CHAPTER III
Cross-Connection Semigroup

The construction of the cross-connection semigroup which we shall consider in this chapter, is an application of the general theory of cross-connections. Here we construct a cross-connection using bilinear form and the corresponding semigroup is expressed in terms of linear transformations and the bilinear form. The principal ingredients in this theory are two finite dimensional vector spaces and a bilinear form defined on their product.

In section 1 we study the local isomorphism induced by a bilinear form on the cartesian product of two vector spaces. Also we cut short the categories involved to use the local isomorphism as a cross-connection. We define adjoint of a morphism in the category and compare this with its transpose in section 2. Finally in section 3, we find the cross-connection semigroup as a subsemigroup of the cartesian product of the semigroups of linear transformations on the vector spaces under consideration.

1. CROSS-CONNECTION INDUCED BY A BILINEAR FORM

If $X$ and $Y$ are two vector spaces over $K$, then a bilinear form $B: X \times Y \to K$ induces two maps one from $X$ to $Y^*$ and the other from $Y$ to $X^*$. This duality gives a local isomorphism. In this section, we describe the cross-connection associated with this local isomorphism.

Throughout this chapter (and the subsequent ones) all vector spaces considered are finite dimensional and defined on the fixed field $K$ of characteristic zero.
1.1 Local isomorphism from a bilinear form

Let $C$ and $D$ be two categories with subobjects. Then recall that a local isomorphism from $C$ to $D$ is a functor $F: C \to D$ such that it is inclusion preserving, fully-faithful and for each $c \in \mathcal{V}C$, $F(c)$ is an isomorphism of the ideal $(c)$ onto $(F(c))$.

First we find out the local isomorphism using a bilinear form $B: X \times Y \to K$ where $X$ and $Y$ are two fixed vector spaces over $K$. Here we denote the linear maps from $X$ to $Y^{*}$ and $Y$ to $X^{*}$ (see Proposition 1.11) by $B_{*}$ and $B^{*}$ respectively. Thus $(x)B_{*} = B(x, -)$ and $(y)B^{*} = B(-, y)$ are linear functionals on $Y$ and $X$ respectively for $x \in X$ and $y \in Y$. Here we show that the linear maps $B_{*}$ and $B^{*}$ are such that one is the adjoint of the other.

**Proposition 1** Let $B: X \times Y \to K$ be a bilinear form. Then $B_{*}: X \to Y^{*}$ and $B^{*}: Y \to X^{*}$ are such that they are adjoints of each other.

**Proof** Adjoint of $B_{*}$ denoted by

$$B_{*}^{*}: Y^{**} \to X^{*}$$

is defined by

$$(y)B_{*}^{*} = B_{*}y$$

(see Equation II.(8))

Here note that $y$ is considered as an element of $Y^{**}$ which is justified since $Y$ is finite dimensional. Hence we get

$$(x)((y)B_{*}^{*}) = (x)(B_{*}y)$$

for all $x \in X$ and $y \in Y$

$$= (xB_{*})y$$

$$= B(x, -)y$$

$$= B(x, y)$$

$$= (x)B(-, y)$$

$$= (x)(yB^{*})$$

for all $x \in X$
Hence \((y)B^* = (y)B^*\) for all \(y \in Y\).

Hence \(B^* = B^*\).

Similarly we can prove that

\[ B^{**} = B_* \]

Hence we conclude the proof. \(\square\)

Here \(B^*: Y \to X^*\) is a linear transformation. Hence its null space denoted by \(N(B^*) \subseteq Y\). We use this to define a subcategory of \(C_Y\). Let \(C'_Y\) be the full subcategory of \(C_Y\) such that

\[ (1) \quad vC'_Y = \{ V \subseteq Y: V \cap N(B^*) = \{0\} \} \]

We have the following

**Proposition 2** \(C'_Y\) defined by Equation (1) is a normal reductive category.

**Proof** \(C'_Y\) is a full subcategory of \(C_Y\) so that \(C'_Y\) is reductive. This is because if \(V\) is an object in \(C'_Y\), it is also in \(C_Y\) and the cone with vertex \(V\) in \(TC_Y\) will also be in \(TC'_Y\). Hence it is sufficient to show that \(C'_Y\) is normal. For this let \(V \in vC'_Y\) and \(V' \subseteq V\). Then \(V' \cap N(B^*) = \{0\}\) so that \(V' \in vC'_Y\). Hence by Lemma 1.1 [12] \(C'_Y\) is also a normal category. \(\square\)

Let \(V\) be a subspace \(Y\). We write

\[ (2) \quad B^*(V) = \{(v)B^*: v \in V\}. \]

This equation leads to the following
**Proposition 3**  If $V \in vC_Y'$, the map $B^*: V \mapsto B^*(V)$ is an isomorphism from $vC_Y'$ into $vC_{X^*}$.

**Proof**  Since $V$ is a subspace of a complement of $N(B^*)$, $B^*$ is an injection from $vC_Y'$ into $vC_{X^*}$. It is a linear transformation also. Hence the proposition follows. \hfill $\square$

We use the map $B^*$ to define a functor between the categories $C_Y'$ and $C_{X^*}$. Let $V$ and $V'$ be two objects in $C_Y'$ and $t: V \to V'$ is a morphism in $C_Y'$. Then there is a unique morphism $t^*$ in $C_{X^*}$ such that the diagram

\[
\begin{array}{ccc}
V & \xrightarrow{t} & V' \\
B^* \downarrow & & \downarrow B^* \\
B^*(V) & \xrightarrow{t^*} & B^*(V')
\end{array}
\]

is commutative. Here note that we can write $t^* = B^{*-1}tB^*$, since $B^*$ on $C_Y'$ is an isomorphism. Define a map $\tilde{\Gamma}: C_Y' \to C_{X^*}$ by the following rule:

\begin{align*}
(3.1) & \quad v\tilde{\Gamma}: V \mapsto B^*(V) \\
(3.2) & \quad \tilde{\Gamma}: t \mapsto t^*
\end{align*}

where $t^*: B^*(V) \to B^*(V')$ is such that $tB^* = B^*t^*$.

Our next proposition gives the property of this map that we need.

**Proposition 4**  Let $X$ and $Y$ be two finite dimensional vector spaces over $K$, then $\tilde{\Gamma}: C_Y' \to C_{X^*}$ is a local isomorphism.

**Proof**  From the definition of $\tilde{\Gamma}$, it is clear that both $v\tilde{\Gamma}$ and $\tilde{\Gamma}$ are well defined. It is a functor because if $t$ and $t'$ are two morphisms from $V$ to $V'$ and $V'$ to $V''$ respectively, then

\[
\tilde{\Gamma}(tt') = B^{*-1}t'tB^* = (B^{*-1}tB^*)(B^{*-1}t'B^*)
\]
\[ \tilde{\Gamma}(t) = \tilde{\Gamma}(t') \]

Again
\[ \tilde{\Gamma}(1_V) = B^{*-1}1_VB^* \]
\[ = 1_{B^*(V)} \]

\( \tilde{\Gamma} \) is full because if for two subspaces \( V \) and \( V' \) in \( vC_Y \), \( g: B^*(V) \to B^*(V') \), then define \( t = B^*gB^{*-1} \) from \( V \) to \( V' \) so that
\[ \tilde{\Gamma}(t) = B^{*-1}(B^*gB^{*-1})B^* \]
\[ = g \]

The functor is faithful because if for any two morphisms \( t \) and \( t' \) from \( V \) to \( V' \), \( \tilde{\Gamma}(t) = \tilde{\Gamma}(t') \), then clearly \( t = t' \). Further, if \( V \) and \( V' \) in \( vC_Y \) are such that \( V \subseteq V' \) and if \( w \in B^*(V) \), then \( w = (v)B^* \) for some \( v \in V \). This element \( v \) belongs to \( V' \) also so that \((v)B^* \in B^*(V') \). Thus \( B^*(V) \subseteq B^*(V') \). Hence \( \tilde{\Gamma} \) preserves inclusion.

Now it is left to show that \( \tilde{\Gamma} \) is an isomorphism on ideals. For this, let \( V \in vC_Y \) and consider \( \tilde{\Gamma}: (V) \to (\tilde{\Gamma}(V)) \), the ideals generated by \( V \) and \( \tilde{\Gamma}(V) \). Let \( V_1 \) and \( V_2 \) be subspaces of \( V \) such that \( \tilde{\Gamma}(V_1) = \tilde{\Gamma}(V_2) \). Hence \( B^*(V_1) = B^*(V_2) \) which gives \( V_1 = V_2 \). Again if \( W \in (\tilde{\Gamma}(V)) \), then since \( \tilde{\Gamma}(V) = B^*(V) \), we get \( W \subseteq B^*(V) \) so that \( B^{*-1}(W) \subseteq V \) and \( \tilde{\Gamma}(B^{*-1}(W)) = W \). Hence we can conclude that \( \tilde{\Gamma} \) is a local isomorphism. \( \square \)

In order to write the detailed expressions for the local isomorphism \( \tilde{\Gamma} \), we need some preliminary results and notations.

If \( B: X \times Y \to K \) is the given bilinear form and \( U \) and \( V \) are subspaces of \( X \) and \( Y \) respectively, we define
\[ (4.1) \quad \hat{V} = \{ x \in X : B(x,v) = 0 \text{ for all } v \in V \} \]
\[ (4.2) \quad \hat{U} = \{ y \in Y : B(u,y) = 0 \text{ for all } u \in U \} \]

It is clear that both \( \hat{U} \) and \( \hat{V} \) are subspaces of the respective vector spaces. We have the following result that relates these newly defined subspaces with the linear map \( B^* \).
**Proposition 5** If $V$ is an object in the category $C'_Y$, then $\bot B^*(V) = \hat{V}$.

**Proof** By Equation 2, we have

\[ B^*(V) = \{(v)B^* : v \in V\} = \{B(-, v) : v \in V\} \]

By Equation II.(6.2), we get

\[ \bot B^*(V) = \{x \in X : (x)f = 0 \forall f \in B^*(V)\} = \{x \in X : B(x, v) = 0 \forall v \in V\} = \hat{V}. \]

Hence we complete the proof. \(\square\)

From Proposition 4 we know that $\tilde{\Gamma} : C'_Y \rightarrow C_X^*$ is a local isomorphism. Also from Proposition II.27 we have the isomorphism $J^{-1} : C_X^* \rightarrow N^*C_X$.

Define $\Gamma' = \tilde{\Gamma} \circ J^{-1}$, as in the following diagram.

\[
\begin{array}{c}
C'_Y \xrightarrow{\tilde{\Gamma}} C_X^* \\
\| \downarrow \quad \uparrow J \\
C'_Y \xrightarrow{\Gamma'} N^*C_X
\end{array}
\]

It is clear that $\Gamma'$ is also a local isomorphism. Using Equations (3.1) and II.(10.1), we get for $V \in uC'_Y$

\[ \Gamma'(V) = H(e, -) \quad \text{where } e \in E(L(X)) \text{ is such that } \]

\[ N(e) = \bot B^*(V) = \hat{V} \quad \text{ (see Proposition 5)} \]

Again if $V$ and $V'$ are two objects in $C'_Y$ and $t : V \rightarrow V'$ is a morphism in $C'_Y$, Equations (3.2) and II.(10.2) give

\[ \Gamma'(t) = \eta_{eC_X}(s, -)\eta_{e'}^{-1} \text{ from } H(e, -) \text{ to } H(e', -) \]

where $e, e'$ in $E(L(X))$ are such that $N(e) = \hat{V}, N(e') = \hat{V}'$. Also $s = \psi'^{-1}u\psi$ where $u = (\omega B^{*-1}tB^*\omega'^{-1})^*$. Consolidating these, we get
**Proposition 6** Let $B: X \times Y \to K$ be a bilinear form. Then there is a local isomorphism $\Gamma': \mathcal{C}_Y \to N^*\mathcal{C}_X$ such that if $V$ and $V'$ are in $\mathcal{C}_Y$ and $t: V \to V'$ is a morphism, then

\begin{align*}
(5.1) & \quad v\Gamma' : V \mapsto H(e, -) \\
(5.2) & \quad \Gamma' : t \mapsto \eta_e \mathcal{C}_X(s, -)\eta_e^{-1}
\end{align*}

where $N(e) = \hat{V}$, $N(e') = \hat{V}'$, $s = \psi'^{-1}u\psi$ is a morphism in $\mathcal{C}_X$ and $u = (\omega B^{-1}tB^*\omega'^{-1})^*$. 

Now we shall find the result that enable us to define a subcategory of $\mathcal{C}_X$.

**Proposition 7** Let $V \in \mathcal{C}_Y$, then $M\Gamma'(V) = \{U \subseteq X : U \oplus \hat{V} = X\}$.

**Proof** From Equation (5.1) we get

$$\Gamma'(V) = H(e, -) \text{ with } N(e) = \hat{V}$$

Hence

$$M\Gamma'(V) = MH(e, -)$$

\begin{align*}
&= \{U \subseteq X : U \oplus N(e) = X\} \text{ (see Proposition II.16)} \\
&= \{U \subseteq X : U \oplus \hat{V} = X\}
\end{align*}

This completes the proof. 

Thus starting with a bilinear form, we could get a local isomorphism between the categories $\mathcal{C}_Y'$ and $N^*\mathcal{C}_X$. But the category $\mathcal{C}_X$ is 'big' compared to $\mathcal{C}_Y'$. Hence we have to cut it into size so that $\Gamma'$ will become a cross-connection.
1.2 Category from $M$–sets

We use the local isomorphism $\Gamma'$ from $\mathcal{C}_X'$ to $N^*\mathcal{C}_X$ to define a subcategory of $\mathcal{C}_X$ in the following way: let

$$v\mathcal{C}_X^0 = \{ U \in v\mathcal{C}_X : U \in M\Gamma'(V) \text{ for some } V \in v\mathcal{C}_Y' \}$$

Consider $\mathcal{C}_X^0$ to be the full subcategory of $\mathcal{C}_X$ with object set $v\mathcal{C}_X^0$.

From the general theory of cross–connections we know that $\mathcal{C}_X^0$, defined like this is an ideal in $\mathcal{C}_X$ and hence it is normal and reductive (see [12], Theorem 4.5). By Equation (1), we can also write,

$$v\mathcal{C}_X' = \{ U \subseteq X : U \cap N(B*) = \{0\} \}$$

We shall now derive a relation between the categories $\mathcal{C}_X'$ and $\mathcal{C}_X^0$.

**Proposition 8** If $B : X \times Y \to K$ is a bilinear form, with the notations introduced above, $\mathcal{C}_X^0$ is a full subcategory of $\mathcal{C}_X'$.

**Proof** Since both $\mathcal{C}_X^0$ and $\mathcal{C}_X'$ are full subcategories of $\mathcal{C}_X$, the claim can be settled if we prove that every object of $\mathcal{C}_X^0$ is in $\mathcal{C}_X'$. For this let $U \in v\mathcal{C}_X^0$, so that $U \in M\Gamma'(V)$ (see Equation (6)) for some $V$ in $v\mathcal{C}_Y'$. In other words $U$ is an algebraic complement of $\tilde{V}$ in $X$. From definition of $\tilde{V}$ (see Equation (4.1)) it is clear that $N(B_*) \subseteq \tilde{V}$ so that $U \oplus \tilde{V} = X$ implies $U \cap N(B_*) = \{0\}$. Hence by Equation (1) $U \in v\mathcal{C}_X'$. Thus we conclude the proof.

Again we are going to show that the above category inclusion is both ways. And that is the harder part of the result.

Let $X'$ and $Y'$ be two fixed algebraic complements of $N(B_*)$ and $N(B^*)$ in $X$ and $Y$ respectively. In other words, we have $X' \oplus N(B_*) = X$ and $Y' \oplus N(B^*) = Y$. First we see the following
PROPOSITION 9 If $X'$ is such that $X' \oplus N(B_*) = X$, then $\hat{X}' = N(B^*)$.

PROOF Let $X'$ be such that $X' \oplus N(B^*) = X$. Also let $y_0 \in \hat{X}'$, so that by Equation (4.2), we have $y_0 \in Y$ and $B(x, y_0) = 0$ for all $x \in X'$. Again by definition of $N(B_*)$, $B(x, y_0) = 0$ for all $x \in N(B_*)$. Let $x \in X$ so that $x = x' + x_0$ where $x' \in X'$ and $x_0 \in N(B_*)$.

Hence

$$B(x, y_0) = B(x' + x_0, y_0) = B(x', y_0) + B(x_0, y_0) = 0$$

for all $x \in X$.

Thus $y_0 \in N(B^*)$. Conversely, let $y_0 \in N(B^*)$ so that $B(x, y_0) = 0$ for all $x \in X$. Again $X' \subseteq X$. So $B(x, y_0) = 0$ for all $x \in X'$ which gives $y_0 \in \hat{X}'$. From these we conclude that $\hat{X}' = N(B^*)$. \qed

Dually we have,

PROPOSITION 10 If $Y'$ is a subspace of $Y$ such that $Y' \oplus N(B^*) = Y$, then $\hat{Y}' = N(B_*)$.

Subspaces of the type $X'$ and $Y'$ considered above play an important role in the coming discussions. So we generalize them in the following

DEFINITION 1 Let $X$ and $Y$ be two vector spaces and $B : X \times Y \rightarrow K$ be a bilinear form. Then subspaces $X'$ and $Y'$ such that $X' \oplus N(B_*) = X$ and $Y' \oplus N(B^*) = Y$ are called maximal subspaces of $X$ and $Y$ respectively relative to $B$.

Though $B$ is an arbitrary bilinear form on the cartesian product of $X$ and $Y$, its restrictions on maximal subspaces have a nice property. Now our aim is to find this.

Let $X'$ and $Y'$ be two maximal subspaces of $X$ and $Y$. Also $B'$ be the restriction of $B$ on $X' \times Y'$. We claim that,
PROPOSITION 11  \( B': X' \times Y' \to K \) is a non-degenerate bilinear form (see p. 140, [18]).

PROOF Let \( x \in X' \) be such that \( B'(x, y') = 0 \) for all \( y' \in Y' \). If \( y \in Y \), then \( y = y' + y_0 \) where \( y' \in Y' \) and \( y_0 \in N(B^*) \).

Now
\[
B(x, y) = B(x, y' + y_0) = B(x, y') + B(x, y_0) = B(x, y') = B'(x, y')
\]
Thus \( B(x, y) = 0 \) for all \( y \in Y \) so that \( x \in N(B^*) \). But \( X' \cap N(B^*) = \{0\} \) so that \( x = 0 \). Similarly, we can see that if \( B'(x, y) = 0 \) for all \( x \in X' \), then \( y = 0 \). □

REMARK 1  \( B': X' \times Y' \to K \) is a non-degenerate bilinear form implies that the dimensions of \( X' \) and \( Y' \) are equal (see [18] Theorem 5.2). Hence maximal subspaces have same dimension.

Consider two maximal subspaces \( X' \) and \( Y' \) of \( X \) and \( Y \) with respect to the bilinear form \( B \). Let \( U \) and \( V \) be subspaces of \( X \) and \( Y \) respectively. We define,

\[
U^+ = \{ y \in Y': B(u, y) = 0 \text{ for all } u \in U \} \quad \text{and} \quad V^+ = \{ x \in X': B(x, v) = 0 \text{ for all } v \in V \}.
\]

By definition it follows that \( U^+ \) and \( V^+ \) are subspaces of \( Y \) and \( X \) respectively.

REMARK 2  In particular if \( U \subseteq X' \) and \( V \subseteq Y' \), then since \( B \) on \( X' \times Y' \) is non-degenerate, we can see that

\[
(U^+)^+ = U \quad \text{and} \quad (V^+)^+ = V \quad \text{(see [18] p.142)}
\]
If \( V \) is a subspace of \( Y' \), we get a relation connecting \( V^+ \) and \( \hat{V} \) in the following

**Proposition 12** If \( V \) is a subspace of \( Y' \), then \( \hat{V} = V^+ \oplus N(B_*) \).

**Proof** Let \( x \in \hat{V} \) so that \( B(x, v) = 0 \) for all \( v \in V \) (see Equation 4.1). Here \( X = X' \oplus N(B_*) \), so that \( x \in \hat{V} \) implies \( x \in X \) and hence \( x = x' + x_0 \) where \( x' \in X' \) and \( x_0 \in N(B_*) \). Now

\[
0 = B(x, v) = B(x' + x_0, v) = B(x', v) + B(x_0, v) = B(x', v)
\]

Hence \( x' \in V^+ \). This shows that every element of \( \hat{V} \) can be expressed as the sum of two elements of which one comes from \( V^+ \) and the other from \( N(B_*) \). Again by definition \( V^+ \subseteq X' \) so that \( V^+ \cap N(B_*) = \{0\} \).

Conversely, let \( x' \in V^+ \) and \( x_0 \in N(B_*) \), then \( B(x' + x_0, v) = B(x', v) + B(x_0, v) = 0 \) for all \( v \in V \). Hence \( x' + x_0 \in \hat{V} \). Thus we get the equality of the two spaces. \( \square \)

The foregoing Proposition has the following consequence.

**Proposition 13** If \( U \subseteq X \) is such that \( U \cap N(B_*) = \{0\} \), then we can find \( V \subseteq Y \) with \( V \cap N(B^*) = \{0\} \) and \( U \oplus \hat{V} = X \).

**Proof** Let \( U \subseteq X \) be such that \( U \cap N(B_*) = \{0\} \). Hence we can find an \( X' \) such that \( X' \oplus N(B_*) = X \) and \( U \subseteq X' \). This can be done by choosing bases for \( N(B_*) \) and \( U \) and filling this to form a basis for \( X \). Then take out the basis elements of \( N(B_*) \). The remaining elements will span \( X' \). Again let \( U_1 \subseteq X' \) be such that \( U \oplus U_1 = X' \). Choose \( Y' \subseteq Y \) such that \( Y' \oplus N(B^*) = Y \). Take \( U_1^+ = \{ y \in Y' : B(u, y) = 0 \text{ for all } u \in U_1 \} \) and put \( V = U_1^+ \). Hence we get
\( V \cap N(B^*) = \{0\} \) and

\[
\hat{V} = V^+ \oplus N(B_*) \\
= (U_1^*)^+ \oplus N(B_*) \\
= U_1 \oplus N(B_*) \\
U \oplus \hat{V} = U \oplus U_1 \oplus N(B_*) \\
= X' \oplus N(B_*) \\
= X.
\]

This completes the proof. \( \square \)

We can now prove the following result that gives the equality of the categories \( C_X^\circ \) and \( C_X'^\circ \).

**Proposition 14** \ The categories \( C_X' \) and \( C_X^\circ \) are the same.

**Proof** \ In view of the Proposition 8, it is sufficient to show that every object in \( C_X' \) is also in \( C_X^\circ \). Let \( U \subseteq X \) be such that \( U \cap N(B_*) = \{0\} \). Then using Proposition 13 there exists \( V \subseteq Y \) such that \( V \cap N(B_*) = \{0\} \) and \( U \oplus \hat{V} = X \). Hence by Equation (6), \( U \in vC_X^\circ \). Note that \( U \oplus \hat{V} = X \) implies \( U \in MT'(V) \) (see Proposition 7).

In the coming section we are going to see that \( C_X' \) and \( C_Y' \) are the categories on which the cross-connection is constructed.

1.3 Cross-connection

If \( B: X \times Y \to K \) is a bilinear form, in Proposition 6 we obtained a local isomorphism \( \Gamma': C_Y' \to N^* C_X \). Hence we get \( \Gamma''': C_X^\circ \to N^* C_Y' \) as a cross-connection (see Theorem IV.15 [13]). But in Proposition 14 we proved that \( C_X^\circ = C_X' \). Thus the cross-connection \( \Gamma''' \) is from \( C_X' \) to \( N^* C_Y' \). If we take its dual (see Theorem IV.26 [13]), we get \( \Gamma'''' : C_Y' \to N^* C_X' \) and this \( \Gamma'''' \) is
nothing but \( \Gamma' \) on the category \( C'_X \). We denote \( \Gamma'' \) by the symbol \( \Gamma \). Hence we have the following

**Theorem 15** If \( B: X \times Y \to K \) is a bilinear form, then there is a cross-connection \( \Gamma: C'_Y \to N^*C'_X \) where \( \Gamma \) is defined by Equations (5.1) and (5.2) respectively.

The foregoing theorem justifies the following

**Definition 2** The cross-connection \( \Gamma: C'_Y \to N^*C'_X \) defined by Equations (5.1) and (5.2) is called the cross-connection induced by the bilinear form \( B \).

Since \( \Gamma: C'_Y \to N^*C'_X \) is a cross-connection, its dual \( \Gamma^*: C'_X \to N^*C'_Y \) is also a cross-connection (see Theorem IV.15 [13]). The object map of \( \Gamma^* \) will be such that

\[
\nu \Gamma^*: U \mapsto H(e', -)
\]

where \( U \in vC'_X \) and \( e' \in E(L(Y)) \) is such that \( N(e') = \hat{U} \). The following result is a dual of Proposition 7.

**Proposition 16** If \( \Gamma: C'_Y \to N^*C'_X \) is the cross-connection induced by the bilinear form \( B \), then for any \( U \in vC'_X \)

\[
M \Gamma^*(U) = \{ V \subseteq Y : \hat{U} \oplus V = Y \}.
\]

**Proof** By Equation (8) we get

\[
\Gamma^*(U) = H(e', -) \text{ with } N(e') = \hat{U}
\]

Hence

\[
M \Gamma^*(U) = MH(e', -) = \{ V \subseteq Y : N(e') \oplus V = Y \} \text{ (see Proposition II.16)}
\]

\[
= \{ V \subseteq Y : \hat{U} \oplus V = Y \}.
\]
This completes the proof of the proposition. □

Propositions 13 and 16 have the following consequence.

**PROPOSITION 17** Let \( U \subseteq X \) be such that \( U \cap N(B^*) = \{0\} \). Then there exists \( V \subseteq Y \) with \( V \cap N(B^*) = \{0\} \) such that

\[
U \oplus \hat{V} = X \text{ if and only if } \hat{U} \oplus V = Y.
\]

**PROOF** Since \( U \subseteq X \) is such that \( U \cap N(B^*) = \{0\} \), by Proposition 13 there exists \( V \subseteq Y \) with \( V \cap N(B^*) = \{0\} \) and such that \( U \oplus \hat{V} = X \). Hence by Proposition 7, \( U \in M\Gamma(V) \). Now by Theorem IV.15 [13] we know that \( U \in M\Gamma(V) \) if and only if \( V \in M\Gamma^*(U) \). Now the rest of the result follows by Proposition 16. □

If \( B: X \times Y \to K \) is a non-degenerate bilinear form, then it is clear that \( N(B^*) = N(B^*) = \{0\} \). Hence in this case \( C'_X = C_X \) and \( C'_Y = C_Y \). Thus we have,

**PROPOSITION 18** If \( B: X \times Y \to K \) is a non-degenerate bilinear form, then there is a cross-connection \( \Gamma: C_Y \to N^*C_X \).

**REMARK 3** It can be seen that the above cross-connection is an isomorphism between the categories. But we are not going to use this result anywhere in the sequel and hence the details are avoided.
2. ADJOINT AND TRANSPOSE

In the previous section, we implemented the theory of cross-connections in vector spaces and linear transformations to get a cross-connection from a bilinear form and called it the cross-connection induced by the bilinear form. Here in this section we are to find the relation between the cross-connection transposes and a particular type of adjoints of morphisms in the categories.

2.1 Transpose of a morphism

Let \( \Gamma \) be the cross-connection induced by the bilinear form \( B \). For a morphism \( t \in C'_Y \), we know \( \Gamma(t) = \eta \sigma_X (s, -) \eta^{-1} \) (see Equation (5.2)). By general theory we know that \( s \) is a transpose of \( t \) and vice versa (see [12], p. 41). Also we know that transpose of a morphism is not unique. In Equation (5.2) \( s \) is expressed in terms of the isomorphisms \( \psi \) and \( \psi' \). These maps depend on the choice of the complements \( R(e) \) and \( R(e') \) where \( N(e) = V \) and \( N(e') = V' \) are fixed (see Equation II.(4)). But all transposes of \( t \) will have the common factor \( (B^* - tB^*)^* \). Hence we can call it the transpose of \( t \) and make the following

**Definition 3** A morphism \( g \) in \( C'_Y \) is called the transpose of the morphism \( f \) in \( C'_X \) under the cross-connection induced by the bilinear form \( B \) if \( fB* = B*g* \).

If \( g \) is the transpose of \( f \), we know that \( f \) is the transpose of \( g \) (see Corollary IV.21 [13]). Hence we call the pair \( (f, g) \) a transpose pair.

2.2 Category adjoint

If \( U \) and \( V \) are two objects and \( t: U \to V \) is a morphism in the category \( C_Y \), the classical vector space adjoint \( t^*: V^* \to U^* \) (see Equation II.(8)) is not a morphism in \( C_{Y^*} \) since \( U^* \) and \( V^* \) are not objects in \( C_{Y^*} \). Hence we find out another type of adjoints for the morphisms in \( C_Y \).
Let \( t: V \to V' \) be a morphism in the category \( C_Y \). Fix two algebraic complements say \( W \) and \( W' \) of \( V \) and \( V' \) such that \( W \oplus V = W' \oplus V' = Y \). We have the two isomorphisms \( \psi \) and \( \psi' \) defined by (Equation II.(4)), such that \( \psi: X/W \to V \) and \( \psi': X/W' \to V' \). Hence for the morphism \( t \), we get a unique morphism (unique because the complements are fixed) \( \psi t \psi'^{-1} \) from \( X/W \) to \( X/W' \). Now since \( \psi t \psi'^{-1} \) is a morphism in \( \tilde{C}_Y \), the isomorphism \( H \) (since \( Y \) is finite dimensional) will take it to \( \omega'(\psi t \psi'^{-1})^* \omega \) from \( W'^\perp \) to \( W^\perp \) (see Equation II(9.2)). We denote this morphism by \( t^\dagger \). We consolidate this in the following

**Proposition 19** Let \( t: V \to V' \) be a morphism in the category \( C_Y \) and \( W \) and \( W' \) be two fixed algebraic complements of \( V \) and \( V' \) in \( Y \). Then there exists a unique morphism \( t^\dagger \) in \( C_{Y^*} \) such that

\[
t^\dagger : W'^\perp \to W^\perp \quad \text{and} \quad t^\dagger = \omega'^{-1}(\psi t \psi'^{-1})^* \omega .
\]

We generalize the morphism obtained in the foregoing Proposition to get the following

**Definition 4** Let \( t: V \to V' \) be a morphism in the category. Then \( t^\dagger : W'^\perp \to W^\perp \) where \( W \) and \( W' \) two algebraic complements of \( V \) and \( V' \) is called a category adjoint of \( t \).

**Remark 4** As in the case of transpose, here also the category adjoint is not unique. It depends on the complements chosen.

### 2.3 Relation between transpose and adjoint

For a morphism \( t: V \to V' \) in \( C_Y \), we have its category adjoint \( t^\dagger \) in \( C_{Y^*} \) such that \( t^\dagger : W'^\perp \to W^\perp \) where \( W \) and \( W' \) are two algebraic complements of \( V \).
and $V'$ respectively. Again the map $J^{-1}$ (see Equation II(10.2)) from $C_Y^*$ to $N^*C_Y$ is such that

$$J^{-1}: t^* \mapsto \eta_e C_Y(s, -) \eta_e^{-1}$$

where $N(e') = \perp(W'^\perp)$, $N(e) = \perp(W^\perp)$, $s = \psi^{-1}u\psi'$ and $u = (\omega' t^* \omega^{-1})^*$. Here note that $\psi$ and $\psi'$ are interchanged because the morphism $t^*$ is from $W'^\perp$ to $W^\perp$. Similarly $\omega, \omega'$; $e, e'$ are also interchanged.

The above expressions give $N(e') = W'$, $N(e) = W$ and substituting the expression for $t^*$, we get $s = t$. Hence we have the following.

**PROPOSITION 20** Let $t: V \to V'$ be a morphism in $C_Y$. Then for two algebraic complements $W$ and $W'$ of $V$ and $V'$ respectively, if $t^*: W'^\perp \to W^\perp$ is a category adjoint of $t$, then the image of $t^*$ in $N^*C_Y$ is $\eta_e C_Y(t, -) \eta_e^{-1}$ where $N(e) = W$ and $N(e') = W'$.

Coming back to cross-connections, if $t: V \to V'$ is a morphism in $C_Y$, then its transpose $s: \hat{W}' \to \hat{W}$ in $C_X'$ where $W$ and $W'$ are two algebraic complements of $V$ and $V'$ has the property

$$\Gamma(t) = \eta_h C_X(s, -) \eta_{h'}^{-1}$$

(see Equation 5.2)

Here $h$ and $h'$ are idempotents in $L(X)$ such that $N(h) = \hat{V}$ and $N(h') = \hat{V}'$. From general theory we know that if $s$ is a transpose of $t$, then $t$ is a transpose of $s$ such that

$$\Gamma^*(s) = \eta_e C_Y(t, -) \eta_e^{-1}$$

where $N(e) = W$ and $N(e') = W'$ (see [12], Proposition 4.6). Collecting all these together we get the following.

**PROPOSITION 21** Let $t: V \to V'$ be a morphism in $C_Y'$ and $W$ and $W'$ be two fixed algebraic complements of $V$ and $V'$ in $Y$, then its transpose $s: \hat{W}' \to \hat{W}$ has the property that

$$\Gamma^*(s) = \eta_e C_Y(t, -) \eta_e^{-1}$$
where $N(e) = W$ and $N(e') = W'$.

Here the categories are such that $C_X' \subseteq C_X$ and $C_Y' \subseteq C_Y$. Hence we can apply both the above Propositions in $C_X'$ and $C_Y'$. As a direct consequence of Propositions 20 and 21, we get

**Theorem 22** Let $t: V \rightarrow V'$ be a morphism in $C_X'$ and $W$ and $W'$ be two fixed algebraic complements of $V$ and $V'$ respectively. Then its transpose $s: \hat{W'} \rightarrow \hat{W}$ and category adjoint $t^*: W'^\perp \rightarrow W^\perp$ have the same image in $N^*C_Y'$.

**Remark 5** Using the isomorphism in Theorem 11.26, we can identify the spaces $C_Y^*$ and $N^*C_Y$. Hence under this identification the foregoing Theorem implies that for any morphism in a category, its category adjoint will be the image of its transpose under the cross-connection.

**3. SEMIGROUP OF LINKED PAIRS**

In the general theory, cross-connection semigroup is the regular semigroup of linked pairs of normal cones (see IV.5.2 [13]). Here our aim is to characterize the linked pairs in terms of linear transformations and the bilinear form.

**3.1 Relation between biordered sets**

Here we shall see a property of the normal cones in the regular semigroup $TC_X$ that is not shared by linear transformation in $L(X)$. Recall that $TC_X'$ is the regular semigroup of all normal cones in $C_X'$.

If $\Gamma: D \rightarrow N^*C$ is a cross-connection, we know that $E_\Gamma = \{(c, d): c \in MT(\Gamma(d))\}$ is a biordered set (see Theorem V.1 [13]). In order to characterize this set, we introduce a new definition.
DEFINITION 5 A pair \((U, V) \in vC'_X \times vC'_Y\) is called a complemented pair if \(U \oplus \hat{V} = X\).

This definition suggests the following

PROPOSITION 23 If \((U, V) \in vC'_X \times vC'_Y\), then the following are equivalent.

(a) \((U, V)\) is a complemented pair

(b) \((U, V) \in E_R\)

(c) \(U \oplus V = Y\)

PROOF \((U, V)\) is a complemented pair if and only if \(U \oplus \hat{V} = X\). By Proposition 7, we know that this is possible if and only if \(U \in M \Gamma(V)\). By definition of \(E_R\), we get \(U \in M \Gamma(V)\) if and only if \((U, V) \in E_R\). Hence we see (a) \iff (b).

Again \((U, V) \in E_R\) is equivalent to \(U \in M \Gamma(V)\). By Theorem IV.15 [13] we get that this is possible if and only if \(V \in M \Gamma^*(U)\) which in turn gives \(\hat{U} \oplus V = Y\) (see Proposition 16). This gives (a) \iff (c).

From the foregoing Proposition we know that if \((U, V)\) is a complemented pair, then \(U \in M \Gamma(V)\). Hence from the general theory (see IV.4.1 [13]) there is a unique idempotent cone \(\gamma(U, V)\) in \(TC'_X\) with vertex \(U\) such that \(\Gamma(V) = H(\gamma(U, V), -)\). Similarly the unique idempotent cone \(\gamma^*(U, V)\) in \(TC'_Y\) is such that \(\Gamma^*(U) = H(\gamma^*(U, V), -)\). We proceed to obtain some of the properties the cones \(\gamma(U, V)\) and \(\gamma^*(U, V)\). If the vertices of such cones are maximal subspaces (see Definition 1), they will have the following property.

PROPOSITION 24 Let \(X'\) be a maximal subspace of \(X\). Then for any two maximal subspaces \(Y'\) and \(Y''\) of \(Y\), \(\gamma(X', Y') = \gamma(X', Y'')\).

PROOF By Proposition 10 we get \(\hat{Y'} = \hat{Y''} = N(B^*)\). Hence \(X' \in M \Gamma(Y')\) and also \(X' \in M \Gamma(Y'')\). Thus we get the cones \(\gamma(X', Y')\) and \(\gamma(X', Y'')\) such
that their vertices are the same and also such that $\Gamma(Y') = H(\gamma(X', Y'), -)$, $\Gamma(Y'') = H(\gamma(X', Y''), -)$. Again by Proposition 7,

$$M\Gamma(Y') = \{X' \subseteq X: X' \oplus \hat{Y'} = X\}$$

$$= \{X' \subseteq X: X' \oplus N(B_*) = X\} \quad \text{(see Proposition 10)}$$

Also $M\Gamma(Y'') = \{X' \subseteq X: X' \oplus N(B_*) = X\}$.

These give $M\Gamma(Y') = M\Gamma(Y'')$ which in turn gives

$$MH(\gamma(X', Y'), -) = MH(\gamma(X', Y''), -)$$

$$\therefore M_{\gamma(X', Y')} = M_{\gamma(X', Y'')} \quad \text{(see Corollary III.8 [13])}$$

$$\therefore \gamma(X', Y') \leq \gamma(X', Y'') \quad \text{(see Theorem III.11(a) [13])}$$

Hence the proof.

As a dual of the foregoing Proposition, we get

**Proposition 25** Let $Y'$ be a maximal subspace of $Y$. Then for any two maximal subspaces $X'$ and $X''$ of $X$, $\gamma^*(X', Y') = \gamma^*(X'', Y')$.

To get the necessary relation between the biordered sets, we need some more results. The first among them is the following
Proposition 26  Corresponding to every element of $E_{\Gamma}$, we get a unique idempotent cone in $E(TC'_X)$ and conversely given a cone in $E(TC'_X)$, we can find an element in $E_{\Gamma}$.

Proof  Let $(U, V)$ be an element in $E_{\Gamma}$ so that we get $U \in M\Gamma(V)$ (see Proposition 23). Hence the unique cone $\gamma(U, V)$ is in $E(TC'_X)$. Conversely, if $\epsilon$ is an idempotent cone in $TC'_X$, assume that $U$ is the vertex of the cone. Hence $U \subseteq X$ is such that $U \cap N(B_*) = \{0\}$. So we can find $V \subseteq Y$ such that $V \cap N(B^*) = \{0\}$ and also $U \oplus \hat{V} = X$ (see Proposition 13). This implies that $(U, V) \in E_{\Gamma}$ (see Proposition 23).

Again we have another result.

Proposition 27  If $(U, V)$ is a complemented pair, then we get a unique idempotent cone $\epsilon$ in $TC_X$ such that $\epsilon \mid C'_X = \gamma(U, V)$ and $\epsilon = \gamma e$ where $e \in E(L(X))$ is such that $N(e) = \hat{V}$ and $R(e) = U$.

Proof  Since $(U, V)$ is a complemented pair, we get $U \oplus \hat{V} = X$ (see Proposition 23). Let $\epsilon$ be the idempotent linear transformation in $L(X)$ such that $R(\epsilon) = U$ and $N(\epsilon) = \hat{V}$. Also let $\epsilon$ be the cone in $TC_X$ defined by $\epsilon(W) = e \mid W$ for $W \subseteq X$. Here the cone $\epsilon$ is such that $\epsilon(U) = e \mid U = 1_U$ and hence $\epsilon$ is an idempotent cone (see Theorem III.2(a) [13]). Further if $\gamma(U, V)$ is the unique cone corresponding to the complemented pair $(U, V)$, we denote this cone by $\gamma$. Hence the vertices of the cones, $\epsilon$ and $\gamma$ are the same. Further,

$$M_{\gamma} = MH(\gamma, -) = M\Gamma(V) = \{W \in vC'_X : W \oplus \hat{V} = X\}$$

But $\hat{V}$ is the null space of $e$ and hence $W$ is a complement of $\hat{V}$ if and only if $e \mid W$ is an isomorphism. Hence we get

$$M_{\gamma} = \{W \in vC'_X : e \mid W \text{ is an isomorphism}\}$$
\[ \{ W \in vC'_X : \epsilon(W) \text{ is an isomorphism} \} = M_{lC'_X} \quad \text{(see Equation III.(1) [13]).} \]

Thus \( \epsilon \mid C'_X \) and \( \gamma \) have the same vertices and \( M \)-sets. Hence they are both \( L \) and \( R \) related and gives \( \gamma = \epsilon \mid C'_X \).

Propositions 26 and 27 lead to the fact that

**Proposition 28**

Given any \( (U, V) \) in \( E(TC'_X) \), we get a unique extension \( \tilde{\gamma}(U, V) \) in \( E(TC_X) \) such that \( \gamma(U, V) = \tilde{\gamma}(U, V) \mid C'_X \). Again \( \tilde{\gamma} \) has the property \( \tilde{\gamma}(W) = e \mid W \) where \( e \) is the idempotent in \( L(X) \) such that \( R(e) = U \) and \( N(e) = \hat{V} \).

**Remark 6**

Extension of a linear transformation to a larger space is not unique. But for normal cones, as seen in the foregoing Proposition unique extensions are possible.

For our convenience, we denote the cones \( \gamma(U, V) \) and \( \tilde{\gamma}(U, V) \) by the same notation \( (U, \hat{V}) \). Hence the map \( \theta \) defined by

\[ \theta : (U, \hat{V}) : (U, V) \quad \text{(9)} \]

is an inclusion from \( E(TC'_X) \) to \( E(TC_X) \). Also it is true that \( \theta \) is a regular bimorphism. We use this \( \theta \) to get a morphism (see Theorem V.7 [13]) between \( TC'_X \) and \( TC_X \).

**3.2 Embedding of \( TC'_X \) in \( TC_X \)**

The inclusion \( \theta \) (see Equation (9)) between the biordered sets of \( TC'_X \) and \( TC_X \) can be used to get an embedding between these semigroups. For this we make a few notations and observations.

Since \( TC'_X \) and \( TC_X \) are regular semigroups, they have their cross-connections (see Theorem IV.17 [13]) say \( I^r_S \) and \( I^l_S \). For these cross-connections, we observe the following.
PROPOSITION 29  The cross-connection $\Gamma_S$ of the regular semigroup $TC_X$ is between $R(TC_X)$ and $N^*C_X$.

PROOF  Generally, we know that the cross-connection of a regular semigroup $S$ is between $R(S)$ and $N^*L(S)$ (see Theorem IV.17 [13]). For any normal reductive category $C$ we know that $C$ is isomorphic to $L(TC)$ (see Theorem III.19 [13]). Hence if we replace $L(TC_X)$ by the isomorphic copy $C_X$, we can conclude the proof.

Here we have one more result on the cross-connection $\Gamma_S$.

PROPOSITION 30  Let $\Gamma_S$ be the cross-connection of the regular semigroup $TC_X$. Then there is a bijective correspondence between $E_{\Gamma_S}$ and $E(TC_X)$.

PROOF  For any cross-connection $\Gamma$, it is true that there is a bijection from $E_{\Gamma}$ onto $E(\hat{S}\Gamma)$ where $\hat{S}\Gamma$ is the cross-connection semigroup of $\Gamma$ (see Remark IV.3 [13]). Again, for any regular semigroup say $S$, $S$ is isomorphic to $\hat{S}\Gamma$ (see Theorem IV.38 [13]). These two together gives the result.

From Proposition 29 we can infer that the cross-connections $\Gamma_S$ and $\Gamma'_S$ are such that

\begin{align}
\Gamma_S: R(TC_X) &\rightarrow N^*C_X \quad \text{and} \\
\Gamma'_S: R(TC'_X) &\rightarrow N^*C'_X
\end{align}

Again by Proposition 30 $E_{\Gamma_S}$ and $E_{\Gamma'_S}$ can be replaced by $E(TC_X)$ and $E(TC'_X)$ respectively. Hence we define

\begin{align}
F: C'_X &\rightarrow C_X \quad \text{and} \\
\theta: E(TC'_X) &\rightarrow E(TC_X)
\end{align}

where $F$ is the inclusion functor and $\theta$ defined by Equation (9). We make all these preliminaries to get the following
PROPOSITION 31 \((\theta, F)\) defines an injective morphism between the cross-connections \(\Gamma'_S\) and \(\Gamma_S\) (see V.7 [13]).

PROOF Let the pairs \((U, V)\) and \((U', V')\) in \(\nu C'_X \times \nu C'_X\) be denoted by \(e\) and \(e'\) respectively so that the cones \(\gamma(U, V)\) and \(\gamma(U', V')\) will be \(\gamma(e)\) and \(\gamma(e')\). Again, \(U^e\) and \(V^e\) denote the first and second co-ordinates of the ordered pair corresponding to \(e\).

Here by our notations

\[
F(\gamma(e)(U^e')) = F(\gamma(U, V)(U')) = \gamma(U, V)(U')
\]

Again

\[
\gamma(\theta(e))(U^{\theta(e')}) = \gamma(U, V)(U').
\]

But by Proposition 27 we have \(\gamma = \bar{\gamma}'| C'_X\) and so \(\gamma(U, V)(U') = \bar{\gamma}(U, V)(U')\). Thus, \(F(\gamma(e)(U^e')) = \gamma(\theta(e))(U^{\theta(e')})\) and hence by Theorem V.7 [13] the pair \((\theta, F)\) defines a unique morphism \(m\) between the cross-connections \(\Gamma'_S\) and \(\Gamma_S\). Also both \(F\) and \(\theta\) are inclusions and hence both are injections. Thus \(m\) is an injective morphism (see Proposition V.8(a) [13]).

Now we shall find the morphism \(m\). If \(\gamma = \gamma(e) \ast f^\circ\) is a cone in \(TC'_X\), then \(m\) defined by \(m(\gamma) = \gamma(\theta(e)) \ast F(f)^\circ\) is a homomorphism from \(TC'_X\) into \(TC_X\) (see Equation V.(9) [13]). This gives \(m(\gamma) = \bar{\gamma}(U, V) \ast f^\circ\). In particular, if \(\gamma = \gamma(U, V) \ast f\) where \(f\) is an isomorphism in \(C'_X\), then \(m(\gamma) = \bar{\gamma}(U, V) \ast f\).

Again, since both \(F\) and \(\theta\) are injections, \(m\) is also an injection. Hence we have

THEOREM 32 There is an injective homomorphism \(m: TC'_X \rightarrow TC_X\) such that if \(\gamma = \gamma(U, V) \ast f^\circ\) is a cone in \(TC'_X\), then

\[
m(\gamma) = \bar{\gamma}(U, V) \ast f^\circ
\]
3.3 A subsemigroup of $L(X)$

Here we shall find a subsemigroup of $L(X)$ using the homomorphism $m$ (see Equation (12)). Further we see that this semigroup depends on the de-generacy of the bilinear form.

If $\Gamma$ is a cross-connection, then recall that $U\Gamma$ is a subsemigroup of the semigroup of normal cones (see Equation IV.25 [13]). Also if $\gamma$ is a cone in $U\Gamma$, then $\gamma = \gamma(U, V) * f$ when $f$ is an isomorphism from $U$ to $C_\gamma$ (see Lemma IV.28 [13]). In the normal category $C'_X$, there is an inclusion from $U\Gamma$ into $TC'_X$. Here this inclusion is denoted by $i$. Hence, we get the diagram

$$
\begin{array}{ccc}
TC'_X & \xrightarrow{m} & TC_X \\
\downarrow{\phi} & & \downarrow{\phi} \\
U\Gamma & \xrightarrow{\bar{m}} & L(X)
\end{array}
$$

Remember that $\phi$ is the isomorphism defined in Proposition II.5. Let

$$\bar{m} = i \circ m \circ \phi$$

From the properties of the factors of this product, it is clear that $\bar{m}$ is an injective homomorphism from $U\Gamma$ into $L(X)$. We shall next find the action of $\bar{m}$ on a cone in $U\Gamma$.

Let $\gamma = \gamma(U, V) * f$ be an arbitrary cone in $U\Gamma$. Now $m(\gamma) = \tilde{\gamma}(U, V) * f$. Again, the isomorphism $\phi$ is such that $\phi(\tilde{\gamma}(U, V) * f) = \tilde{\gamma}(U, V)(X)\bar{f}$ (see Proposition II.5) where $\tilde{\gamma}(U, V)(X)$ is the component of $\tilde{\gamma}(U, V)$ at $X$. We summarize the results obtained thus far in

**Proposition 33** There is an injective homomorphism $\bar{m}$ from $U\Gamma$ into $L(X)$ such that if $\gamma = \gamma(U, V) * f$ in $U\Gamma$,

$$(13) \quad \bar{m}(\gamma) = \tilde{\gamma}(U, V)(X)\bar{f}$$
where $\tilde{\gamma}(U, V)$ is the unique cone in $TC_X$ determined by the complemented pair $(U, V)$.

Since $\tilde{m}$ is an injective homomorphism, the image of $UT$ under this map in $TC_X$ is a regular subsemigroup of $L(X)$. We denote this image by the notation $L'(X)$.

For applications, it is necessary to have an internal description of the semigroup $L'(X)$. For this we shall require the following

**Proposition 34** Let $\gamma \in UT$ and $X'$ be a maximal subspace of $X$. Then $\gamma = \epsilon(X', Y') \ast \gamma(X')$ where $Y'$ is a maximal subspace of $Y$ and $\epsilon(X', Y')$ is the unique idempotent cone in $TC_X$ determined by the complemented pair $(X', Y')$ such that

$$\Gamma(Y') = H(\epsilon(X', Y'), -).$$

**Proof** Let $\gamma \in UT$ so that $\gamma = \gamma(U', V) \ast f$ where $\gamma(U', V)$ is the unique cone determined by the complemented pair $(U', V)$ and $f: U' \to C$ is an isomorphism in $C_X$ (see Lemma IV.28 [13]). Let $X'$ be a maximal subspace of $X$. Hence if $Y'$ is any maximal subspace of $Y$, we know that $(X', Y')$ is a complemented pair (see Propositions 9 and 10) and the uniquely determined cone be $\epsilon(X', Y')$. Again $\epsilon(X', Y') \ast 1_X = \epsilon(X', Y')$ and hence $\epsilon(X', Y')$ is also in $UT$ (see Lemma IV.28 [13]). Now fix $Y'$ in such a way that $V \subseteq Y'$. Thus we have $\Gamma(Y') = H(\epsilon(X', Y'), -)$ and $\Gamma(V) = H(\gamma(U', V), -)$ (see Definitions of $\epsilon(X', Y')$ and $\gamma(U', V)$). Again, $\Gamma$ preserves inclusion so that $\Gamma(V) \subseteq \Gamma(Y')$ which gives $H(\gamma(U', V), -) \subseteq H(\epsilon(X', Y'), -)$. This gives that $\gamma(U', V) \omega \epsilon(X', Y')$ (see Theorem III.11(b) [13]). Hence by definition,

$$\epsilon(X', Y') \gamma(U', V) = \gamma(U', V)$$

$$\therefore (\epsilon(X', Y') \gamma(U', V)) \ast f = \gamma(U', V) \ast f$$

$$\epsilon(X', Y') \ast (\gamma(U', V) \ast f)(X') = \gamma(U', V) \ast f$$

i.e.,

$$\epsilon(X', Y') \ast \gamma(X') = \gamma$$
III.3 Semigroup of linked pairs

This completes the proof.

Note that we use the notation $e(X', Y')$ to distinguish the idempotents with vertices maximal subspaces.

Here the choice of $Y'$ can also be made arbitrary as seen in the following

**Corollary 35** If $(X', Y')$ is a complemented pair of maximal subspaces and $\gamma \in UT$, then $\gamma = e(X', Y') * \gamma(X')$.

**Proof** This is nothing but a consequence of the foregoing Proposition and Proposition 24.

The regular semigroup $L'(X)$ is described in the following

**Theorem 36** If $L'(X)$ is the homomorphic image of $UT$ under $\bar{m}$ in $L(X)$, then

$$L'(X) = \{ f \in L(X): N(B_*) \subseteq N(f) \text{ and } R(f) \cap N(B_*) = \{0\} \}$$

**Proof** Let $f \in L'(X)$ so that by definition of $L'(X), f = \bar{m}(\gamma)$ where $\gamma \in UT$. Let $\gamma = e(X', Y') * \gamma(X')$ (see Corollary 35). Hence $\bar{m}(\gamma) = e\gamma(X')j$ where $e \in E(L(X))$ is such that $N(e) = N(B_*)$ and $R(e) = X'$. Also, $j$ is the corresponding inclusion. But, here the expression for $\gamma$ is not unique. Hence if $\gamma = e(X'', Y'') * \gamma(X'')$, then the corresponding $\bar{m}(\gamma) = e'\gamma(X'')j$ with $R(e') = X''$ and $N(e') = N(B_*)$. Again

$$\gamma = e(X'', Y'') * \gamma(X'') \text{ gives }$$

$$\gamma(X') = e(X'', Y'')(X')\gamma(X'')$$

$$= \bar{e}(X'', Y'')(X')\gamma(X'') \text{ since in } C'_X, \bar{e}(X'', Y'') = e(X'', Y'')$$

But in, $C'_X \bar{e}(X'', Y'')(X') = e'$ which gives $\gamma(X') = e'\gamma(X'')$.

Further, $N(e) = N(e') = N(B_*)$ so that $eRe'$ (see Corollary 1.9). Therefore, $e' = ee'$ and hence $e'\gamma(X'') = ee'\gamma(X'') = e\gamma(X'')$. Thus $f = e\gamma(X')j = e'\gamma(X'')j$. Again for this $f$, $N(f) = N(e\gamma(X'))$ and $N(e) \subseteq N(e\gamma(X'))$. Thus
\( N(e) \subseteq N(f) \) and \( N(e) = N(B_\ast) \) gives \( N(B_\ast) \subseteq N(f) \). Further, since \( \gamma \) is a cone in \( C'_X \), we have \( R(f) = C_\gamma \) is in \( vC'_X \) and this gives \( R(f) \cap N(B_\ast) = \{0\} \).

Conversely, if \( f \in L(X) \) is such that \( N(B_\ast) \subseteq N(f) \) and \( R(f) \cap N(B_\ast) = \{0\} \), let \( f = eu\bar{j} \) be a normal factorization. Assume that \( W = N(e) = N(f) \) and \( R(e) = U \) which gives \( U \oplus W = X \). Construct a subspace \( V \) in \( vC'_Y \) such that \( \hat{V} = W \) for which if \( W' \oplus N(B_\ast) = W \), take \( V = W'^{+} \) so that

\[
\hat{V} = V^{+} \oplus N(B_\ast) \quad \text{(see Proposition 12)}
\]
\[
= W' \oplus N(B_\ast)
\]
\[
= W.
\]

Let \( \gamma(U, V) \) be the unique cone determined by the pair \( (U, V) \) in \( C'_X \). Let \( \gamma = \gamma(U, V) \ast u \) so that \( \gamma \in UT \) (see Lemma IV.28 [13]). Further,

\[
\tilde{m}(\gamma) = (\tilde{\gamma}(U, V) \ast u)(X)j
\]
\[
= \tilde{\gamma}(U, V)(X)uj
\]
\[
= euj
\]
\[
= f
\]

Thus we get the equality in the expression for \( L'(X) \). \( \square \)

Dually we have,

**Theorem 37** \( \text{If } B : X \times Y \to K \text{ is a bilinear form, then we can find an injective homomorphism } \bar{n} : UT^* \to L(Y) \text{ such that the image of } UT^* \text{ in } L(Y) \) denoted by

\( L'(Y) = \{ g \in L(Y) : N(B^*) \subseteq N(g) \text{ and } R(g) \cap N(B^*) = \{0\} \}. \) \( \square \)
Remark 7 If the bilinear form $B$ is non-degenerate, we know that $N(B^*) = N(B_*) = \{0\}$. Hence in this case, the semigroup $L'(X)$ and $L(X)$ will be the same. Similarly, we will get $L'(Y)$ and $L(Y)$ also equal.

3.4 Bilinear form semigroup

We shall conclude this chapter by applying the theory of cross-connections to get a semigroup which is a replica of $\hat{S}\Gamma$ (see Equation IV.(30) [13]) in $L(X) \times L(Y)$. In this subsection, we assume that $B: X \times Y \to K$ is a bilinear form and $\Gamma$ is the induced cross-connection. Recall that a pair of cones $(\rho, \lambda)$ in $U\Gamma \times U\Gamma^{*op}$ is a linked pair if $\chi_{\Gamma}(\rho) = \lambda$. Also $\hat{S}\Gamma$ is the collection of all linked pairs. Here we make a new definition similar to that of transpose pair (see Definition 3) for a pair of linear transformations in $L(X) \times L(Y)$.

Definition 6 A pair $(f, g) \in L(X) \times L(Y)$ is called an adjoint pair if $B(f(x), y) = B(x, g(y))$ for all $(x, y) \in X \times Y$.

This definition has the following consequence.

Proposition 38 If $(f, g) \in L(X) \times L(Y)$, then the following are equivalent

(a) $(f, g)$ is an adjoint pair
(b) $fB^* = B_*g^*$
(c) $gB^* = B^*f^*$.

Proof Let $(f, g)$ be an adjoint pair so that

$B(f(x), y) = B(x, g(y)) \forall (x, y) \in X \times Y$. Hence we have

$yB(f(x), -) = g(y)B(x, -) \forall y \in Y$

i.e., $B(f(x), -) = gB(x, -) = B(x, -)g^*$ (see Equation II.(8))
i.e., \[ f(x) B_* = x(B_g) \forall x \in X \]
i.e., \[ f B_* = B_g \]

Hence (a) \(\Rightarrow\) (b) and reversing the steps we get (b) \(\Rightarrow\) (a). (b) \(\Leftrightarrow\) (c) is a consequence of Proposition 1.

We can now give an alternative expression for a transpose pair.

**PROPOSITION 39** Let \(f \in C'_X\) and \(g \in C'_Y\) be such that \(f \in C_X(U', U)\) and \(g \in C_Y(V', V)\). Then the following are equivalent.

(a) \(g\) is a transpose of \(f\).

(b) \(B(f(x), y) = B(x, g(y))\) for all \((x, y) \in U' \times V'\).

(c) \((U', V), (U, V')\) are in \(\mathcal{E}_f\) and the cones \(\rho = \gamma(U', V) \ast f^o\) and \(\lambda = \gamma^*(U, V') \ast g^o\) are linked.

**PROOF** Let \(g\) be a transpose of \(f\) so that \(f B_* = B_* g^o\) on \(U'\) (see Definition 3). Hence for all \(x \in U'\),

\[(xf) B_* = (xB_*) g^o\]
i.e., \[B(f(x), -) = B(x, -) g^o\]
i.e., \[B(f(x), -) = g B(x, -)\] (See Equation II.(8))

So for all \(y \in V'\)

\[B(f(x), y) = g(y) B(x, -) = B(x, g(y)).\]
i.e., \[B(f(x), y) = B(x, g(y)) \forall (x, y) \in U' \times V'.\]

Converse follows by reversing the steps.

Now to show that (a) \(\Rightarrow\) (c), let \(f\) and \(g\) be transposes. Since \(f \in C_X(U', U)\), its transpose \(g\) will be from a complement of \(\hat{U}\) to a complement
of $\hat{U}'$. Hence $V'$ and $V$ are complements of $\hat{U}$ and $\hat{U}'$. From these we get $\hat{U} \oplus V' = Y$ and $\hat{U}' \oplus V = Y$. These conditions give $(U', V)$ and $(U, V')$ belong to $E_\Gamma$ (see Proposition 23). Hence we can consider the cones $\rho = \gamma(U', V) * f^\circ$ and $\lambda = \gamma^*(U, V') * g^\circ$. Also $\rho \in \Gamma(U, V), \lambda \in \Gamma^*(U, V)$ and $\chi_r(U, V)(\rho) = \lambda$ (see Equation IV.29 [13]). Hence $(\rho, \lambda)$ is a linked pair. Converse follows from the definition of linked pairs (see Equation IV.(30) [13]).

The following Proposition is also immediate.

**Proposition 40** Let $(f, g) \in L'(X) \times L'(Y)$ be such that it is an adjoint pair. Then $N(f) = \overline{R(g)}$ and $N(g) = \overline{R(f)}$.

**Proof** $(f, g)$ is an adjoint pair so that $B(f(x), y) = B(x, g(y))$ for all $(x, y) \in X \times Y$. Let $x \in N(f)$ which gives $f(x) = 0$. Hence $B(x, g(y)) = 0$ for all $y \in Y$. This implies that $x \in \overline{R(g)}$ (see Equation (4.1)). Conversely, if $x \in \overline{R(g)}$ then $B(x, g(y)) = 0$ for all $y \in Y$. This gives $B(f(x), y) = 0$ for all $y$ so that $f(x) \in N(B^*_*)$. But $f \in L'(X)$ which implies $f(x)$ is in an algebraic complement of $N(B^*_*)$. These give $f(x) = 0$ or $x \in N(f)$. Combining the two results we get $N(f) = \overline{R(g)}$. Similarly the other result also follows. \[\square\]

Again, we have the following characterization for adjoint pairs in $L'(X) \times L'(Y)$.

**Theorem 41** Let $(f, g) \in L'(X) \times L'(Y)$. Then $(f, g)$ is an adjoint pair if and only if

(i) $N(f) = \overline{R(g)}, N(g) = \overline{R(f)}$

(ii) if $U'$ is a complement of $N(f)$ and $V'$ is a complement of $N(g)$, then $t = f^\circ | U'$ and $s = g^\circ | V'$ are transposes.

**Proof** Let $(f, g) \in L'(X) \times L'(Y)$ be such that it is an adjoint pair. Then Proposition 40 gives (i). Let $f = e\tilde{t}$ and $g = e'\tilde{s}$ be normal factorizations such that $t$ and $s$ are defined on the complements $U'$ and $V'$ of $N(f)$ and $N(g)$ respectively. Here we have $f^\circ = et, g^\circ = e's$ and $f^\circ | U' = t, g^\circ | V' = s$. 
Further, $B(f(x), y) = B(x, g(y))$ for all $(x, y) \in X \times Y$ implies $B(t(x), y) = B(x, s(y))$ for all $(x, y) \in U' \times V'$. Hence by Proposition 39 $t$ and $s$ are transposes. Thus the direct part of the theorem follows.

To prove the converse part, let $(f, g) \in L'(X) \times L'(Y)$ be such that it satisfies (i) and (ii). Let $N(f) = U$ and $N(g) = V$ so that $U \subseteq U' = X$, $V \oplus V' = Y$. Also let $e$ and $e'$ be idempotents such that $N(e) = U$, $R(e) = U'$, $N(e') = V$ and $R(e') = V'$. Consider the normal factorization of $f$ and $g$ in the form $f = e\hat{f}$ and $g = e'\hat{g}$. Hence by (ii) $t$ and $s$ are transposes.

Let $x \in X$ and $y \in Y$ so that $x = x_0 + x'$ and $y = y_0 + y'$. Further, $x_0 \in U$, $x' \in U'$, $y_0 \in V$ and $y' \in V'$.

$$
B(f(x), y) = B(f(x_0 + x'), y)
$$
$$
= B(f(x_0) + f(x'), y)
$$
$$
= B(f(x'), y)
$$
$$
= B(f(x'), y_0 + y')
$$
$$
= B(f(x'), y_0) + B(f(x'), y')
$$

But $y_0 \in N(g)$ implies $y_0 \in \hat{R}(f)$. Also $f(x') \in R(f)$. These two give $B(f(x'), y_0) = 0$. Hence

$$
B(f(x), y) = B(f(x'), y')
$$
$$
= B(t(x'), y')
$$

Similarly we can see $B(x, g(y)) = B(x', s(y'))$. But $t$ and $s$ are transposes so that $B(t(x'), y') = B(x', s(y'))$ for all $(x', y') \in U' \times V'$. Thus we get $B(f(x), y) = B(x, g(y))$ for all $(x, y) \in X \times Y$. This proves the theorem. □

There is another result that gives a characterization for the set of adjoint pairs in $L'(X) \times L'(Y)$. But we need this separately.
Theorem 42. Let \((f, g) \in L'(X) \times L'(Y)\) be such that \(R(f) = U\) and \(R(g) = V\). Also let \(U'\) and \(V'\) be two algebraic complements of \(N(f)\) and \(N(g)\) respectively. If \((f, g)\) is an adjoint pair then \((U', V), (U, V')\) belong to \(E_T\) and the cones \(\rho = \gamma(U', V) \ast t, \lambda = \gamma^*(U, V') \ast s\) are linked where \(t = f^\circ | U', s = g^\circ | V'\). Also in this case \(\tilde{m} \rho = f\) and \(\tilde{n} \lambda = g\).

Proof. Since \((f, g)\) is an adjoint pair, by Theorem 41, we get \(t = f^\circ | U'\) and \(s = g^\circ | V'\) are transposes. Further using Proposition 39, we get \((U', V)\) and \((U, V')\) are in \(E_T\) because \(U = R(f) = R(t), V = R(g) = R(s)\). Also \(\rho = \gamma(U', V) \ast f^\circ = \gamma(U', V) \ast f^\circ | U' = \gamma(U', V) \ast t\) and \(\lambda = \gamma^*(U, V') \ast g^\circ = \gamma^*(U', V') \ast g^\circ | V' = \gamma^*(U, V') \ast s\) are linked.

Again, here

\[
\tilde{m} \rho = \tilde{\gamma}(U', V)(X) \tilde{t} \quad (\text{see Proposition 33})
\]

where \(N(e) = \hat{V}, R(e) = U'\). Since \((f, g)\) is an adjoint pair by Proposition 40, we get \(N(f) = \widehat{R(g)} = \hat{V} = N(e)\). Also \(t = f^\circ | U'\) where \(U' \oplus \hat{V} = X\). Hence \(e \tilde{t}\) is the normal factorization of \(f\) on the fixed subspaces (see Proposition II.2). Similarly, we can prove \(\tilde{n} \lambda = g\).

The foregoing theorem has a converse which is also true. We prove this in the following

Theorem 43. If \(\rho = \gamma(U', V) \ast t\) and \(\lambda = \gamma^*(U, V') \ast s\) are linked cones, then we can find a unique adjoint pair \((f, g) \in L'(X) \times L'(Y)\) such that \(t = f^\circ | U'\) and \(s = g^\circ | V'\).

Proof. Let \(f = \tilde{m} \rho\) and \(g = \tilde{n} \lambda\) so that \(f = e \tilde{t}\) and \(g = e' \tilde{s}\) with \(N(e) = \hat{V}, R(e) = U', N(e') = \hat{U}\) and \(R(e') = V'\). Again from definition of linked pairs we get \(t = f^\circ | U'\) and \(s = g^\circ | V'\) are transposes. Also \(N(f) = N(e) = \hat{V} = \widehat{R(s)} = \widehat{R(g)}\) and similarly, we will get \(N(g) = \widehat{R(f)}\). Hence \((f, g)\) is an adjoint pair (see Theorem 41).
If \( \rho = \gamma(U_1, V) * t_1 \) is another representation for the cone, then we get \( \bar{m} \rho = e_1 f_1 = f_1 \). Again \( \rho = \gamma(U', V) * t \) gives \( \rho(U') = \gamma(U', V)(U')t = t \), since \( U' \) is the vertex of the idempotent cone \( \gamma(U', V) \). In a similar way we get \( \rho(U_1) = t_1 \).

Now \( \rho = \gamma(U_1, V) * t_1 = \gamma(U_1, V) * \rho(U_1) \)

Hence
\[
\rho(U') = \gamma(U_1, V)(U')\rho(U_1) = \gamma(U_1, V)(U')\rho(U_1) = e_1 \rho(U_1)
\]

Further \( N(e) = N(e_1) = \hat{V} \) which gives \( e \mathcal{R} e_1 \) so that \( e_1 = ee_1 \). Now
\[
e_1 \rho(U_1) = ee_1 \rho(U_1) = e \rho(U').
\]

This leads to \( e_1 f_1 = e f \) or equivalently \( f = f_1 \). Similarly, we will get the uniqueness in \( L'(Y) \) and the proof is complete.

We are in a position to complete the construction of the regular subsemigroup associated with the cross-connection. Consider the set of all adjoint pairs in \( L'(X) \times L'(Y) \) and denote it by \( \hat{SB} \). Hence
\[
\hat{SB} = \{(f, g) \in L'(X) \times L'(Y)^{op}: B(f(x), y) = B(x, g(y)) \ \forall (x, y) \in X \times Y \}
\]

We are going to show that \( \hat{SB} \) is an isomorphic copy of \( \hat{ST} \). For this consider a linked pair \( (\rho, \lambda) \) in \( \hat{ST} \). Define a map \( \tau: \hat{ST} \to \hat{SB} \) by the rule
\[
\tau: (\rho, \lambda) \mapsto (\bar{m} \rho, \bar{n} \lambda).
\]

This map is well defined (see Theorem 43). Also \( \tau \) is a surjection (see Theorem 42). It is an injection because both \( \bar{m} \) and \( \bar{n} \) are injections (see Proposition 33 and Theorem 37). Again \( \tau \) is a homomorphism since \( \bar{m} \) and \( \bar{n} \) are homomorphisms. Hence we can state the following
THEOREM 44 Let $X$ and $Y$ be two vector spaces and $B: X \times Y \rightarrow K$ be a bilinear form. Then if $\Gamma$ is the induced cross-connection, $\hat{S}\Gamma$ is isomorphic to $\hat{S}B$ and hence $\hat{S}B$ is a regular subsemigroup of $L(X) \times L(Y)^{op}$.

The semigroup $\hat{S}B$ will be of much importance in our coming discussions. Hence we label it in the following

DEFINITION 7 Let $S$ be a subsemigroup of $L(X) \times L(Y)^{op}$. Then $S$ is called a bilinear form semigroup if there is a bilinear form $B: X \times Y \rightarrow K$ such that $S$ is isomorphic to $\hat{S}B$. 