CHAPTER I

Preliminaries

In this chapter on preliminaries, an account of some of the definitions and results that are used in the sequel are given. Mainly these include results from theory of semigroups, categories and linear algebra. The necessary preliminaries are presented, as far as possible, as and when they are needed. But those results that may look odd in between are presented here and referred to in the course of discussion. Also, the preliminary notations and conventions that are to be followed in the sequel are set up here.

As in [9] here also we write the composition of mappings in the order in which they appear in commutative diagrams. For those notations or terminology that are not explicitly defined here, the reader should refer to [1,3,4,9] or [13].

We begin with the results from the theory of semigroups.

1. SEMIGROUPS

We present some of those definitions and results in semigroup theory that are essential. Elementary notions of this theory as in Chapters I and II of [4] are assumed without reference. First we mention the definition of regular semigroups.

1.1 Regular Semigroups

Recall that an element $x$ in a semigroup $S$ is regular if there is an element $x'$ in $S$ such that $xx'x = x$ (see p. 26 [1]). A semigroup $S$ is called regular if all its elements are regular.
Regular subsemigroups of regular semigroups are defined in the same way as subsystems of other algebraic systems are defined.

If \( x \) is an element in a regular semigroup \( S \) and if \( x' \in S \) is such that \( xx'x = x \), the element \( xx' \) has the property \( (xx')^2 = xx' \). Same result applies to \( x'x \) also. Generalizing this property we get

**Definition 1** An element \( e \) in a regular semigroup \( S \) is called idempotent if \( e^2 = e \).

Again, if \( T \) is a subset of a regular semigroup \( S \), the set of idempotents in \( T \) is denoted by \( E(T) \). If \( S \) is a regular semigroup, we know that \( E(S) \) is non-empty (see p.26 [1]).

For the definition of Green's relations in an arbitrary semigroup \( S \), we refer to (p. 38, [4]). The following result in semigroups concerning the \( D \)-class structure is from [1], Theorem 2.17.

**Theorem 1** (Clifford–Miller) If \( a \) and \( b \) are the elements of a semigroup \( S \), then \( ab \in Ra \cap Lb \) if and only if \( Rb \cap La \) contains an idempotent.

Here note that \( Ra \) and \( Lb \) are respectively the \( R \)-class and the \( L \)-class containing \( a \) and \( b \) in \( S \).

### 1.2 Biordered Sets and Related Concepts

The concept of biordered sets introduced in [14] has frequent reference in the sequel. So we list the definition and some of the related concepts here.

Let \( E \) be a partial algebra, viz., the set \( E \) together with a partial binary operation on \( E \). The domain of the partial binary operation \( E \) is denoted by \( DE \). On \( E \) the relations \( \omega^I \), \( \omega^R \) and \( \omega \) are defined by

**Definition 2** \( \omega^I \) if and only if \( ef = e \), \( \omega^R \) if and only if \( fe = e \) where \( e, f \in E \) and \( \omega = \omega^I \cap \omega^R \).
Again, recall that an abstract biordered set is a partial algebra \( E \) satisfying the following axioms and their duals for \( e, f, g \in E \) (see p.2 [14]).

\[(B1) \ \omega^r \text{ and } \omega^l \text{ are quasiorders on } E \text{ and } D_E = (\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1} \]

\[(B21) \ f \in \omega^r(e) \implies f \leq_R fe \omega e.\]

\[(B22) \ g \omega^l f, f, g \in \omega^r(e) \implies g \omega^l f e.\]

\[(B31) \ g \omega^r f \omega^r e \implies g f = (ge)f.\]

\[(B32) \ g \omega^l f, f, g \in \omega^r(e) \implies (fg)e = (fe)(ge).\]

For \( e, f \in E \), the sandwich set \( S(e,f) \) is defined by

\[(1) \ S(e,f) = \{ h \in M(e,f) : g \prec h \text{ for all } g \in M(e,f) \} \]

where \( M(e,f) \) denotes the set \( \omega^l(e) \cap \omega^r(f) \) with the quasiorder defined by \( g \prec h \) if and only if \( eg \omega^r eh, g \omega^l hf. \)

The biordered set \( E \) is said to be regular if \( S(e,f) \neq \emptyset \) for all \( e, f \in E \). It has been proved that (see Corollary 1.4 [14])

**Proposition 2** The biordered set \( E \) is regular if and only if \( E \) can be embedded as the set of idempotents \( E(S) \) of a regular semigroup \( S \). \( \Box \)

The following is from [15].

**Proposition 3** If \( S \) is a regular semigroup, then for any two idempotents \( e, f \in S \), \( S(e,f) \neq \emptyset \). \( \Box \)

Again, recall that if \( E \) is a biordered set, then \( D_E \) is a relation on \( E \) and \( (e,f) \in D_E \) if and only if the product \( ef \) exists in \( E \) (see p. 1, [14]).

Using this, a map is introduced between two biordered sets.

**Definition 3** Let \( E \) and \( E' \) be biordered sets and \( \theta : E \to E' \) be a mapping. Then \( \theta \) is called a bimorphism if it satisfies the following axiom.

\[(2) \ (e,f) \in D_E \implies (e\theta, f\theta) \in D_{E'} \text{ and } \]


(ef)θ = (eθ)(fθ).

Again θ is called a regular bimorphism if further

\[ S(e, f)θ \subseteq S'(eθ, fθ) \quad \text{and} \quad S(e, f) \neq \emptyset \iff S'(eθ, fθ) \neq \emptyset \]

for all e, f ∈ E. Here \( S'(eθ, fθ) \) denotes the sandwich set in \( E' \).

It has been proved that (see Theorem I.1, [14])

**Proposition 4** If \( S \) and \( S' \) are two regular semigroups and \( θ : E(S) → E(S') \) is a homomorphism, then θ is a regular bimorphism.

A biorder isomorphism is a partial algebra isomorphism of biordered sets (see V.1.1, [13]).

2. VECTOR SPACES AND LINEAR TRANSFORMATIONS

Basic definitions and properties of vector spaces and linear transformations as in [3] are taken for granted. In this section we note only those results and terminology that are either not unique or need some modifications for our use.

2.1 Basic Notations and Properties

Let \( X \) and \( Y \) be two vector spaces and \( t : X → Y \) be a linear transformation. We call \( X \) and \( Y \) respectively the **domain** and **codomain** of the linear transformation. Again the **range** of \( t \) denoted by \( R(t) = \{(x)t : x ∈ X\} \) and **null space** of \( t \) denoted by \( N(t) = \{x ∈ X : (x)t = 0\} \). Here since the composition of mappings is in the diagram order, we write the elements before the linear transformations.

Often, we find the complement of a subspace in a vector space. Hence we specify this here in the following
DEFINITION 4 Let $X$ be a vector space and $A$, $B$ be subspaces of $X$. If $A + B = X$ and $A \cap B = \{0\}$, then $A$ and $B$ are called algebraic complements of each other.

In this case, we write $X = A \oplus B$ and preserve this notation throughout the sequel. The following result is proved in [20], (see Section 44, Theorem B).

THEOREM 5 Let $X$ be a vector space. If $e: X \to X$ is an idempotent linear transformation, then $X = N(e) \oplus R(e)$. Conversely, if $U$ and $V$ are subspaces of $X$ with $X = U \oplus V$, then there exists a unique idempotent linear transformation $e: X \to X$ with $R(e) = U$ and $N(e) = V$. \qed

For an idempotent linear transformation there is a result that relates its range and null space which we use. This is given in [5] (Problem I.F).

PROPOSITION 6 If $e$ is an idempotent linear transformation on $X$, then $X/N(e)$ is isomorphic to $R(e)$. \qed

For any vector space $X$, we reserve the notation $L(X)$ for the semigroup of all linear transformations on $X$. It can be seen that $L(X)$ is a regular semigroup (see [1] Chapter 2.2, Problem 6). We note this in the following

PROPOSITION 7 $L(X)$ is a regular semigroup for the composition defined by $(x)st = (xs)t$ for all $s, t \in L(X)$ and $x \in X$. \qed

Here we may state and prove a result that characterizes the $\omega$-relations (see Definition 2) in $E(L(X))$ in terms of the ranges and null spaces of idempotent linear transformations. Though this result has been proved in [7] we cannot use it as such because, there composition of mappings is in the composition order.
**Proposition 8**  
Let $e$ and $f$ be two elements in $E(L(X))$, then

(a) $e \omega^1 f \iff R(e) \subseteq R(f)$ and
(b) $e \omega^r f \iff N(f) \subseteq N(e)$.

**Proof**  
Let $e \omega^1 f$ so that by Definition 2 we get $ef = e$. Hence $R(e) = R(ef) \subseteq R(f)$.

Conversely, let $R(e) \subseteq R(f)$. For any idempotent linear transformation $f$, it is easy to see that $R(f) = N(1 - f)$ where $1$ is the identity linear transformation on $X$. Hence $R(e) \subseteq N(1 - f)$ so that $e(1 - f) = 0$. This gives $e = ef$ and hence $e \omega^1 f$.

Further let $e \omega^r f$ so that by Definition 2 we get $fe = e$. Now $N(f) \subseteq N(fe) = N(e)$ which gives $N(f) \subseteq N(e)$. Conversely if $N(f) \subseteq N(e)$, then $N(f) = R(1 - f) \subseteq N(e)$. This gives $(1 - f)e = 0$ so that $e = ef$ which implies $e \omega^r f$.

Definitions of $\mathcal{L}$ and $\mathcal{R}$ on $E(L(X))$ give the following

**Corollary 9**  
$e \mathcal{L} f$ if and only if $R(e) = R(f)$ and $e \mathcal{R} f$ if and only if $N(e) = N(f)$.

One of the fundamental results in linear algebra that we use in the sequel is the induced map theorem. We use the following form of the theorem as given in [5].

**Theorem 10**  
Let $t$ be a linear transformation from $U$ into $V$ and $s$ be a linear transformation from $U$ into $W$ where $U$, $V$ and $W$ are three vector spaces over the same field. If the null space of $t$ contains that of $s$, then there is a unique linear transformation $u$ from $W$ into $V$ such that $t = su$. Function $u$ is one-to-one if and only if the null spaces of $s$ and $t$ coincide.

$$
\begin{array}{ccc}
U & \xrightarrow{t} & V \\
W & \xrightarrow{s} & W \\
\end{array}
$$

$u$
2.2 Bilinear Forms

Another concept used here from this part is that of a bilinear form. If $X$ and $Y$ are two vector spaces over a field $K$, a bilinear form (see P. 327 [17]) $B: X \times Y \to K$ is such that

\begin{align*}
B(x + x', y) &= B(x, y) + B(x', y), \\
B(\alpha x, y) &= \alpha B(x, y) \\
B(x, y + y') &= B(x, y) + B(x, y'), \\
B(x, \alpha y) &= \alpha B(x, y)
\end{align*}

for $x, x' \in X$, $y, y' \in Y$ and $\alpha \in K$.

Again if there is a bilinear form $B: X \times Y \to K$, then we have two linear transformations $B^*: X \to Y^*$ and $B^*: Y \to X^*$ (see [18] p. 138). These maps are such that

\textbf{Proposition 11} If $B: X \times Y \to K$ is a bilinear form, there exist two maps $B^*: X \to Y^*$ and $B^*: Y \to X^*$ such that $(x)B^* = B(x, -) \in Y^*$ and $(y)B^* = B(-, y) \in X^*$.

If $X$ is a finite dimensional vector space, it is known that $X^*$ separates points of $X$. In other words, given $x \in X$, such that $x \neq 0$, we can find an $f \in X^*$ such that $f(x) \neq 0$ (see Lemma 4.3.2 [4]). This process can be carried over to infinite dimensional vector spaces also and is done by constructing \textit{total subspaces} of $X^*$. Those subspaces of $X^*$ that separates points of $X$ are called \textit{total subspaces}. It has been observed that total subspaces can be constructed for any vector space $X$ (see p. 251, [18]). Hence we get

\textbf{Proposition 12} Let $X$ be a vector space and $x$ be a non-zero element of $X$. Then we can choose $f \in X^*$ such that $f(x)$ is non-zero.
3. CATEGORIES

In this section, we mention some elementary results and language of category theory. Moreover, a quick review of the concepts that are necessary to introduce cross-connections is made. For notations or terminology not explicitly mentioned here, the reader should refer to [9] or [13].

3.1 Definitions and Notations

A small category means a category in which the set of objects and set of arrows are both small sets (see p. 22 [9]). Categories that involve in our constructions are small so that we assume the freedom in using the terminology that are applicable to small categories.

If $C$ is a category, $vC$ denotes the object class of $C$. For us, $vC$ is always a set. $C$ may also be used to denote the set of morphisms in the category $C$. For two objects $U$ and $V$ in $C$, $C(U, V)$ is used to represent the set of all morphisms with domain $U$ and codomain $V$. $1_U$ denotes the identity morphism of the object $U$.

Here a functor means a covariant functor. For any category $C$, $C^{op}$ denotes the category with the same object set and morphisms reversed. If $F: C^{op} \to D$ is a covariant functor, then $F: C \to D$ is a contravariant functor.

In the following we consider a particular type of functor that is used in the sequel.

**DEFINITION 5** Let $C$ be a small category. For a fixed $U$ in $vC$ and $f \in C(U', U'')$, let $C(U, f)$ denote the functor from $C(U', U'')$ to $C(U, U'')$ defined by

$$C(U, f)(g) = gf \quad \text{for every } g \in C(U, U').$$

Then the assignment $U' \mapsto C(U, U')$ and $f \mapsto C(U, f)$ defines a functor $C(U, -)$ from $C$ to $\textbf{Set}$. This is called the covariant hom-functor determined by $U$. 
Here note that **Set** is the category where the set of small sets is the object class and set of all small functions is the morphism class. Also **Set** is not a small category (see p. 23 [9]).

If $F: C \to D$ is a functor between the categories $C$ and $D$, $vF$ denotes the restriction of the functor on $vC$. $F$ is called $v$-injective if $vF$ is injective and $F$ is called $v$-surjective if $vF$ is surjective. $F$ is said to be **faithful** if the morphism map of $F$ is injective on each hom set of $C$.

**Definition 6**  
A functor $F: C \to D$ is called injective if it is faithful and $v$-injective.

We shall say that a functor $F$ is **full** if its morphism map is surjective on each horn-set of $C$. Then we have

**Definition 7**  
A functor $F: C \to D$ is said to be surjective if its morphism map is surjective.

Again we have to use the concepts of isomorphisms and embeddings in the sequel. For reference, we note them in the following

**Definition 8**  
A functor $F: C \to D$ is called an embedding if it is fully-faithful and $v$-injective.

An embedding in which $vF$ is a bijection is called an isomorphism.

We have occasions to use natural transformations in the sequel. So here we give the definition and afterwards some constructions using this.

Let $F$ and $G$ be two functors from the category $C$ to the category $D$. A **natural transformation** $\eta: F \to G$ is a map $U \mapsto \eta(U)$ from the object class of $C$ to the morphism class of $D$ such that for each $U \in vC$, $\eta(U): F(U) \to G(U)$ is a morphism in $D$ (called the component of $\eta$ at $U$) and the following diagram

One of our main results strongly depends on Yoneda Lemma (see [9], p. 59-62). Hence in this category, the composition is defined component wise. In other words, if
commutes for all \( f: U \rightarrow V \) in \( C \).

\[
\begin{array}{ccc}
F(U) & \xrightarrow{\eta(U)} & G(U) \\
\downarrow F(f) & & \downarrow G(f) \\
F(V) & \xrightarrow{\eta(V)} & G(V)
\end{array}
\]

We have the following

**Definition 9** If each component of a natural transformation is an isomorphism, then it is called a natural isomorphism.

If there is a natural isomorphism \( \eta: F \rightarrow G \), then \( F \) and \( G \) are said to be naturally equivalent.

If \( C \) and \( D \) are two small categories, we consider the category \([C, D]\) whose objects are functors from \( C \) to \( D \) and morphisms are natural transformations. Any subcategory of \([C, D]\) will be called a functor category. For any category \( C \), the functor category \([C, \text{Set}]\) is denoted by \( C^* \).

If \( S \) and \( T \) are two functors from \( C \) to \( D \), we use the notation \( \text{Nat}(S, T) \) to denote the set of all morphisms in the functor category \([C, D]\) from \( S \) to \( T \). In this category, the composition is defined component wise. In other words, if \( \eta \in \text{Nat}(S, T), \zeta \in \text{Nat}(T, U) \), then \( \eta \zeta \in \text{Nat}(S, U) \) is a natural transformation defined by

\[(\eta \zeta)(U) = \eta(U)\zeta(U) \quad \text{for all} \quad U \in vC.
\]

One of our main results strongly depends on Yoneda Lemma (see [9], p. 59–62). Hence we give the necessary details here.

### 3.2 Representable Functors

There are several equivalent definitions of representable functors. First we have
Let $F \in C^*$, then the pair $(U, X)$ where $U \in \nu C$ and $X \in F(U)$ is called a universal element for $F$ if for all $U' \in \nu C$ and $Y \in F(U')$, there is a unique map $f: U \to U'$ such that $F(f)(X) = Y$.

Again, a functor $F \in C^*$ is said to be representable if $F$ is naturally isomorphic to a covariant hom–functor $C(U, -)$ determined by $U$ (see Definition 5). The following Proposition is a consequence of Yoneda Lemma (see I.3.2 [13] and p. 59–62 [9]).

**Proposition 13** A functor $F \in C^*$ is representable if and only if $F$ has a universal element.

The foregoing Proposition gives that if $F \in C^*$ has a universal element $(U, X)$, then $F$ is naturally isomorphic to the covariant hom–functor $C(U, -)$ determined by $U$.

Now we shall find this natural isomorphism between $F$ and $C(U, -)$. From (I.3.1, [13]), it follows that if $(U, X)$ is a universal element, the natural isomorphism between $F$ and $C(U, -)$ is $\zeta^X$ defined by

$$\zeta^X_{U'}(f) = F(f)(X) \text{ where } f \in C(U, U')$$

Another result that we need in our category constructions is the following (see [9] Corollary p. 61).

**Proposition 14** For objects $U, U'$ in $\nu C$, each natural transformation from $C(U, -)$ to $C(U', -)$ has the form $C(h, -)$ for a unique arrow $h: U' \to U$ and $C(h, -)$ is the natural transformation induced by $h$.

### 3.3 Normal Category

We consider a subobject as an object in the category rather than an equivalence class of monomorphisms (see [9] p. 122). A choice of subobjects is a cross section of the equivalence class of monomorphisms satisfying certain conditions.
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(see [8]). If we specify a choice of subobjects, we can replace equivalence classes of monomorphisms by objects, as usual in concrete categories like Set, Grp, etc. We shall say that the category $C$ is a category with subobjects if a choice of subobjects has been specified (see [8]). Again if $U$ and $V$ are in $vC$, we say that $U$ is a subobject of $V$ written as $U \subseteq V$ if there is a unique morphism $j_U^V: U \to V$. In this case $j_U^V$ is called the inclusion morphism of $U$ in $V$. Note that $vC$ together with the relation $\subseteq$ defined above is a partially ordered set which determines a subcategory of $C$. This subcategory is called the preorder of inclusions (see [13]). If $C$ and $D$ are two categories with subobjects, a functor $\Gamma: C \to D$ preserves inclusion (or inclusion preserving) if $\Gamma \mid vC$ is a functor from the preorder of inclusions into $vD$.

The concept of inclusion morphism defined in the category makes the following definition possible.

**Definition 11** A morphism in a category is called a retraction if it is the right inverse of an inclusion.

Let a morphism $f$ in a category $C$ be expressible in the form $euj$ where $e$ is a retraction, $u$ an isomorphism and $j$ an inclusion. Such factorization for a morphism is called normal factorization. In such factorizations, $eu$ is an epimorphism (see p. 19 [9]) and is denoted by $f^o$ throughout. The following result on the uniqueness of the epi part of a morphism in a category is from Lemma I.1 [12].

**Proposition 15** Let $f = euj = e'u'j'$ be two normal factorizations of a morphism $f$. Then $j = j'$ and $eu = e'u'$.

All these concepts are introduced in order to define a particular type of category on which the whole theory of cross-connection is built.

**Definition 12** Let $C$ be a small category with subobjects. Then $C$ is called a normal category if
(a) every inclusion morphism has a right inverse and
(b) each morphism in $\mathcal{C}$ has a normal factorization.

By a cone $\gamma$ from the base $\nu \mathcal{C}$ to the vertex $C_{\gamma} \in \nu \mathcal{C}$, we mean a natural transformation from the identity functor $1_{\nu \mathcal{C}}$ on the preorder of inclusions $\nu \mathcal{C}$ to the constant functor with value $C_{\gamma}$; that is, a mapping $\gamma : \nu \mathcal{C} \to \mathcal{C}$ such that

(i) $\gamma(U) \in \mathcal{C}(U, C_{\gamma})$ for all $U \in \nu \mathcal{C}$ and
(ii) if $U \subseteq V$, then $j_{U}^{V} \gamma(V) = \gamma(U)$.

Note that $\gamma(U)$ is the component of the cone $\gamma$ at $U$.

A normal cone $\gamma$ is a cone from the base $\nu \mathcal{C}$ to the vertex $C_{\gamma}$ in which at least one component $\gamma(U)$ is an isomorphism.

Let $T \mathcal{C}$ denotes the set of all normal cones in the normal category $\mathcal{C}$. For $\gamma \in T \mathcal{C}$ and $f \in \mathcal{C}(C_{\gamma}, U)$ (where $C_{\gamma}$ is the vertex of $\gamma$), define the cone $\gamma * f^{\circ}$ by

$$(\gamma * f^{\circ})(U) = \gamma(U) \circ_{\gamma(U)} f^{\circ}$$

It is easy to check that $\gamma * f^{\circ} \in T \mathcal{C}$. We have (see Theorem 3.3 [12])

**Theorem 16** Let $\mathcal{C}$ be a normal category. Then the set $T \mathcal{C}$ of all normal cones is a semigroup with binary operation defined by

$$\gamma \gamma' = \gamma * (\gamma'(C_{\gamma}))^{\circ} \text{ where } \gamma, \gamma' \in T \mathcal{C}.$$  

Moreover, $T \mathcal{C}$ is a regular semigroup if for every $C \in \nu \mathcal{C}$, there is $\gamma \in T \mathcal{C}$ with $C = C_{\gamma}$. 

A normal category satisfying the conditions of the foregoing theorem is said to be reductive. Thus $T \mathcal{C}$ is a regular semigroup if $\mathcal{C}$ is reductive (see Remark 3.2 [12]). In the following we shall consider those normal categories that are reductive.
Let $\gamma \in TC$ where $C$ is normal (and reductive). Define $H(\gamma, -) : C \to \text{Set}$ as follows:

$$H(\gamma, C) = \{\gamma \ast f^\circ : f \in C(C, C')\}$$

and for $g \in C(C, C')$

$$H(\gamma, g) : \gamma \ast f^\circ \mapsto \gamma \ast (fg)^\circ$$

Then $H(\gamma, -) \subseteq vC^*$ and it can be shown that $H(\gamma, -)$ is representable with $C_\gamma$ as the representing object; that is, the natural transformation $\eta_\gamma : H(\gamma, -) \to C(C_\gamma, -)$ sending $\gamma \mapsto 1_{C_\gamma}$ is a natural isomorphism. We denote the full subcategory of $C^*$ with vertex set $\{H(\gamma, -) : \gamma \in TC\}$ by $N^*C$. Then $N^*C$ is a normal category (see Theorem III.25 [13]) with inclusion as the inclusion of the set-valued functors; that is, $H(\gamma, -) \subseteq H(\gamma', -)$ if and only if $H(\gamma, C) \subseteq H(\gamma', C)$ for all $C \in vC$ and the inclusion $H(\gamma, C) \subseteq H(\gamma', C)$ is a natural transformation $j : H(\gamma, -) \to H(\gamma', -)$. The normal category $N^*C$ is called the normal dual of $C$ (see Definition III.4 [13]).

Let $C$ and $D$ be categories with subobjects. An inclusion preserving functor $\Gamma : C \to D$ is a local isomorphism if $\Gamma$ is fully faithful and such that $v\Gamma : vC \to vD$ induces an order isomorphism on each principal order ideal of $vC$; that is, if $C \in vC$, then $\Gamma \mid (C) : (C) \to (\Gamma(C))$ is an isomorphism of categories. Here $(C)$ denotes the ideal in $C$ generated by the object $C$; that is, the full subcategory of $C$ with $v(C) = \{C' : C' \subseteq C\}$ (see Definition IV.1 [13]). Finally, using this we have

**Definition 13** A cross-connection between two normal reductive categories $C$ and $D$ is a local isomorphism $\Gamma : D \to N^*C$ such that the image of $\Gamma$ is total in $N^*C$, where $N^*C$ is the normal dual of $C$ (see p (iii), [13]).