CHAPTER II

OPTIMAL REPAIR RATE IN SOME MAINTAINED SYSTEMS
2.1 INTRODUCTION

Maintained systems are quite prevalent and useful in modern defence/industrial systems. During last three decades a large number of papers have appeared to analyse such systems under different conditions. These generalizations (contributions) are motivated to the following objectives:

(i) To generalize the underlying distributions. These distributions may be for failure time, repair-time and/or for other features of the system units [Nakagawa and Osaki (1974a), Srinivasan and Gopalan (1973a), Garg (1963b)] etc.

(ii) To incorporate certain new parameters in existing models. These new parameters may be of interest to system designers and maintenance engineers [Kumar and Lal (1979), Arnold (1973), Chow (1975), Gopalan and Saxena (1977), Khalil and Bougas (1975), Kapoor and Kapur (1975a), Kumar and Jain (1977)] etc.

(iii) To obtain and analyse new system parameters which may serve useful purpose while making evaluation studies [Kumar (1976), Mine and Kawai (1974)] etc.

Garg (1963b) has taken a general distribution for repair-time and analysed a redundant system model. For quite a long time, workers have been assuming at least one of the two distributions: failure-time and repair-time as exponential. Srinivasan and Gopalan (1973a) have analysed a 2-unit system model assuming both the distributions as general. They obtained reliability and availability using regenerative properties. Later, Nakagawa and Osaki (1974a) developed a unique modification of MRP and discussed in detail
reliability characteristics of a 2-unit cold standby system.

As regards inclusion of new parameters of interest to system designers/analysts, some of the interesting parameters include ‘delayed repair’ Khalil and Bougas (1975), Kapoor and Kapur (1975a), Gopalan and Saxena (1977), Kumar and Jain (1977) : ‘Intermittent repair’ and ‘contact failure’ Kumar and Lal (1979), ‘switch failure modes’ Nakagawa (1977a), ‘automatic recovery’ Arnold (1973) etc.

Further, Kumar (1976) obtained expected profit for 2-unit systems operating under different conditions. This parameter has been suggested as the measure of system effectiveness and optimal preventive maintenance policies are also discussed. Mine and Kawai (1974) considered a 2-unit parallel system with good, degraded and failed states and discussed a preventive maintenance policy that maximizes the expected profit rate of the system over an infinite time span. Mine and Kawai (1975) discussed optimal replacement policy for a unit which assumes any one of several Markov states. The problem was formulated as a Markov decision process with a modified Policy Improvement Routine and evaluation function is the expected cost per unit time over an infinite time span.

Above review reveals that mostly the contributions are concerned with determination of system parameters and/or inclusion of new features of units constituting the system. Nevertheless, efforts have been in the direction of helping system analysts and decision makers to take optimal decisions as and when they are needed. These papers generally provide answer to the following broad question :
What should be the optimal preventive maintenance policy to maintain the system so that the overall system performance (availability, reliability, cost, profit etc.) is optimised?

The purpose of the present chapter is two fold:

(a) To develop cost equations for a maintained system and obtain thereby expected profit of the system in a given time interval.

(b) To obtain optimal values for repair rates which may help in better management of a maintained system.

It may be worthwhile to point out here that in all the papers Kumar (1976, 1977a, b, c, d) derived expected profit by direct substitution of system parameters in Howard's (1971) expression for \( g \) (expected profit). Whereas we write basic equations for two models to derive the expression for \( g \). These equations help in better understanding of the system's economic behaviour and provide better insight into solution method. However, the result for the other two models have been derived by using Howard's (1971) technique.

We discuss following models in the subsequent sections:

Model 1 :- Profit evaluation in a single unit system

Model 2 :- Profit evaluation in a single unit system with a less productive state

Model 3 :- Profit considerations in a single unit system with minor repair

Model 4 :- Profit evaluation in a 2-unit parallel redundant system.
2.2 Model 1 :- PROFIT EVALUATION IN A SINGLE UNIT SYSTEM

System Model

(i) There is a 1-unit system.

(ii) Failure-time distribution of the system is exponential with parameter $\beta$.

(iii) Whenever the system fails, it undergoes repair.

(iv) Repair-time distribution of the system is exponential with parameter $\mu$.

(v) Repair facility always exists.

(vi) After the repair, the system acts as new one.

(vii) All probability distributions are s-independent.

(viii) The system earns at a fixed rate so long as it is operable.

(ix) The system loses at a fixed rate for the time it is under repair.

(x) There is a fixed transition reward (cost) whenever the system changes its state viz., operable (failed) to failed (operable).

System states and transitions

The system can be in one of the following 2 states :

0 : operating

1 : failed

To identify the system at any instant, let

$S_0$ : system is operating
\[ S_1 \] : system is in failed state and is under repair.

**Notation**

\[ \beta \] : constant failure rate of the system

\[ \mu \] : constant repair rate of the system

\[ r_{ii} \] : cost per unit time or earning rate in \( S_i \) (\( i = 0, 1 \))

\[ r_{ij} \] : transition cost for a transition from state \( S_i \) to \( S_j \), \( i \neq j \)

\( (i, j = 0, 1) \)

\[ V_i(t) \] : total expected earning of the system for a future period of length ‘t’ given that at \( t=0 \) the system was in \( S_i \) (\( i = 0, 1 \))

\[ \Delta \] : small interval

\[ \bar{f}(s) \] : Laplace transform of \( f(t) \), i.e.,

\[ \bar{f}(s) = \int_0^\infty e^{-st} f(t) \, dt, \quad \text{Re} \, (s) > 0. \]

**Cost Equations and System Profit**

Following Howard (1960), the following cost equations can be written

\[ V_0(t+\Delta) = [r_{00}\Delta+V_0(t)][1-\beta\Delta]+[\tau_{01}+V_1(t)]\beta\Delta \quad (2.2.1) \]

\[ V_1(t+\Delta) = [r_{11}\Delta+V_1(t)][1-\mu\Delta]+[\tau_{10}+V_0(t)]\mu\Delta \quad (2.2.2) \]

Initially \( V_i(0) = 0 \) \( (i = 0, 1). \)

The equation 2.2.1 is interpreted in the following manner.

During the short time interval \( \Delta \), the system which started in ‘0’ state at \( t=0 \) may remain in ‘0’ state or make a transition to state ‘1'. If it remains
in ‘0’ state for time $\Delta$, the system’s earning will be $r_{00}\Delta$ plus the expected earning it will make in the remaining time of ‘t’ units viz., $V_0(t)$. The probability that it remains in state ‘0’ for time $\Delta$ is 1 minus the probability that it makes a transition to state ‘1’ in $\Delta$, i.e., $1 - \beta \Delta$. In case the system makes a transition to state ‘1’ during time interval $\Delta$, the system’s earning will be $r_{01}$ plus the expected earning it will make when the system starts in state ‘1’ with time $t$ remaining viz., $V_1(t)$. The sum of the products of earnings with their respective probabilities gives the total expected earning. Equation 2.2.2 can be interpreted in the same manner.

Expanding equations 2.2.1 and 2.2.2 we get

$$V_0(t+\Delta) = r_{00}\Delta + V_0(t) - \beta V_0(t)\Delta + \beta r_{01}\Delta + \beta V_1(t)\Delta + 0(\Delta) \tag{2.2.3}$$

$$V_1(t+\Delta) = r_{11}\Delta + V_1(t) - \mu V_1(t)\Delta + \mu r_{10}\Delta + \mu V_0(t)\Delta + 0(\Delta) \tag{2.2.4}$$

Neglecting terms of higher order of $\Delta$ and letting $\Delta \to 0$, we obtain the following system of differential equations

$$\left[ \frac{d}{dt} + \beta \right] V_0(t) = r_{00} + \beta r_{01} + \beta V_1(t) \tag{2.2.5}$$

$$\left[ \frac{d}{dt} + \mu \right] V_1(t) = r_{11} + \mu r_{10} + \mu V_0(t) \tag{2.2.6}$$

Applying Laplace transforms to the set of equations 2.2.5 and 2.2.6 and employing initial conditions $V_i(0) = 0$, ($i = 0, 1$) we get

$$\overline{V}_0(s) = \frac{r_{00} + \beta r_{01}}{s(s + \beta)} + \frac{\beta}{s + \beta} \overline{V}_1(s) \tag{2.2.7}$$

$$\overline{V}_1(s) = \frac{r_{11} + \mu r_{10}}{s(s + \mu)} + \frac{\mu}{s + \mu} \overline{V}_0(s) \tag{2.2.8}$$
On substituting the value of $\overline{V}_1(s)$ form (2.2.8) in (2.2.7), we get

$$\overline{V}_0(s) = \frac{(r_{00} + \beta r_{01})s + \mu(r_{00} + \beta r_{01}) + \beta(r_{11} + \mu r_{10})}{s^2(s + \beta + \mu)} \quad (2.2.9)$$

On substituting the value of $\overline{V}_0(s)$ from (2.2.7) in (2.2.8), we get

$$\overline{V}_1(s) = \frac{(r_{11} + \mu r_{10})s + \beta(r_{11} + \mu r_{10}) + \mu(r_{00} + \beta r_{01})}{s^2(s + \beta + \mu)} \quad (2.2.10)$$

On inverting (2.2.9) and (2.2.10), we get

$$V_0(t) = \frac{1}{\beta + \mu} \left[ (r_{00} + \beta r_{01}) \mu + (r_{11} + \mu r_{10}) \beta \right] t + (r_{00} + \beta r_{01})$$

$$- \frac{1}{(\beta + \mu)^2} \left[ (r_{00} + \beta r_{01}) \mu + (r_{11} + \mu r_{10}) \beta \right]$$

$$- \beta(r_{11} + \mu r_{10} - r_{00} - \beta r_{01}) e^{-(\beta + \mu)t} \quad (2.2.11)$$

$$V_1(t) = \frac{1}{\beta + \mu} \left[ (r_{00} + \beta r_{01}) \mu + (r_{11} + \mu r_{10}) \beta \right] t + (r_{11} + \mu r_{10})$$

$$- \frac{1}{(\beta + \mu)^2} \left[ (r_{00} + \beta r_{01}) \mu + (r_{11} + \mu r_{10}) \beta \right]$$

$$- \mu(r_{00} + \beta r_{01} - r_{11} - \mu r_{10}) e^{-(\beta + \mu)t} \quad (2.2.12)$$

In order to exhibit transient behaviour of the system and thus usage of results arrived above in (2.2.11) and (2.2.12), we give below an illustration.

**Illustration**

Let us specify various parameters for the purpose of numerical illustration. We give the results in the subsequent tables (2.2.1-2.2.4) for different values of $\beta$, $\mu$, $r_{00}$, $r_{11}$, $r_{01}$ and $r_{10}$.
Define $V_{0}^{l}(t) = V_{0}(t+1) - V_{0}(t)$

and $V_{1}^{l}(t) = V_{1}(t+1) - V_{1}(t)$

From table 2.2.1, it may be observed if the system starts from operable state at $t=0$ it achieves steady-state after 13 units of time and the steady-state expected profit is 3.5238. If the system starts from failed state it achieves steady-state after 14 units of time and the steady-state expected profit is 3.5238.

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$\beta = 0.25$, $\mu = 0.8$, $r_{00} = 20.0$, $r_{11} = -50.0$, $r_{01} = -3.0$, $r_{10} = 4.0$

Table 2.2.1

It may be observed that initially there is steep fall in $V_{0}^{l}(t)$, whereas, as the time progresses, this decrease reduces and stabilizes to a fixed value
which is called steady-state expected profit of the system, Kumar (1976). Further, in the beginning $V_1(t)$ increases rapidly and then it stabilizes to the same fixed value viz., expected steady-state profit.

From table 2.2.2, it may be observed that steady-state is achieved after 11 units of time if system starts in operable state and the expected profit is 5.9048. Further, it may be noted that when earning in the failed state increases (cost spent on repairs decreases), expected profit increases and a reduction of 20% in repairs ensures 67.5% increase in the expected profit.

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$\beta = 0.25, \mu = 0.8, r_{00} = 20.0, r_{11} = -40.0, r_{01} = -3.0, r_{10} = 4.0$

Table 2.2.2
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$\beta = 0.25, \; \mu = 0.8, \; r_{00} = 25.0, \; r_{11} = -50.0, \; r_{01} = -3.0, \; r_{10} = 4.0$

**Table 2.2.3**

In this case, steady-state is achieved at $t=13$. Further, it may be noted that an increase of 25% in the earning rate of the system while it is operative, there is 108.1% increase in steady-state profit of the system.
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<td>6.2003</td>
<td>7.8419</td>
<td>6.1986</td>
</tr>
<tr>
<td>13</td>
<td>91.0400</td>
<td>6.2000</td>
<td>38.8400</td>
<td>6.2000</td>
</tr>
<tr>
<td>15</td>
<td>103.4400</td>
<td>6.2000</td>
<td>51.2400</td>
<td></td>
</tr>
</tbody>
</table>

$\beta = 0.25, \mu = 1.0, r_{00} = 20.0, r_{11} = -50.0, r_{01} = -3.0, r_{10} = 4.0$

**Table 2.2.4**

Table 2.2.4 shows that faster repair results in increase in profit. An increase in repair rate by 25% gives an increase of about 76% in steady-state profit of the system.

We now depict transient behaviour of the economics of the system so as to bring into light some interesting facts which may be helpful for systems which are operated for a short time span. Behaviour of $V_0(t)$ with respect to variation in $t$ has been given in Fig. 2.2.1. Some of the observations are as under:

(i) The total earning of the system increases initially. As time progresses, the earning reaches a maximum limit and then starts
FIGURE 2.2.1

Graph showing the behaviour of total expected earning $V_0(t)$ against $t$ when $t$ is not large enough.
decreasing. At first sight it may look ridiculous that earning of the system in time $t$ decreases as the time interval $t$ increases. But, a close look in the system structure parameters viz., failure rate, repair rate etc. reveals that the earning does not solely depend upon the time but it is influenced by several other factors including failure and repair rates of the system. When the earning in a larger interval reduces, this merely implies that a major share of the money is spent on repairs of the system.

(ii) Further, just after 6 units of time, the system starts running in loss and it continues to be in this region in steady-state. More elaborately, after initial adjustments, the system runs in loss. This happens because of the choice of cost parameters and system parameters.

**Expected Profit Optimization**

The expected profit per unit time denoted by $g$ from (2.2.11) or (2.2.12) after allowing $t$ tend to infinity can be easily obtained. Thus, we get

$$g = \frac{(r_{00} + \beta r_{01}) \mu + (r_{11} + \mu r_{10}) \beta}{\beta + \mu} \quad (2.2.13)$$

The expected profit $g$, thus depends upon failure rate $\beta$, repair rate $\mu$ and $r_{ij}$ ($i, j = 0, 1$). It may be observed that it will be of interest to obtain the value for the repair rate which maximizes the expected profit when $r_{ij}$ and $\beta$ are given (fixed). For the purpose, let us consider

$$r_{00} = f_1(\beta), \ r_{01} = f_2(\beta), \ r_{10} = f_3(\mu), \ r_{11} = f_4(\mu) \quad (2.2.14)$$
Further, if we assume

\[ r_{00} = f_1(\beta) = \frac{c_1}{\beta}, \quad c_1 \geq 0, \beta > 0 \]
\[ r_{01} = f_2(\beta) = A, \quad A \leq 0 \]
\[ r_{10} = f_3(\mu) = B, \quad B \geq 0 \]
\[ r_{11} = f_4(\mu) = \alpha + c_2 \mu + c_3 \mu^2, \quad c_2, c_3 \leq 0, \quad \alpha \text{ is a constant} \quad (2.2.15) \]

On making use of (2.2.15) in (2.2.13) the expected profit \( g \) takes the form

\[ g = \frac{a \mu^2 + b \mu + c}{\mu + d} \quad (2.2.16) \]

where

\[ a = \beta c_3, \quad b = \frac{c_1}{\beta} + \beta (A + B + c_2), \quad c = \alpha \beta, \quad d = \beta. \]

The value of \( \mu \) which maximizes the expected profit in (2.2.16) is obtained by differentiating \( g \) in (2.2.16) with respect to \( \mu \) and equating it to zero i.e.,

\[ \begin{align*}
\frac{dg}{d\mu} &= \frac{a \mu^2 + 2ad \mu + bd - c}{(\mu + d)^2} = 0 \\
\Rightarrow a \mu^2 + 2ad \mu + bd - c &= 0 \quad (2.2.17)
\end{align*} \]

The optimum value of repair rate \( \mu \) denoted by \( \mu^* \) which maximizes \( g \) denoted by \( g^* \) is given by

\[ \mu^* = \frac{-2ad \pm \sqrt{(2ad)^2 - 4a(bd - c)}}{2a} = \frac{-ad \pm \sqrt{\omega}}{a} \quad (2.2.18) \]
where

\[ \omega = (ad)^2 - a(bd - c) \]

Since \( \mu^* \) is finite and cannot be negative, being repair rate, to decide \( \mu^* \) gives maximum we have to further examine the sign of \( \frac{d^2 g}{d \mu^2} \bigg|_{\mu = \mu^*} \)

(i) If \( \mu^* > 0 \) and \( \frac{d^2 g}{d \mu^2} \bigg|_{\mu = \mu^*} < 0 \), \( \mu^* \) provides maximum g i.e.,

the sufficient condition for \( \mu^* > 0 \) to be absolute maximum point for \( g \) is

\[ \beta^3 c_3 + \beta^2 (A + B + c_2) + \alpha \beta - c_1 < 0. \]

(ii) However, if \( \mu^* > 0 \) and \( \frac{d^2 g}{d \mu^2} \bigg|_{\mu = \mu^*} > 0 \), \( \mu^* \) is the minimum point for \( g \) i.e., when

\[ \beta^3 c_3 + \beta^2 (A + B + c_2) + \alpha \beta - c_1 > 0 \]

and absolute maximum is achieved at \( \mu^* = 0 \) for which \( g^* = \alpha \).

We now proceed to illustrate the use of results in (2.2.18). We present optimum repair rate \( \mu^* \) against various values of failure rates, \( \beta \) for different sets of constants \( \alpha, c_1, c_2, c_3, A \) and \( B \) (where \( \alpha = 0 \)) in the Figure 2.2.2. Some of observations are:

(i) It may be observed that in general \( \mu^* \) decreases as \( \beta \) increases.

Further the decrease in the value of \( \mu^* \) for small (initial) values of \( \beta \) is quite substantial as compared to the rate of decrease in the value of \( \mu^* \) for large values of \( \beta \). To elaborate this point

45
LEGEND

<table>
<thead>
<tr>
<th>CURVE</th>
<th>A</th>
<th>B</th>
<th>C₁</th>
<th>C₂</th>
<th>C₃</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-4</td>
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<td>-10</td>
<td></td>
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<td>-4</td>
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<td>-5</td>
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</tr>
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<td>5</td>
<td>-4</td>
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<td>10</td>
<td>-10</td>
<td>-5</td>
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<tr>
<td>6</td>
<td>-2</td>
<td>6</td>
<td>10</td>
<td>-10</td>
<td>-5</td>
</tr>
</tbody>
</table>

FIGURE 2:2:2: GRAPH SHOWING THE VARIATION OF OPTIMUM REPAIR RATE (α̂) AGAINST FAILURE RATE (β̂)
further, we present the results in table 2.2.4a corresponding to curve 1 in Figure 2.2.2. From the same table we observe that the expected profit $g^*$ also behaves in similar manner. Behaviour of other curves also follow the same pattern.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\mu^*$</th>
<th>$\mu^<em>(\beta)-\mu^</em>(\beta+.1)$</th>
<th>$g^*$</th>
<th>$g^<em>(\beta)-g^</em>(\beta+.1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>6.27</td>
<td>2.80</td>
<td>196.47</td>
<td>136.21</td>
</tr>
<tr>
<td>0.15</td>
<td>3.47</td>
<td>0.95</td>
<td>60.26</td>
<td>28.56</td>
</tr>
<tr>
<td>0.25</td>
<td>2.52</td>
<td>0.57</td>
<td>31.70</td>
<td>12.74</td>
</tr>
<tr>
<td>0.35</td>
<td>1.95</td>
<td>0.42</td>
<td>18.96</td>
<td>7.23</td>
</tr>
<tr>
<td>0.45</td>
<td>1.53</td>
<td>0.33</td>
<td>11.73</td>
<td>4.54</td>
</tr>
<tr>
<td>0.55</td>
<td>1.20</td>
<td>0.28</td>
<td>7.19</td>
<td>2.97</td>
</tr>
<tr>
<td>0.65</td>
<td>0.92</td>
<td>0.25</td>
<td>4.22</td>
<td>1.95</td>
</tr>
<tr>
<td>0.75</td>
<td>0.67</td>
<td>0.21</td>
<td>2.27</td>
<td>1.21</td>
</tr>
<tr>
<td>0.85</td>
<td>0.46</td>
<td>0.19</td>
<td>1.06</td>
<td>0.70</td>
</tr>
<tr>
<td>0.95</td>
<td>0.27</td>
<td>0.17</td>
<td>0.36</td>
<td>0.31</td>
</tr>
<tr>
<td>1.05</td>
<td>0.10</td>
<td></td>
<td>0.05</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 2.2.4a

(ii) With the increase in $c_1$, the value of $\mu^*$ increases but shows a steep increase for smaller values of $\beta$, for example

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\mu^*$ from curve (1)</th>
<th>$\mu^*$ from curve (2)</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>2.51</td>
<td>3.16</td>
<td>0.65</td>
</tr>
<tr>
<td>1.00</td>
<td>0.18</td>
<td>0.55</td>
<td>0.37</td>
</tr>
</tbody>
</table>

(iii) With the increase in $c_2$, the value of $\mu^*$ first increases slowly and then significantly, for example

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\mu^*$ from curve (1)</th>
<th>$\mu^*$ from curve (3)</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>2.51</td>
<td>2.55</td>
<td>0.04</td>
</tr>
<tr>
<td>1.00</td>
<td>0.18</td>
<td>0.48</td>
<td>0.30</td>
</tr>
</tbody>
</table>

(iv) With the increase in $c_3$, the value of $\mu^*$ first increases significantly
and then slowly, for example

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>2.51</td>
<td>3.30</td>
<td>0.79</td>
</tr>
<tr>
<td>1.00</td>
<td>0.18</td>
<td>0.29</td>
<td>0.11</td>
</tr>
</tbody>
</table>

(v) With the increase in \( A \), the value of \( \mu^* \) first increases slowly and then significantly, for example

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>2.51</td>
<td>2.54</td>
<td>0.03</td>
</tr>
<tr>
<td>1.00</td>
<td>0.18</td>
<td>0.34</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Further, if one is confronted with the problem of making a choice regarding installing of a production system whose various parameters like failure rate \( \beta \), repair rate \( \mu \) and the earning rates in operable state and failed state, transition costs for making a transition from operable state to failed state and vice versa i.e., \( r_{ij} \) are specified. Then one should choose a system which gives more expected profit than others making use of (2.2.13). In what follows we present the table 2.2.5 for given system under consideration.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>1.0</td>
<td>0.5</td>
<td>0.8</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>( \mu )</td>
<td>1.5</td>
<td>1.3</td>
<td>1.8</td>
<td>1.5</td>
<td>1.5</td>
<td>2.0</td>
<td>2.0</td>
<td>2.5</td>
</tr>
<tr>
<td>( r_{00} )</td>
<td>20.0</td>
<td>25.0</td>
<td>30.0</td>
<td>40.0</td>
<td>35.0</td>
<td>50.0</td>
<td>55.0</td>
<td>55.0</td>
</tr>
<tr>
<td>( r_{11} )</td>
<td>-30.0</td>
<td>-30.0</td>
<td>-35.0</td>
<td>-50.0</td>
<td>-40.0</td>
<td>-65.0</td>
<td>-70.0</td>
<td>-75.0</td>
</tr>
<tr>
<td>( r_{01} )</td>
<td>-4.0</td>
<td>-4.0</td>
<td>-2.0</td>
<td>-2.0</td>
<td>-3.0</td>
<td>-2.0</td>
<td>-3.0</td>
<td>-4.0</td>
</tr>
<tr>
<td>( r_{10} )</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>4.0</td>
<td>5.0</td>
<td>4.0</td>
<td>2.0</td>
</tr>
<tr>
<td>( g )</td>
<td>7.12</td>
<td>7.22</td>
<td>12.30</td>
<td>4.60</td>
<td>16.62</td>
<td>18.85</td>
<td>14.00</td>
<td>16.42</td>
</tr>
</tbody>
</table>

Table 2.2.5

Thus we find from the above table that the expected profit \( g \) is maximum for the \( 6^{th} \) system and therefore, management should opt for this system.
2.3 Model 2: PROFIT EVALUATION IN A SINGLE UNIT SYSTEM WITH A LESS PRODUCTIVE STATE

In this section we discuss a model of a single unit consisting of two modules. The failure of one module brings the system in a less productive state whereas the failure in the other module brings the system to failed state.

System Model

(i) There is 1-unit system consisting of type I module and type II module.

(ii) From operable state, system goes to failed state on the failure of type I module.

(iii) From operable state, system goes to less productive state on the failure of type II module.

(iv) From less productive state system goes to failed state on the failure of type I module.

(v) In the failed state, repairs are carried out to bring the system back to operable state.

(vi) Failure time distributions of type I and type II modules are assumed to be exponential.

(vii) Repair time distribution is assumed to be general.

(viii) In failed state, if there is any failed module of type II, it is repaired free of cost together with repair of type I module.

(ix) After repair, system acts as new one.
(x) All probability distributions are assumed to be s-independent.

(xi) The system earns (loses) at a fixed rate in each state which can be different for each state. There is a fixed transition reward (cost) whenever the system changes its state.

Define the following system states to identify the system at any instant.

System states and Transitions

\( S_0 \) : Operable or fully productive.

\( S_1 \) : Less productive, type II module is in failed state.

\( S_2 \) : Failed state, type I module or type I and type II modules are in failed state and the system is under repair.

The system is up in \( S_0 \) and \( S_1 \), and it is down in \( S_2 \). The transitions between states are given in Figure 2.3.1.

![Figure 2.3.1. Transitions between various states](image-url)
NOTATION

$\beta_1$ : constant failure rate of Type I module.

$\beta_2$ : constant failure rate of Type II module.

$g(t)$ : pdf of repair time

$\mu$ : expected time for repair

$\bar{T}_i$ : mean unconditional waiting time in $S_i$ ($i = 0, 1, 2$)

$p_{ij}$ : one step transitional probability from $S_i$ to $S_j$

$\pi_i$ : steady-state probability in $S_i$ ($i = 0, 1, 2$)

$r_{ij}$ : transition cost for a transition from $S_i$ to $S_j$ (for $i \neq j$).

$r_{ii}$ : cost per unit time or earning rate in $S_i$

$g$ : total expected earning per unit time in steady-state or steady-state expected profit.

Expected Profit of the System

Observing state transitions from figure 2.3.1, various state transition probabilities can be written as:

\[
\begin{align*}
 p_{01} &= \int_0^\infty \beta_2 e^{-\beta_2 t} e^{-\beta_1 t} \, dt = \frac{\beta_2}{\beta_1 + \beta_2} \\
 p_{02} &= \int_0^\infty \beta_1 e^{-\beta_1 t} e^{-\beta_2 t} \, dt = \frac{\beta_1}{\beta_1 + \beta_2} \\
 p_{20} &= \int_0^\infty g(t) \, dt = 1 \\
 p_{12} &= \int_0^\infty \beta_1 e^{-\beta_1 t} \, dt = 1
\end{align*}
\]

(2.3.1)
Further, it is easy to see that

\[ \overline{T}_0 = \int_0^\infty e^{-\beta_1 t} e^{-\beta_2 t} \, dt = \frac{1}{\beta_1 + \beta_2} \]

\[ \overline{T}_1 = \int_0^\infty e^{-\beta_1 t} \, dt = \frac{1}{\beta_1} \]  \hspace{1cm} (2.3.2)

\[ \overline{T}_2 = \frac{1}{\mu} \]

Steady-state probabilities \( \pi_i \) can be obtained by following relations of Howard (1971) and \( p_{ij} \) from (2.3.1)

\[ \pi_j = \sum_i p_{ij} \pi_i \]  \hspace{1cm} (2.3.3)

and

\[ \sum_i \pi_i = 1 \]  \hspace{1cm} (2.3.4)

Therefore,

\[ \pi_0 = \frac{\beta_1 + \beta_2}{2\beta_1 + 3\beta_2}, \quad \pi_1 = \frac{\beta_2}{2\beta_1 + 3\beta_2}, \quad \pi_2 = \frac{\beta_1 + \beta_2}{2\beta_1 + 3\beta_2} \]  \hspace{1cm} (2.3.5)

Following Howard (1971) expected profit \( g \) can be written as

\[ g = \frac{\sum \pi_i T_i q_i}{\sum \pi_i T_i} \]  \hspace{1cm} (2.3.6)

where

\[ q_i = r_{ii} + \frac{1}{T_i} \sum_j p_{ij} r_{ij} \]
Substituting values of $p_{ij}$ from (2.3.1), $\overline{T}_i$ from (2.3.2) and $\pi_i$ (2.3.5) in (2.3.6) and after simplification $g$ can be obtained as

$$g = \frac{\mu \beta_1 (r_{00} + \beta_2 r_{01} + \beta_1 r_{02}) + \mu \beta_2 (r_{11} + \beta_1 r_{12}) + (\beta_1 + \beta_2) \beta_1 (r_{22} + \mu r_{20})}{\mu (\beta_1 + \beta_2) + \beta_1 (\beta_1 + \beta_2)}$$  

(2.3.7)

**Profit Optimization**

Let the repair rate $\mu$ of the system is controllable variable, expected profit, $g$ can be optimized for two cases as under:

**Case I**

When the earning in the failed state is a continuous function of repair rate. Let us specify $r_{ij}$ and $r_{ii}$ as follows:

$$r_{00} = A, \quad A > 0,$$

$$r_{11} = B, \quad A > B > 0$$

$$r_{22} = \alpha + C_1 \mu + C_2 \mu^2, \quad \alpha \leq 0; \quad C_1, C_2 < 0$$  

(2.3.8)

and

$$r_{ij} = 0 \text{ for } i \neq j$$

Substituting these value of $r_{00}, r_{11}, r_{22}$ from (2.3.8) in (2.3.7) and after simplification we get

$$g = \frac{K \mu^2 + I \mu + N}{P \mu + Y}$$  

(2.3.9)

where
\[ K = (\beta_1 + \beta_2)\beta_1 C_2 \]
\[ L = \beta_1 A + \beta_2 B + (\beta_1 + \beta_2)\beta_1 C_1 \]
\[ N = (\beta_1 + \beta_2)\beta_1 \alpha \]
\[ P = (\beta_1 + \beta_2) \]
\[ Y = (\beta_1 + \beta_2)\beta_1 \]

(2.3.10)

To determine the value of \( \mu \) which maximizes \( g \) for given \( \beta_1, \beta_2 \) and known values of cost parameters, we differentiate \( g \) in (2.3.9) with respect to \( \mu \) and equate it to zero i.e., \( \frac{dg}{d\mu} = 0 \), which gives the value of \( \mu \) denoted by \( \mu^* \) maximizing \( g \) denoted by \( g^* \), so

\[ \mu^* = \frac{-KY \pm \sqrt{K^2Y^2 - KP(LY- NP)}}{KP} \]  
(2.3.11)

Since \( \mu^* \) is finite and cannot be negative, being repair rate, to decide if \( \mu^* \) gives maximum, we have to further examine the sign of \( \frac{d^2g}{d\mu^2} \) at \( \mu = \mu^* \)

(i) If \( \mu^* > 0 \) and \( \frac{d^2g}{d\mu^2} \bigg|_{\mu = \mu^*} < 0 \), \( \mu^* \) provides absolute maximum \( g^* \) i.e., the sufficient condition for \( \mu^* > 0 \) to be absolute maximum point for \( g \) is \( (\beta_1 + \beta_2)(\beta_1^2C_2 - \beta_1 C_1 + \alpha) < A\beta_1 + B\beta_2 \)

(ii) However, if \( \mu^* > 0 \) and \( \frac{d^2g}{d\mu^2} \bigg|_{\mu = \mu^*} > 0 \), \( \mu^* \) is the minimum point i.e., when \( (\beta_1 + \beta_2)(\beta_1^2C_2 - \beta_1 C_1 + \alpha) > A\beta_1 + B\beta_2 \), \( \mu^* \) is the minimum point and absolute maximum is achieved at \( \mu^* = 0 \)

54
for which \( g^* = \alpha \).

**Illustration**

In what follows we shall illustrate the results with the help of numerical example. Let us consider a system for which the various values of parameters are given as below:

\[
\begin{align*}
\beta_1 &= 0.1, \quad A = 50.0, \quad \alpha = -50.0, \quad C_2 = -2.0 \\
\beta_2 &= 0.2, \quad B = 40.0, \quad C_1 = -4.0
\end{align*}
\]

On substituting these values \( \beta_1, \beta_2, A, B, \alpha, C_1 \) and \( C_2 \) in (2.3.10) and (2.3.11), we obtain \( \mu^* = 6.916 \)

For \( \mu^* = 6.916 \), \( g^* \) can be obtained from (2.3.9) i.e.,

\[ g^* = 40.245. \]

**Case II**

When the earning in the failed state is a discrete function of repair rate.

In many practical situations the earning of the system in failed state i.e., \( r_{22} \) is not a continuous function of the repair rate \( \mu \) and in fact \( r_{22} \) is discrete function of repair rate \( \mu \).

Let the management have the option to choose a repair policy out of a number of given repair policies (repair rates) and their corresponding earning rates (\( r_{22} \)) in failed states.

In such cases, \( g \) is computed for each \( \mu \) and corresponding \( r_{22} \) from
(2.3.7) and that repair policy is chosen which gives more expected profit than others.

In case more then one repair policy gives the same amount of expected profit, any one of them can be selected.

Illustration

In what follows we shall illustrate with a numerical example. Let the values of the various parameters of the system under consideration are specified as below:

\[ r_{00} = 50.0, \; \beta_1 = 0.1, \; r_{01} = -2.0, \; r_{12} = -2.0 \]

\[ r_{11} = 40.0, \; \beta_2 = 0.2, \; r_{02} = -3.0, \; r_{20} = 4.0. \]

Then the values of \( g \) computed from (2.3.7) for known values of repair rates, \( \mu \) and \( r_{22} \) are presented in the table 2.3.1.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>1.0</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>4.5</th>
<th>5.0</th>
<th>7.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_{22} )</td>
<td>-60.0</td>
<td>-68.0</td>
<td>-75.0</td>
<td>-85.0</td>
<td>-88.0</td>
<td>-105.0</td>
<td>-160.0</td>
</tr>
<tr>
<td>( g )</td>
<td>33.969</td>
<td>38.063</td>
<td>39.548</td>
<td>40.235</td>
<td>40.510</td>
<td>40.457</td>
<td>40.502</td>
</tr>
</tbody>
</table>

Table 2.3.1

From the above table we observe that for repair rate, \( \mu = 4.5 \), \( g \) is maximum and hence the management should follow this repair policy.

2.4 Model 3: PROFIT CONSIDERATIONS IN A SINGLE UNIT SYSTEM WITH MINOR REPAIR

System Model

(i) There is a 1-unit system. The system can go to either failed
state or less productive state.

(ii) When the system goes to less productive state, the management can opt:
(a) To carry out minor repair in order to bring the system back to fully productive state.
(b) To allow the system to fail.

(iii) In the failed state repairs are carried out to bring the system back to operable state.

(iv) Failure-time distributions namely going from operable to failed state and less productive state are assumed to be exponential.

(v) Repair-time distributions are assumed to be general.

(vi) All probability distributions are assumed to be s-independent.

(vii) The system earns (loses) at a fixed rate in each state which can be different for each state. There is a fixed transition reward (cost) whenever the system changes its state.

Define the following system states to identify the system at any instant.

**System states and Transitions**

- $S_0$: Operable or fully productive
- $S_1$: Less productive
- $S_2$: Failed and is under repair

We consider two cases - one under which the system is sent for minor repairs as soon as it enters $S_1$ and second when the system is allowed to
enter $S_2$ in due course of time. This is depicted in transitional diagrams (Figure 2.4.1 and 2.4.2).

![Diagram](image)

Fig. 2.4.1 Transitions between various states

**NOTATION**

- $\beta_1$ : Constant rate of going from $S_0$ to $S_1$
- $\beta_2$ : Constant rate of going from $S_0$ to $S_2$
- $\beta_3$ : Constant rate of going from $S_1$ to $S_2$
- $g_1(t)$ : pdf of minor repair time
- $g_2(t)$ : pdf of repair time
- $\mu_1$ : expected time for minor repair
- $\mu_2$ : expected time for repair
- $\bar{T}_i$ : mean unconditional waiting time in $S_i$ (i=0, 1, 2)
- $p_{ij}$ : one step transitional probability from $S_i$ to $S_j$
\( \pi_i \) : steady-state probability in \( S_i \) (\( i = 0, 1, 2 \))

\( r_{ij} \) : transition reward for a transition from \( S_i \) to \( S_j \)

\( r_{ii} \) : earning rate in \( S_i \) (\( i = 0, 2 \))

\( r_{11}^M \) : earning rate in \( S_1 \) when minor repairs are being carried out

\( r_{11}^N \) : earning rate in \( S_1 \) when minor repairs are not being carried out

\( g_M \) : expected profit per unit time when the system is under minor repair

\( g_N \) : expected profit per unit time when minor repairs are not carried out in \( S_1 \).

**Expected Profit of the System**

**Case I**

Observing state transitions from figure 2.4.1, various transition probabilities can be written as

\[
\begin{align*}
\frac{p_{01}}{t} &= \int_0^\infty \beta_1 e^{-\beta_1 t} e^{-\beta_2 t} dt = \frac{\beta_1}{\beta_1 + \beta_2} \\
\frac{p_{02}}{t} &= \int_0^\infty \beta_2 e^{-\beta_2 t} e^{-\beta_1 t} dt = \frac{\beta_2}{\beta_1 + \beta_2} \\
p_{10} &= \int_0^\infty g_1(t) dt = 1 \\
p_{20} &= \int_0^\infty g_2(t) dt = 1
\end{align*}
\]

(2.4.1)

Further, it is easy to see that
\[
\overline{T}_0 = \int_0^\infty e^{-\beta_1 t} e^{-\beta_2 t} \, dt = \frac{1}{\beta_1 + \beta_2}
\]

\[
\overline{T}_1 = \frac{1}{\mu_1}
\]

\[
\overline{T}_2 = \frac{1}{\mu_2}
\]

Also steady-state probabilities \( \pi_i \) can be obtained by following relations of Howard (1971) and \( p_{ij} \) from (2.4.1)

\[
\pi_j = \sum_i p_{ji} \pi_i
\]

(2.4.3)

\[
\sum_i \pi_i = 1
\]

(2.4.4)

Therefore,

\[
\pi_0 = \frac{1}{2}, \quad \pi_1 = \frac{\beta_1}{2(\beta_1 + \beta_2)}, \quad \pi_2 = \frac{\beta_2}{2(\beta_1 + \beta_2)}
\]

(2.4.5)

Following Howard (1971), expected profit per unit time, \( g \) can be written as

\[
g = \frac{\sum_i \pi_i \overline{T}_i q_i}{\sum_i \pi_i \overline{T}_i}
\]

(2.4.6)

where \( q_i = r_{ii} + \frac{1}{\overline{T}_i} \sum_j p_{ij} r_{ij} \)

Substituting values of \( p_{ij} \) from (2.4.1), \( \overline{T}_i \) from (2.4.2) and \( \pi_i \) from (2.4.5) in (2.4.6) and after simplification, \( g_M \) can be obtained as
\[ g_M = \frac{\mu_1 \mu_2 (r_{00} + \beta_1 r_{01} + \beta_2 r_{02}) + \beta_1 \mu_2 (r_{10}^M + \mu_1 r_{10}) + \beta_2 \mu_1 (r_{22} + \mu_2 r_{20})}{\mu_1 \mu_2 + \beta_1 \mu_2 + \beta_2 \mu_1} \]

This can also be written as

\[ g_M = \frac{\mu_1 A + B}{\mu_1 D + E} \]  

(2.4.7)

where

\[ A = \mu_2 (r_{00} + \beta_1 r_{01} + \beta_2 r_{02}) + \beta_1 \mu_2 r_{10}^M + \beta_2 r_{22} + \beta_2 \mu_2 r_{20} \]

\[ B = \beta_1 \mu_2 r_{11}^M \]

\[ D = \beta_2 + \mu_2 \]

\[ E = \beta_2 \mu_2 \]

Fig. 2.4.2 Transitions between various states

Case II

Observing state transitions from figure 2.4.2, various transition probabilities can be written as
\[
p_{01} = \int_0^\infty \beta_1 e^{-\beta_1 t} e^{-\beta_2 t} \, dt = \frac{\beta_1}{\beta_1 + \beta_2}
\]
\[
p_{02} = \int_0^\infty \beta_2 e^{-\beta_2 t} e^{-\beta_1 t} \, dt = \frac{\beta_2}{\beta_1 + \beta_2}
\]
\[
p_{12} = \int_0^\infty \beta_3 e^{-\beta_3 t} \, dt = 1
\]
\[
p_{20} = \int_0^\infty g_2(t) \, dt = 1
\]

Further, it is easy to see that

\[
\overline{T}_0 = \int_0^\infty e^{-\beta_1 t} e^{-\beta_2 t} \, dt = \frac{1}{\beta_1 + \beta_2}
\]
\[
\overline{T}_1 = \int_0^\infty e^{-\beta_3 t} \, dt = \frac{1}{\beta_3}
\]
\[
\overline{T}_2 = \frac{1}{\mu_2}
\]

Also steady-state probabilities \( \pi_i \) can be obtained by making use of (2.4.3), (2.4.4) and \( p_{ij} \) from (2.4.8).

\[
\pi_0 = \frac{\beta_1 + \beta_2}{3\beta_1 + 2\beta_2}, \quad \pi_1 = \frac{\beta_1}{3\beta_1 + 2\beta_2}, \quad \pi_2 = \frac{\beta_1 + \beta_2}{3\beta_1 + 2\beta_2}
\]

Substituting these values of \( p_{ij} \) from (2.4.8), \( \overline{T}_i \) from (2.4.9) and \( \pi_i \) from (2.4.10) in (2.4.6) and after simplification we get,

\[
g_N = \frac{\mu_b^3 (r_0 + \beta_1 r_1 + \beta_2 r_2) + \mu_b (r_1 + \beta_3 (r_{11} + \beta_3 r_{12}) + \beta_3 (r_{22} + \mu_2 r_{20}))}{\mu_2^3 \beta_3 + \beta_2 \mu_2 + \beta_3 \beta_1 + \beta_2}
\]

This can also be written as

\[
g_N = \frac{\beta_3^2 F + G}{\beta_3 H + E}
\]
where

\[ F = \mu_2(R_{00} + \beta_1 R_{01} + \beta_2 R_{02} + \beta_1 R_{12}) + (\beta_1 + \beta_2)(R_{22} + \mu_2 R_{20}) \]
\[ G = \beta_1 \mu_2 R_{11}^N \]
\[ H = \beta_1 + \beta_2 + \mu_2 \]
\[ E = \beta_1 \mu_2 \]

**Decision Criterion Regarding Profit**

Now it is evident that when \( g_M > g_N \), the management of the system should follow procedures to carry out minor repairs for acquiring more profit.

When \( g_M = g_N \), it is immaterial whether minor repairs are carried out in less productive state or the system is allowed to fail.

When \( g_M < g_N \), it is advisable, not to follow minor repair procedures.

For \( g_M \geq g_N \),

Then, from (2.4.7) and (2.4.11)

\[ \frac{\mu_A + B}{\mu_D + E} \geq \frac{F + G}{H + E} \]

\[ \Rightarrow \mu_2 \beta_3 (AH - FD) + \mu_1 (EA - GD) + \beta_3 (BH - FE) \geq GE - BE \quad (2.4.12) \]

Let

\[ K = \frac{\mu_1}{\beta_3} \]

Therefore, (2.4.12) reduce to

\[ K[\beta_3^2 (AH - FD) + \beta_3 (EA - GD)] \geq GE - BE - \beta_3 (BH - FE). \]
Then, we get critical value of $K$ denoted by $K^*$ as

$$K^* = \frac{(GE - BE) - \beta_3(BH - FE)}{\beta_3^2(AH - FD) + \beta_3(EA - GD)}$$  \hspace{1cm} (2.4.13)$$

For this value of $K^*$, $g_M = g_N$.

Further, notice if $K < K^*$, it is profitable not to carry out minor repairs, since this leads to less profit.

However, when $K > K^*$, it is profitable to carry out minor repairs.

**ILLUSTRATION**

Let us consider a system for which the various values of parameters are given as below except $\mu_1$ and $\beta_3$

$\beta_1 = 0.6$, $\ r_0 = 10.0$, $\ r_{11}^M = -5.0$, $\ r_{12} = -3.0$

$\beta_2 = 0.4$, $\ r_{01} = -2.0$, $\ r_{11}^N = 5.0$, $\ r_{20} = 5.0$

$\mu_2 = 3.0$, $\ r_{02} = -4.0$, $\ r_{10} = 3.0$, $\ r_{22} = -15.0$.

Substituting these values in (2.4.13), the critical values of $K^*$ are given in Table 2.4.1 for various values of $\beta_3$

<table>
<thead>
<tr>
<th>$\beta_3$</th>
<th>$K^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>22.505</td>
</tr>
<tr>
<td>0.2</td>
<td>10.058</td>
</tr>
<tr>
<td>0.3</td>
<td>6.211</td>
</tr>
<tr>
<td>0.4</td>
<td>4.406</td>
</tr>
<tr>
<td>0.5</td>
<td>3.379</td>
</tr>
<tr>
<td>1.0</td>
<td>1.502</td>
</tr>
<tr>
<td>1.5</td>
<td>0.949</td>
</tr>
<tr>
<td>2.0</td>
<td>0.690</td>
</tr>
<tr>
<td>2.5</td>
<td>0.541</td>
</tr>
</tbody>
</table>

**Table 2.4.1**

64
Now suppose in the given problem $\beta_3 = 1.0$ and $\mu_1 = 1.0$, then, $K = 1.0$ which is less than $K^* (K^* = 1.502$ from the above table) and therefore, it is profitable if minor repairs are not carried out.

Now suppose $\beta_3 = 1.0$ and $\mu_1 = 2.0$ then $K = 2.0$ which is greater than $K^*$ and therefore, it is profitable to carry out minor repair.

2.5 Model 4: PROFIT EVALUATION IN A 2-UNIT PARALLEL REDUNDANT SYSTEM

System Model

(i) The system consists of two units, I and II. Both the units operate simultaneously.

(ii) Failure-time distributions of units I and II are exponential with rates $\beta_1$ and $\beta_2$ respectively.

(iii) Whenever one of the two units fails, the operative unit continues to operate whereas the failed unit undergoes repair.

(iv) On failure of both units, they undergo repair.

(v) Repair-time distributions of units I and II are exponential with rates $\mu_1$ and $\mu_2$ respectively.

(vi) All probability distributions are assumed to be statistically independent.

(vii) After repair, units act as new ones.

(viii) The system earns (loses) at a fixed rate in each state which can be different for each state. There is a fixed transition reward
(cost) whenever the system changes its state.

Define the following system states to identify the system at any instant.

System States and Transitions

- $S_0$ : Both the units are operating.
- $S_1$ : Unit II is operating while unit I fails and undergoes repair.
- $S_2$ : Unit I is operating while unit II fails and undergoes repair.
- $S_3$ : Both the units are in failed state and are under repair.

The system is up in $S_0$, $S_1$ and $S_2$ and it is down in $S_3$. Transitions between various states are given in Figure 2.5.1.

![Figure 2.5.1: Transitions between various states](image)

Figure 2.5.1: Transitions between various states
NOTATION

\[\beta_1\] : constant failure rate of Unit I.
\[\beta_2\] : constant failure rate of Unit II.
\[\mu_1\] : constant repair rate of Unit I.
\[\mu_2\] : constant repair rate of Unit II.
\[r_{ii}\] : cost per unit time or earning rate in state \(S_i\) \((i = 0, 1, 2, 3)\).
\[r_{ij}\] : transition cost for a transition from state \(S_i\) to \(S_j\) \((i, j = 0, 1, 2, 3)\).
\[V_i(t)\] : total expected earning of the system for a future period of length \(t\) given that at \(t = 0\), the system was in \(S_i\) \((i = 0, 1, 2, 3)\).

Cost Equations and System Profit

Following Howard (1960), we can write the following cost equations:

\[V_0(t+\Delta) = [r_{00}\Delta+V_0(t)][1-\beta_1\Delta][1-\beta_2\Delta]+[r_{01}+V_1(t)]
\times [1-\beta_2\Delta]\beta_1\Delta+[r_{02}+V_2(t)][1-\beta_1\Delta]\beta_2\Delta\]  
(2.5.1)

\[V_1(t+\Delta) = [r_{11}\Delta+V_1(t)][1-\mu_1\Delta][1-\beta_2\Delta]+[r_{10}+V_0(t)]
\times [1-\beta_2\Delta]\mu_1\Delta+[r_{13}+V_3(t)][1-\mu_1\Delta]\beta_2\Delta\]  
(2.5.2)

\[V_2(t+\Delta) = [r_{22}\Delta+V_2(t)][1-\mu_2\Delta][1-\beta_1\Delta]+[r_{20}+V_0(t)]
\times [1-\beta_1\Delta]\mu_2\Delta+[r_{23}+V_3(t)][1-\mu_2\Delta]\beta_1\Delta\]  
(2.5.3)

\[V_3(t+\Delta) = [r_{33}\Delta+V_3(t)][1-\mu_1\Delta][1-\mu_2\Delta]+[r_{31}+V_1(t)]
\times [1-\mu_1\Delta]\mu_2\Delta+[r_{32}+V_2(t)][1-\mu_2\Delta]\mu_1\Delta\]  
(2.5.4)
where $\Delta$ is a small interval.

Initially $V_i(0) = 0$ for $i = 0, 1, 2, 3$. \hfill (2.5.5)

Neglecting terms of order of 2 and higher of $\Delta$, and letting $\Delta \to 0$, we obtain following system of differential equations from (2.5.1) to (2.5.4).

\[
\frac{d}{dt} + (\beta_1 + \beta_2) V_0(t) = r_{00} + \beta_1 r_{01} + \beta_2 r_{02} + \beta_1 V_1(t) + \beta_2 V_2(t) \tag{2.5.6}
\]

\[
\frac{d}{dt} + (\mu_1 + \beta_2) V_1(t) = r_{11} + \mu_1 r_{10} + \beta_2 r_{13} + \mu_1 V_0(t) + \beta_2 V_3(t) \tag{2.5.7}
\]

\[
\frac{d}{dt} + (\beta_1 + \mu_2) V_2(t) = r_{22} + \mu_2 r_{20} + \beta_1 r_{23} + \mu_2 V_0(t) + \beta_1 V_3(t) \tag{2.5.8}
\]

\[
\frac{d}{dt} + (\mu_1 + \mu_2) V_3(t) = r_{33} + \mu_2 r_{31} + \mu_1 r_{32} + \mu_2 V_1(t) + \mu_1 V_2(t) \tag{2.5.9}
\]

Let $\tilde{f}(s)$ denote the Laplace transform of $f(t)$, e.g.,

\[
\tilde{f}(s) = \int_0^\infty e^{-st}f(t)\,dt, \quad \text{Re}(s)>0.
\]

Taking Laplace transform of (2.5.6) to (2.5.9) and employing initial conditions (2.5.5), we get after simplification :

\[
\mathcal{V}_0(s) = \frac{r_{00} + \beta_1 r_{01} + \beta_2 r_{02}}{s(s + \beta_1 + \beta_2)} + \frac{\beta_1 \mathcal{V}_1(s) + \beta_2 \mathcal{V}_2(s)}{(s + \beta_1 + \beta_2)} \tag{2.5.10}
\]

\[
\mathcal{V}_1(s) = \frac{r_{11} + \mu_1 r_{10} + \beta_2 r_{13}}{s(s + \mu_1 + \beta_2)} + \frac{\mu_1 \mathcal{V}_0(s) + \beta_2 \mathcal{V}_3(s)}{(s + \mu_1 + \beta_2)} \tag{2.5.11}
\]

\[
\mathcal{V}_2(s) = \frac{r_{22} + \mu_2 r_{20} + \beta_1 r_{23}}{s(s + \beta_1 + \mu_2)} + \frac{\mu_2 \mathcal{V}_0(s) + \beta_1 \mathcal{V}_3(s)}{(s + \beta_1 + \mu_2)} \tag{2.5.12}
\]

\[
\mathcal{V}_3(s) = \frac{r_{33} + \mu_2 r_{31} + \mu_1 r_{32}}{s(s + \mu_1 + \mu_2)} + \frac{\mu_2 \mathcal{V}_1(s) + \mu_1 \mathcal{V}_2(s)}{(s + \mu_1 + \mu_2)} \tag{2.5.13}
\]
Further simplification of (2.5.10) to (2.5.13) gives

\[ \nabla_v(s) = \frac{R_0 A_1 + \beta_1 R_1 A_2 + \beta_2 R_2 A_3 + \beta_3 R_3 A_4}{s^2(s+\beta_1+\mu_1)(s+\beta_2+\mu_2)(s+\beta_1+\mu_1+\beta_2+\mu_2)} \] (2.5.14)

where

\[ R_0 = r_{00} + \beta_1 r_{01} + \beta_2 r_{02} \]
\[ R_1 = r_{11} + \mu_1 r_{10} + \beta_2 r_{13} \]
\[ R_2 = r_{22} + \mu_2 r_{20} + \beta_1 r_{23} \]
\[ R_3 = r_{33} + \mu_2 r_{31} + \mu_1 r_{32} \]

and

\[ A_1 = (s+\mu_1+\beta_2)(s+\beta_1+\mu_2)(s+\mu_1+\mu_2) - \beta_1 \mu_1(s+\mu_1+\beta_2) - \beta_2 \mu_2(s+\beta_1+\mu_2) \]
\[ A_2 = (s+\beta_1+\mu_2)(s+\mu_1+\mu_2) - \beta_2 \mu_1 + \beta_2 \mu_2 \]
\[ A_3 = (s+\beta_2+\mu_1)(s+\mu_1+\mu_2) + \beta_2 \mu_1 - \beta_2 \mu_2 \]
\[ A_4 = 2s + \beta_1 + \mu_1 + \beta_2 + \mu_2 . \]

On inverting (2.5.14) we get

\[
V_0(t) = R_0 \left[ \frac{a_1 e^{-a_1 t}}{a_2 a_5} + \frac{a_2 e^{-a_2 t}}{-a_1 a_5} - \frac{a_3 e^{-a_3 t}}{a_4} \right] + B_1 \left[ \frac{e^{-a_1 t}}{a_2 a_5} + \frac{e^{-a_2 t}}{a_1 a_5} + \frac{e^{-a_3 t}}{a_4} \right] \\
+ B_2 \left[ \frac{1-e^{-a_3 t}}{a_1 a_2 a_3} + \frac{e^{-a_1 t}}{a_1 a_2 a_5} + \frac{e^{-a_2 t}}{-a_1 a_2 a_5} \right] \\
+ B_3 \left[ \frac{a_t^2 - a_3}{a_4^2} + \frac{a_2 e^{-a_1 t} - a_1 e^{-a_2 t}}{a_5} \right] \] (2.5.15)
where

\[ B_1 = (a_3 + \mu_1 + \mu_2)R_0 + \beta_1 R_1 + \beta_2 R_2 \]
\[ B_2 = [a_3(\mu_1 + \mu_2) + \beta_1 \beta_2]R_0 + (a_1 + \mu_2)\beta_1 R_1 + (a_2 + \mu_1)\beta_2 R_2 + \beta_1 \beta_2 R_3 \]
\[ B_3 = R_0 \mu_1 \mu_2 + R_1 \beta_1 \mu_2 + R_2 \mu_1 \beta_2 + R_3 \beta_1 \beta_2 \]

and

\[ a_1 = \beta_1 + \mu_1, \quad a_2 = \beta_2 + \mu_2, \quad a_3 = a_1 + a_2, \quad a_4 = a_1 a_2, \quad a_5 = a_1 - a_2. \]

Similarly expressions for \( V_1(t) \), \( V_2(t) \) and \( V_3(t) \) can be obtained. The total expected earning of the system per unit time in steady-state or the expected steady-state profit denoted by \( g \) may be obtained after allowing \( t \) tend to infinity from (2.5.15).

Therefore

\[ g = \frac{R_0 \mu_1 \mu_2 + R_1 \beta_1 \mu_2 + R_2 \mu_1 \beta_2 + R_3 \beta_1 \beta_2}{(\beta_1 + \mu_1)(\beta_2 + \mu_2)} \] (2.5.16)

In order to examine economic behaviour of the system in a finite interval of time \( t \) (\( t \) is not large enough) let us specify various parameters as below:

\[ \beta_1 = 0.050, \quad \beta_2 = 0.025, \quad \mu_1 = 1.500, \quad \mu_2 = 1.000, \]
\[ r_{00} = 175.0, \quad r_{11} = 50.0, \quad r_{22} = 75.0, \quad r_{33} = -215.0, \]
\[ r_{ij} = 0 \text{ for } i \neq j. \]

Substituting these values in (2.5.15) we obtain \( V_0(t) \) for various values of \( t \) as given in table 2.5.1:
\[
\begin{array}{|c|c|c|c|}
\hline
\text{t} & V_0(t) & \text{t} & V_0(t) \\
\hline
0.0 & 0.0 & 8.0 & 1352.33147 \\
0.5 & 86.61606 & 9.0 & 1520.73082 \\
1.0 & 172.07458 & 10.0 & 1689.12988 \\
2.0 & 341.49547 & 11.0 & 1857.52884 \\
3.0 & 510.19627 & 12.0 & 2025.92776 \\
4.0 & 678.68960 & 13.0 & 2194.32666 \\
5.0 & 847.11942 & 14.0 & 2362.72556 \\
6.0 & 1015.52878 & 15.0 & 2531.12446 \\
7.0 & 1183.93130 & & \\
\hline
\end{array}
\]

Table 2.5.1

From above table we see that for \( t \geq 12 \) the increase in value of \( V_0(t) \) for every unit increase in value of \( t \) is 168.39890 which is nothing but the steady-state expected profit of the system.

Profit Optimization :- Further if one is posed with the problem of selecting failure rate \( \beta \) and repair rate \( \mu \) from a given set of these parameters along with cost parameters, we should proceed as follows.

(a) Specify cost matrix \((r_{ij})\), values of \( \beta \) and \( \mu \) corresponding to each alternative.

(b) Develop cost equations for the system model under consideration and obtain expression for \( g \).

(c) Choose that alternative taking values of \( \beta \) and \( \mu \) as decision variables for which \( g \) is maximum.

In what follows, we present the results in table 2.5.2 for the system under consideration.
<table>
<thead>
<tr>
<th>Pair No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parameter</td>
</tr>
<tr>
<td>$\beta_1$</td>
</tr>
<tr>
<td>$\beta_2$</td>
</tr>
<tr>
<td>$\mu_1$</td>
</tr>
<tr>
<td>$\mu_2$</td>
</tr>
<tr>
<td>$r_{00}$</td>
</tr>
<tr>
<td>$r_{11}$</td>
</tr>
<tr>
<td>$r_{22}$</td>
</tr>
<tr>
<td>$r_{33}$</td>
</tr>
<tr>
<td>$r_{01}$</td>
</tr>
<tr>
<td>$r_{02}$</td>
</tr>
<tr>
<td>$r_{10}$</td>
</tr>
<tr>
<td>$r_{20}$</td>
</tr>
<tr>
<td>$r_{13}$</td>
</tr>
<tr>
<td>$r_{23}$</td>
</tr>
<tr>
<td>$r_{31}$</td>
</tr>
<tr>
<td>$r_{32}$</td>
</tr>
</tbody>
</table>

Table 2.5.2

From the above table we observe that for the sixth pair of units steady-state expected profit is maximum.

**Particular Case**

We now discuss a particular case of the model when both the units are identical. Let the failure and repair rates be $\beta$ and $\mu$ respectively. Define the following three states

<table>
<thead>
<tr>
<th>State</th>
<th>Condition of the system</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>Both the units are operating.</td>
</tr>
<tr>
<td>$S_1$</td>
<td>One unit operates, the other is in failed state and is under repair.</td>
</tr>
<tr>
<td>$S_2$</td>
<td>Both the units are in failed state and are under repair.</td>
</tr>
</tbody>
</table>
The corresponding cost matrix for such a system can be given by

\[ R = (r_{ij}) = \begin{bmatrix}
  r_{00} & r_{01} & r_{02} \\
  r_{10} & r_{11} & r_{12} \\
  r_{20} & r_{21} & r_{22}
\end{bmatrix} \]

Obviously \( r_{02} = r_{20} = 0 \). Since these transitions are not possible, results for this particular case can easily be obtained for the model discussed above after making the following observation.

<table>
<thead>
<tr>
<th>Original model</th>
<th>Particular case</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1, \beta_2 )</td>
<td>( \beta )</td>
</tr>
<tr>
<td>( \mu_1, \mu_2 )</td>
<td>( \mu )</td>
</tr>
<tr>
<td>( r_{01}, r_{02} )</td>
<td>( r_{01} )</td>
</tr>
<tr>
<td>( r_{10}, r_{20} )</td>
<td>( r_{10} )</td>
</tr>
<tr>
<td>( r_{31}, r_{32} )</td>
<td>( r_{21} )</td>
</tr>
<tr>
<td>( r_{13}, r_{23} )</td>
<td>( r_{12} )</td>
</tr>
<tr>
<td>( r_{11}, r_{22} )</td>
<td>( r_{11} )</td>
</tr>
<tr>
<td>( r_{33} )</td>
<td>( r_{22} )</td>
</tr>
</tbody>
</table>

Therefore

\[ g = \frac{1}{(\beta + \mu)^2} \{ \mu^2 (r_{00} + 2\beta r_{01}) + 2\beta \mu (r_{11} + \mu r_{10} + \beta r_{12}) + \beta^2 (r_{22} + 2\mu r_{21}) \} \]  \hspace{1cm} (2.5.17)

To see the analytic behaviour of \( g \) in (2.5.17), let us consider the following cost structure:

\[ r_{00} = A, \quad A > 0 \]
\[ r_{11} = f_1(\mu) = \alpha_1 + c_1 \mu, \quad \alpha_1 > 0, c_1 < 0 \]
\[ r_{22} = f_2(\mu) = \alpha_2 + c_2 \mu, \quad \alpha_2 \geq 0, c_2 < 0 \]

and
\[ r_{01} = r_{10} = r_{12} = r_{21} = 0. \]

So

\[ g = \frac{1}{(\beta + \mu)^2} [\mu^2(A + 2\beta c_1) + \mu(2\beta \alpha_1 + \beta^2 c_2) + \beta^2 \alpha_2] \] (2.5.18)

To determine that value of \( \mu \) which maximizes expected profit, \( g \) in (2.5.18) for a given failure rate and known values of cost parameters, we differentiate \( g \) with respect to \( \mu \) and put equal to zero, i.e., \( \frac{dg}{d\mu} = 0 \), which gives the value of \( \mu \) denoted by \( \mu^* \) which maximizes expected profit, \( g \) denoted by \( g^* \)

\[ \mu^* = \frac{2\beta \alpha_2 - \beta(2\alpha_1 + \beta c_2)}{2(A + 2\beta c_1) - (2\alpha_1 + \beta c_2)} \]

since \( \mu^* \) is finite and cannot be negative, being repair rate, to decide \( \mu^* \) gives maximum we have to further examine the sign of \( \frac{d^2g}{d\mu^2} \bigg| _{\mu = \mu^*} \)

(i) \( \text{If } \mu^* > 0 \text{ and } \frac{d^2g}{d\mu^2} \bigg| _{\mu = \mu^*} < 0, \mu^* \text{ is a maximum point for } g \text{ i.e., the sufficient condition for } \mu^* > 0 \text{ to be absolute maximum point for } g \text{ is} \)
\[ (A + 2\beta c_1) - (2\alpha_1 + \beta c_2) + \alpha_2 < 0. \]

(ii) \( \text{However, if } \mu^* > 0 \text{ and } \frac{d^2g}{d\mu^2} \bigg| _{\mu = \mu^*} > 0, \mu^* \text{ is the minimum point for } g \text{ i.e., when } (A + 2\beta c_1) - (2\alpha_1 + \beta c_2) + \alpha_2 > 0 \)
and absolute maximum is achieved at \( \mu^* = 0 \) for which \( g^* = \alpha_2. \)