Preface

In [2] Grillet introduced the theory of cross-connections in order to obtain a representation of fundamental regular semigroups. He characterized the partially ordered sets of principal left and right ideals of a regular semigroup as regular partially ordered sets. Also he determined the relation between two given regular partially ordered sets $\Lambda$ and $I$ so that they are order isomorphic to $\Lambda(S)$ and $I(S)$ respectively where $\Lambda(S)$ and $I(S)$ are partially ordered sets of left and right ideals of a regular semigroup $S$. His characterization amounts to a 'weak identification' (a local isomorphism) of $I(S)$ as a partially ordered subset of the 'dual' of $\Lambda(S)$.

Nambooripad in [13] considers an extension of Grillet’s theory to arbitrary regular semigroups. Here the regular partially ordered sets $\Lambda(S)$ and $I(S)$ of left and right ideals in Grillet’s theory are replaced by categories $\mathcal{L}(S)$ and $\mathcal{R}(S)$ of left and right ideals and morphisms appropriate translations. The categories $\mathcal{L}(S)$ and $\mathcal{R}(S)$ are abstractly characterized as normal categories and the normal dual $\mathcal{N}^*C$ of a normal category $C$ is characterized as a suitable subcategory of $C^*$ (the category of set-valued functors on $C$) which is also normal. Again, the cross-connection in Grillet’s theory is replaced by a local isomorphism $\Gamma : \mathcal{D} \to \mathcal{N}^*\mathcal{C}$ of the normal category $\mathcal{D}$ into the normal dual of $\mathcal{C}$. Now the functor $\Gamma : \mathcal{D} \to \mathcal{N}^*\mathcal{C}$ induces a bifunctor $\Gamma(-,-) : \mathcal{C} \times \mathcal{D} \to \text{Set}$ which gives rise to a duality between normal categories $\mathcal{C}$ and $\mathcal{D}$. We shall refer to this duality as the cross-connection duality induced by the given cross-connection $\Gamma : \mathcal{D} \to \mathcal{N}^*\mathcal{C}$.

This extended theory of cross-connections provides a unified framework for studying various classes of regular semigroups. In particular, this applies
to the study of regular semigroup of matrices (or endomorphisms of finite dimensional vector spaces) and related semigroups. In such contexts, it is natural to interpret the normal categories as certain categories of subspaces and associated duality in terms of the classical duality in vector spaces. Thus, it is of interest to study the relation between cross-connection duality induced by a given cross-connection between normal categories of vector spaces and the classical duality relevant to the context.

We remark that the duality in vector spaces (both algebraic and topological) is often specified in terms of a bilinear form relevant to the context. When the vector spaces under consideration are finite dimensional, the duality can be specified in terms of a bilinear form determined by a matrix. Our principal aim here is to study the cross-connection determined by a bilinear form defined on a pair of finite dimensional vector spaces. In simplest applications, the bilinear form is non-degenerate; but there exist contexts (especially when dealing with dualities between infinite dimensional vector spaces) where duality specified by degenerate bilinear forms are also of interest. We therefore study the cross-connection and associated semigroup determined by an arbitrary bilinear form defined on a pair of finite dimensional vector spaces.

The material presented in Chapter I is not complete on its own. We have only 'listed' a few results and terminology that are around with representation from each of the three major areas which are involved. Since we are using [13] extensively, references are made directly then and there except for a few definitions listed in this chapter. For notations and terminology we have referred to both [12] and [13] depending on our convenience. Since there is a slight variation in the terminology in them, we stick on to the one that suits our context.

Chapter II deals with the basic tools for the construction of cross-connection, viz., the normal reductive category $C_X$ of all subspaces of the vector space $X$ (see Theorem II.4) and its normal dual $N^*C_X$ consisting of $H$–functors and their natural transformations (see Theorem II.14). The fact
that the regular semigroup of all normal cones $TC_X$ is isomorphic to the linear transformation semigroup $L(X)$ is established here (see Proposition II.5). Aim of the chapter is to get an identification between $N^*C_X$ and the category $C_X^*$ of all subspaces of the dual space $X^*$ of $X$. As an intermediate step, we define a category $\hat{C}_X$ whose objects are quotient spaces of $X$ and we prove that there is an embedding from $\hat{C}_X$ into $C_X^*$ (see Proposition II.23). Now using this, we obtain an embedding of $N^*C_X$ in $C_X^*$ (see Theorem II.24) and if $X$ is finite dimensional, we see them isomorphic (see Theorem II.26). Hence we conclude that in this interpretation the concepts of normal dual and the vector space dual coincide.

In Chapter III, we first establish the existence of a local isomorphism between the categories $C'_Y$ and $C_X^*$ using a bilinear form $B: X \times Y \to K$ where $X$ and $Y$ are vector spaces over $K$ and $C'_Y$ is a subcategory of $C_Y$ (see Proposition III.4). This local isomorphism in turn gives a cross-connection $\Gamma: C'_Y \to N^*C'_X$ where $C'_X$ is a subcategory of $C_X$ (see Theorem III.15). Transpose of a morphism (see Definition III.3) and category adjoint (see Definition III.4) are obtained and further we prove that for a morphism category adjoint is the image of its transpose under the cross-connection (see Remark III.5). This justifies the term 'transpose' in the theory of cross-connections. Again this cross-connection gives two regular subsemigroups $L'(X)$ and $L'(Y)$ of $L(X)$ and $L(Y)$ respectively (see Equations III (14) and III (15)) and the cross-connection semigroup, denoted by $\hat{S}B$, consists of pairs $(f, g)$ in $L'(X) \times L'(Y)^{op}$ such that $B(f(x), y) = B(x, g(y))$ for all $(x, y)$ in $X \times Y$. We call any semigroup isomorphic to $\hat{S}B$, a bilinear form semigroup (see Definition III.7).

Properties of the bilinear form semigroups are discussed in Chapter IV. We see that this semigroup is isomorphic to $L(X)$ if and only if the bilinear form is right non-degenerate (see Theorem IV.9). $\omega$-relations and Green's relations are characterized and expression for sandwich set is obtained (see Proposition IV.15). We prove that the maximal idempotents in this semigroup form a rectangular band and that the maximum $\mathcal{D}$-class is a completely
simple orthodox semigroup (see Theorem IV.32). A pair of compatible linear transformations (see Definition IV.3) will induce an injective homomorphism between bilinear form semigroups if they satisfy the axioms B1 and B2 (see Proposition IV.44). At the end of the chapter it has been proved that the constructed homomorphism preserves both primitive and maximal idempotents in the bilinear form semigroups.

In the last chapter we establish the structure of the bilinear form semigroups. The concept of the rank of an idempotent is introduced and it has been proved that every idempotent in the bilinear form semigroup has a rank that is equal to the common dimension of the vector spaces in the pair corresponding to it (see Corollary V.9). Again we construct a Rees matrix semigroup over a regular semigroup and conclude that the bilinear form semigroup is isomorphic to a quotient of such semigroups (see Theorem V.18). The homomorphism that exists between the constructed Rees matrix semigroup and the bilinear form semigroup is idempotent pure (see Proposition V.19) and the inverse image of an idempotent under this homomorphism is a rectangular band. We conclude this chapter by making some remarks and discussing a few instances where the bilinear form semigroups occur naturally.