} = 0, \quad n \geq 1. \tag{4.3.4}$$

Here $f(u) = u$, $\alpha = 3$, $\beta = 1$, $a_n = n^3$, $q_n = n^2$, *and* $\sigma(n) = n - 3$. *By taking* $\xi(n) = n + 1$, *we see that all conditions of Corollary 4.2.3 are satisfied and hence every solution of equation* (4.3.4) *is oscillatory.*

Example 4.3.4. *Consider the difference equation*

$$\Delta\left(n\Delta\left(\Delta^2 x_n\right)^{1/3}\right) + n x_{n-3} = 0, \quad n \geq 1. \tag{4.3.5}$$

Here $f(u) = u$, $\alpha = \frac{1}{3}$, $\beta = 1$, $a_n = n$, $q_n = n$, *and* $\sigma(n) = n - 3$. *By taking* $\xi(n) = n + 1$, *we see that all conditions of Corollary 4.2.4 are satisfied and hence every solution of equation* (4.3.5) *is oscillatory.*

We conclude this chapter with the following remark.

Remark 4.3.1. *It would be interesting to extend the results of this chapter to the equation* (4.1.1) *when* $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\alpha}} < \infty$, *and also for the case* $\{q_n\}$ *is negative.*