CHAPTER 6

Resolvent equations and co-resolvent equations

6.1 Introduction

The concept of resolvent equations was given by Noor [60]. He developed equivalence between variational inclusions and resolvent equations. The resolvent equation technique is quite general and flexible. This technique has been used to develop some numerical methods for solving the mixed variational inequalities and variational inclusions.

In section 6.2, we define $H(\cdot, \cdot)$-mixed relaxed co-$\eta$-monotone mapping and used it to solve a resolvent equation problem in Hilbert spaces. We also developed a Mann-type iterative algorithm to approximate the solution of resolvent equation problem. Finally we discuss the convergence of iterative sequences generated by the iterative algorithm.

In section 6.3, we study a co-variational inequality problem and a co-resolvent equation problem and further we established an equivalence relation between them. An iterative algorithm is also suggested for solving a co-resolvent equation problem.

6.2 $H(\cdot, \cdot)$-mixed relaxed co-$\eta$ monotone mapping with an application

In this section, we define $H(\cdot, \cdot)$-mixed relaxed co-$\eta$ monotone mapping and prove some of its properties.
6.2. $H(\cdot,\cdot)$-mixed relaxed co-$\eta$ monotone mapping with an application

**Definition 6.2.1.** Let $A, B : X \rightarrow X$ and $H, \eta : X \times X \rightarrow X$ be the single valued mappings such that $H$ is $\mu$-relaxed $\eta$-cocoercive with respect to $A$ and $\alpha$-$\xi$-relaxed $\eta$-cocoercive with respect to $B$. The multi-valued mapping $M : X \rightarrow 2^X$ is said to be $H(\cdot,\cdot)$-mixed relaxed co-$\eta$-monotone mapping with respect to $A$ and $B$, if $M$ is $m$-relaxed $\eta$-monotone and $[(H(A, B) + \lambda M)](X) = X$, for every $\lambda > 0$.

Now, we state some properties of $H(\cdot,\cdot)$-mixed relaxed co-$\eta$ monotone mapping

**Theorem 6.2.1.** Let $H(\cdot,\cdot)$ be a $\mu$-relaxed $\eta$-cocoercive with respect to $A$ and $\alpha$-$\xi$-relaxed $\eta$-cocoercive with respect to $B$, $A$ is $\beta$-Lipschitz continuous and $B$ is $\gamma$-Lipschitz continuous. Let $M$ be $H(\cdot,\cdot)$-mixed relaxed co-$\eta$-monotone with respect to $A$ and $B$. Then, the operator $[H(A, B) + \lambda M]^{-1}$ is single-valued for $0 < \lambda < \frac{\xi - (\mu \beta^2 + \alpha \gamma^2)}{m}$.

**Proof.** For any given $u \in X$, let $x, y \in [H(A, B) + \lambda M]^{-1}(u)$. It follows that

$$-H(Ax, Bx) + u \in \lambda Mx,$$

$$-H(Ay, By) + u \in \lambda My. \tag{6.2.1}$$

As $M$ is $m$-relaxed $\eta$-monotone, we have

$$-m\|x - y\|^2 \leq \frac{1}{\lambda} \langle -H(Ax, Bx) + u - (-H(Ay, By) + u), \eta(x, y) \rangle$$

$$= -\frac{1}{\lambda} \langle H(Ax, Bx) - H(Ay, By), \eta(x, y) \rangle$$

$$= -\frac{1}{\lambda} \langle H(Ax, Bx) - H(Ay, Bx) + H(Ay, Bx) - H(Ay, By), \eta(x, y) \rangle$$

$$= -\frac{1}{\lambda} \langle H(Ax, Bx) - H(Ay, Bx), \eta(x, y) \rangle$$

$$- \frac{1}{\lambda} \langle H(Ay, Bx) - H(Ay, By), \eta(x, y) \rangle. \tag{6.2.2}$$

Since $H$ is $\mu$-relaxed $\eta$-cocoercive with respect to $A$ and $\alpha$-$\xi$-relaxed $\eta$-cocoercive with respect to $B$, $A$ is $\beta$-Lipschitz continuous and $B$ is $\gamma$-Lipschitz continuous, thus (6.2.2) becomes

$$-m\lambda\|x - y\|^2 \leq \mu \beta^2 \|x - y\|^2 + \alpha \gamma^2 \|x - y\|^2 - \xi \|x - y\|^2$$

$$= (\mu \beta^2 + \alpha \gamma^2 - \xi)\|x - y\|^2,$$
which implies that
\[ m\lambda \|x - y\|^2 \geq -(\mu\beta^2 + \alpha\gamma^2 - \xi)\|x - y\|^2. \] (6.2.3)

If \( x \neq y \), then \( \lambda \geq \frac{\xi - \mu\beta^2 - \alpha\gamma^2}{m} \), which contradicts that \( 0 < \lambda < \frac{\xi - (\mu\beta^2 + \alpha\gamma^2)}{m} \).

Thus, we have \( x = y \), i.e., \([H(A, B) + \lambda M]^{-1}\) is single-valued.

**Definition 6.2.2.** Let \( H(A, B) \) be a \( \mu \)-relaxed \( \eta \)-cocoercive with respect to \( A \) and \( \alpha \)-\( \xi \)-relaxed \( \eta \)-cocoercive with respect to \( B \), \( A \) is \( \beta \)-Lipschitz continuous and \( B \) is \( \gamma \)-Lipschitz continuous. Let \( M: X \to 2^X \) be an \( H(\cdot, \cdot) \)-mixed relaxed co-\( \eta \)-monotone mapping with respect to \( A \) and \( B \). The resolvent operator \( R^{H(\cdot, \cdot)-\eta}_{\lambda, M}: X \to X \) associated with \( H \) and \( M \) is defined by
\[ R^{H(\cdot, \cdot)-\eta}_{\lambda, M}(u) = [H(A, B) + \lambda M]^{-1}(u), \forall u \in X. \] (6.2.4)

**Theorem 6.2.2.** Let \( H(A, B) \) be a \( \mu \)-relaxed \( \eta \)-cocoercive with respect to \( A \) and \( \alpha \)-\( \xi \)-relaxed \( \eta \)-cocoercive with respect to \( B \), \( \eta: X \times X \to X \) be \( \sigma \)-Lipschitz continuous, \( A \) is \( \beta \)-Lipschitz continuous and \( B \) is \( \gamma \)-Lipschitz continuous. Let \( M: X \to 2^X \) be an \( H(\cdot, \cdot) \)-mixed relaxed co-\( \eta \)-monotone mapping with respect to \( A \) and \( B \). Then the resolvent operator defined by (6.2.4) is \( \sigma \)-Lipschitz continuous for \( 0 < \lambda < \frac{\xi - (\mu\beta^2 + \alpha\gamma^2)}{m} \), i.e.,
\[ \|R^{H(\cdot, \cdot)-\eta}_{\lambda, M}(u) - R^{H(\cdot, \cdot)-\eta}_{\lambda, M}(v)\| \leq \frac{\sigma}{[\gamma - m\lambda - (\mu\beta^2 + \alpha\gamma^2)]}\|u - v\|, \forall u, v \in X. \] (6.2.5)

**Proof.** Let \( u \) and \( v \) be any given points in \( X \). It follows from (6.2.4) that
\[ R^{H(\cdot, \cdot)-\eta}_{\lambda, M}(u) = [H(A, B) + \lambda M]^{-1}(u), \]
\[ R^{H(\cdot, \cdot)-\eta}_{\lambda, M}(v) = [H(A, B) + \lambda M]^{-1}(v). \] (6.2.6)

For the sake of clarity, we denote
\[ t_1 = R^{H(\cdot, \cdot)-\eta}_{\lambda, M}(u), \quad t_2 = R^{H(\cdot, \cdot)-\eta}_{\lambda, M}(v). \]
6.2. $H(\cdot, \cdot)$-mixed relaxed co-$\eta$ monotone mapping with an application

Then (6.2.2) implies that

\[
\frac{1}{\lambda} \left( u - H(A(t_1), B(t_1)) \right) \in M(t_1),
\]

\[
\frac{1}{\lambda} \left( v - H(A(t_2), B(t_2)) \right) \in M(t_2).
\]

(6.2.7)

Since $M$ is $m$-relaxed $\eta$-monotone, we have

\[
-m\|t_1 - t_2\|^2 \leq \frac{1}{\lambda} \left\langle \left( u - H(A(t_1), B(t_1)) \right) - \left( v - H(A(t_2), B(t_2)) \right), \eta(t_1, t_2) \right\rangle
\]

\[
= \frac{1}{\lambda} \left\langle u - v - H(A(t_1), B(t_1)) + H(A(t_2), B(t_2)), \eta(t_1, t_2) \right\rangle,
\]

which implies that

\[
-m\|t_1 - t_2\|^2 \leq \langle u - v, \eta(t_1, t_2) \rangle + \left\langle -H(A(t_1), B(t_1)) + H(A(t_2), B(t_2)), \eta(t_1, t_2) \right\rangle.
\]

(6.2.8)

By Cauchy-Schwartz inequality and (6.2.8), we have

\[
\|u - v\| \|\eta(t_1, t_2)\| \geq \langle u - v, \eta(t_1, t_2) \rangle
\]

\[
\geq -\left\langle -H(A(t_1), B(t_1)) + H(A(t_2), B(t_2)), \eta(t_1, t_2) \right\rangle
\]

\[
- m\|t_1 - t_2\|^2
\]

\[
= \left\langle H(A(t_1), B(t_1)) - H(A(t_2), B(t_2)), \eta(t_1, t_2) \right\rangle
\]

\[
- m\|t_1 - t_2\|^2
\]

\[
= \left\langle H(A(t_1), B(t_1)) - H(A(t_2), B(t_2)), \eta(t_1, t_2) \right\rangle - m\|t_1 - t_2\|^2
\]

\[
+ \left\langle H(A(t_2), B(t_1)) - H(A(t_2), B(t_2)), t_1 - t_2 \right\rangle.
\]

(6.2.9)

As $H$ is $\mu$-relaxed $\eta$-cocoercive with respect to $A$, $\alpha$-$\xi$-relaxed $\eta$-cocoercive with respect to $B$, $\eta : X \times X \to X$ is $\sigma$-Lipschitz continuous, $A$ is $\beta$-Lipschitz continuous and $B$ is $\gamma$-Lipschitz continuous, we have

\[
\sigma\|u - v\|\|t_1 - t_2\| \geq [-\mu\beta^2 - \alpha\gamma^2 + \gamma - m\lambda]\|t_1 - t_2\|^2.
\]
Thus, we have
\[ \|t_1 - t_2\| \leq \frac{\sigma}{[\gamma - m\lambda - (\mu\beta^2 + \alpha\gamma^2)]}\|u - v\|, \]
i.e.,
\[ \|R_{\lambda,M}^{H(\cdot,-\eta)}(u) - R_{\lambda,M}^{H(\cdot,-\eta)}(v)\| \leq \theta_1\|u - v\|, \forall u, v \in X, \]
where \( \theta_1 = \frac{\sigma}{[\gamma - m\lambda - (\mu\beta^2 + \alpha\gamma^2)]} \). This completes the proof.

Now, we discuss a variational inclusion problem and its corresponding resolvent equation problem. We further define an iterative algorithm to approximate the solution of the resolvent equation problem.

In connection with the variational inclusion problem (2.2.10) discussed in chapter 2, we consider its corresponding resolvent equation problem:
Find \( z, u \in X, x \in T(u), y \in F(u) \) such that
\[ S(x,y) + \lambda^{-1}J_{M,\lambda}^{H^{\cdot,-\eta}}(z) = 0, \]
where \( \lambda > 0 \) is a constant.\(^{111}\)

**Lemma 6.2.1.** The triplet \( (u, x, y) \), where \( u \in X, x \in T(u), y \in F(u) \), is a solution of variational inclusion problem (2.2.10) if and only if it satisfies the equation:
\[ g(u) = R_{\lambda,M}^{H^{\cdot,-\eta}}[H(A(g(u)), B(g(u))) - \lambda S(x,y)], \]
where \( \lambda > 0 \) is a constant.

**Proof:** By using the definition of resolvent operator \( R_{\lambda,M}^{H^{\cdot,-\eta}} \), the conclusion follows directly.

Based on Lemma (6.2.1), we prove the following Lemma which ensures the equivalence of variational inclusion problem and resolvent equation problem (6.2.10).
Lemma 6.2.2. The variational inclusion problem (2.2.10) has a solution \((u, x, y)\), where \(u \in X, x \in T(u), y \in F(u)\), if and only if the resolvent equation problem (6.2.10) has a solution \((z, u, x, y)\), where \(z, u \in X, x \in T(u), y \in F(u)\), where

\[
g(u) = R_{X,M}^{H(\cdot, \cdot) - \eta}(z) \tag{6.2.12}
\]

and \(z = H(A(g(u)), B(g(u))) - \lambda S(x, y)\).

Proof. Let \((u, x, y)\) where \(u \in X, x \in T(u), y \in F(u)\) is a solution of variational inclusion problem. Then by Lemma (6.2.1), we have

\[
g(u) = R_{X,M}^{H(\cdot, \cdot) - \eta}[H(A(g(u)), B(g(u))) - \lambda S(x, y)].
\]

Using the fact \(J_{M,\lambda}^{H(\cdot, \cdot) - \eta} = I - H(A(R_{X,M}^{H(\cdot, \cdot) - \eta}), B(R_{X,M}^{H(\cdot, \cdot) - \eta}))\) and equation (6.2.12), we obtain

\[
J_{M,\lambda}^{H(\cdot, \cdot) - \eta}[H(A(g(u)), B(g(u))) - \lambda S(x, y)]
\]

\[
= H(A(g(u)), B(g(u))) - \lambda S(x, y)
\]

\[
- H[A(R_{X,M}^{H(\cdot, \cdot) - \eta}(H(A(g(u)), B(g(u))) - \lambda S(x, y))),
\]

\[
B(R_{X,M}^{H(\cdot, \cdot) - \eta}(H(A(g(u)), B(g(u))) - \lambda S(x, y)))]
\]

\[
= - \lambda S(x, y),
\]

which implies that \(S(x, y) + \lambda^{-1} J_{M,\lambda}^{H(\cdot, \cdot) - \eta}(z) = 0\) with \(z = H(A(g(u)), B(g(u))) - \lambda S(x, y)\).

That is \((z, u, x, y)\) is a solution of resolvent equation problem (6.2.10).

Conversely, let \((z, u, x, y)\) is a solution of resolvent equation problem (6.2.10), then

\[
S(x, y) + \lambda^{-1} J_{M,\lambda}^{H(\cdot, \cdot) - \eta}(z) = 0,
\]

i.e., \(J_{M,\lambda}^{H(\cdot, \cdot) - \eta}(z) = - \lambda S(x, y)\).

Using the definition of \(J_{M,\lambda}^{H(\cdot, \cdot) - \eta}\), we have

\[
[I - H(A(R_{X,M}^{H(\cdot, \cdot) - \eta}), B(R_{X,M}^{H(\cdot, \cdot) - \eta}))](z) = - \lambda S(x, y),
\]

112
Algorithm 6.2.1. For any initial points \((z_0, u_0, x_0, y_0)\), where \(z_0, u_0 \in X, x_0 \in T(u_0), y_0 \in F(u_0)\), compute the sequences \(\{z_n\}, \{u_n\}\{x_n\}\) and \(\{y_n\}\) by the iterative scheme:

\[
\begin{align*}
(i) \quad & g(u_n) = R_{\lambda, M}^{H(\cdot, \cdot)} - \eta, \\
(ii) \quad & \|x_n - x_{n+1}\| \leq D(T(u_n), T(u_{n+1})), x_n \in T(u_n), x_{n+1} \in T(u_{n+1}), \\
(iii) \quad & \|y_n - y_{n+1}\| \leq D(F(u_n), F(u_{n+1})), y_n \in F(u_n), y_{n+1} \in F(u_{n+1}), \\
(iv) \quad & z_{n+1} = H(A(g(u_n)), B(g(u_n))) - \lambda S(x_n, y_n), \text{ where } n = 0, 1, 2, ..., \text{ and } \lambda > 0 \text{ is a constant, } D(\cdot, \cdot) \text{ is the Hausdorff metric on } CB(X).
\]

Theorem 6.2.3. Let \(X\) be a real Hilbert space and \(A, B, g : X \rightarrow X, S, H : X \times X \rightarrow X, \eta : X \times X \rightarrow X\) be the single valued mappings. Let \(T, F : X \rightarrow CB(X)\) be the multi-valued mappings and \(M : X \rightarrow 2^X\) be \(H(\cdot, \cdot)\)-mixed relaxed co-\(\eta\)-monotone mapping. Assume that

\[
\begin{align*}
(i) \quad & H(A, B) \text{ is } \mu\text{-relaxed } \eta\text{-cocoercive with respect to } A \text{ and } \alpha\text{-}\xi\text{-relaxed } \eta\text{-cocoercive with respect to } B, r_1\text{-Lipschitz continuous with respect to } A \text{ and } r_2\text{-Lipschitz continuous with respect to } B, \\
(ii) \quad & T \text{ and } F \text{ are } D\text{-Lipschitz continuous mappings with constant } \delta_T \text{ and } \delta_F, \text{ respectively,} \\
(iii) \quad & A \text{ is } \beta\text{-Lipschitz continuous and } B \text{ is } \gamma\text{-Lipschitz continuous},
\end{align*}
\]
(iv) \( g \) is \( \lambda_g \)-Lipschitz continuous and \( t \)-strongly monotone,

(v) \( S \) is \( \lambda_{s_1} \)-Lipschitz continuous in first argument and \( \lambda_{s_2} \)-Lipschitz continuous in second argument.

If for some \( \lambda > 0 \), the following condition is satisfied:

\[
\sqrt{1 - 2t + \lambda^2 g} < [1 - \theta_1],
\]

where \( \theta = \frac{\mu\eta - \alpha \delta}{\mu \alpha + \alpha \gamma \delta} \), \( \theta_1 = [(r_1 + r_2)\lambda + \lambda (\lambda_{s_1} \delta T + \lambda_{s_2} \delta F)] \).

Then, there exist \( z, u \in X, x \in T(u), y \in F(u) \) satisfying resolvent equation problem (6.2.10) and the iterative sequences \( \{u_n\}, \{z_n\}, \{x_n\}, \{y_n\} \) generated by the Algorithm (6.2.1) strongly converge to \( u, z, x \) and \( y \), respectively.

Proof. From (iv) of Algorithm (6.2.1), we have

\[
\|z_{n+1} - z_n\| = \|[H(Ag(u_n), Bg(u_n)) - \lambda S(x_n, y_n)] - [H(Ag(u_{n-1})),
B(g(u_{n-1})) - \lambda S(x_{n-1}, y_{n-1})]\|
\leq \||H(Ag(u_n), Bg(u_n)) - H(Ag(u_{n-1}), B(g(u_{n-1})))||
- \lambda ||S(x_n, y_n) - S(x_{n-1}, y_{n-1})||
\leq \||H(Ag(u_n)), B(g(u_n))) - H(Ag(u_{n-1}), B(g(u_{n-1})))||
+ \|H(Ag(u_{n-1}), B(g(u_{n-1}))) - H(Ag(u_{n-1}), B(g(u_{n-1}))))||
+ \lambda ||S(x_n, y_n) - S(x_{n-1}, y_n)||
+ \lambda ||S(x_{n-1}, y_n) - S(x_{n-1}, y_{n-1})||.
\]

(6.2.14)

As \( H(A, B) \) is \( r_1 \)-Lipschitz continuous with respect to \( A \) and \( r_2 \)-Lipschitz with respect to \( B \) and \( g \) is \( \lambda_g \)-Lipschitz continuous, we have

\[
\|H(Ag(u_n)), B(g(u_n))) - H(Ag(u_{n-1}), B(g(u_{n-1})))||
+ \|H(Ag(u_{n-1}), B(g(u_{n-1})))||
- \|H(Ag(u_{n-1}), B(g(u_{n-1}))))||
\leq r_1 \lambda_g ||u_n - u_{n-1}||.
\]

(6.2.15)
Since $S$ is $\lambda_s$-Lipschitz continuous in the first argument and $\lambda_s^2$-Lipschitz continuous in the second argument, $T$ is $D$-Lipschitz continuous with constant $\delta_T$ and $F$ is $D$-Lipschitz continuous with constant $\delta_T$ and $F$ is $D$-Lipschitz continuous constant $\delta_F$, we have

\[
\|S(x_n, y_n) - S(x_{n-1}, y_n)\| + \|S(x_{n-1}, y_n) - S(x_{n-1}, y_{n-1})\| \\
\leq \lambda_s \|x_n - x_{n-1}\| + \lambda_s^2 \|y_n - y_{n-1}\| \\
\leq \lambda_s \delta_T \|u_n - u_{n-1}\| + \lambda_s^2 \delta_F \|u_n - u_{n-1}\|. \tag{6.2.16}
\]

Using (6.2.15) and (6.2.16), (6.2.14) becomes

\[
\|z_{n+1} - z_n\| \leq \|r_1 \lambda_g + r_2 \lambda_g + \lambda \lambda_s \delta_F\|u_n - u_{n-1}\| \\
= \theta_1 \|u_n - u_{n-1}\|, \tag{6.2.17}
\]

where $\theta_1 = [r_1 + r_2] \lambda_g + \lambda (\lambda_s \delta_T + \lambda_s^2 \delta_F)$. By (i) of Algorithm (6.2.1), we have

\[
\|u_n - u_{n-1}\| = \|u_n - u_{n-1} - g(u_n) + g(u_{n-1}) + R_{\lambda, M}^{H(\cdot), \eta}(z_n) - R_{\lambda, M}^{H(\cdot), \eta}(z_{n-1})\| \\
\leq \|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| + \|R_{\lambda, M}^{H(\cdot), \eta}(z_n) - R_{\lambda, M}^{H(\cdot), \eta}(z_{n-1})\|. \tag{6.2.18}
\]

Since $g$ is $t$-strongly monotone and $\lambda_g$-Lipschitz continuous, by using technique of Noor [60], it follows that

\[
\|u_n - u_{n-1} - (g(u_n) - g(u_{n-1}))\| \leq \sqrt{1 - 2t + \lambda_g^2} \|u_n - u_{n-1}\|. \tag{6.2.19}
\]

Using equation (6.2.19) and Lipschitz continuity of the resolvent operator $R_{\lambda, M}^{H(\cdot), \eta}$, equation (6.2.18) becomes

\[
\|u_n - u_{n-1}\| \leq \sqrt{1 - 2t + \lambda_g^2} \|u_n - u_{n-1}\| + \theta \|z_n - z_{n-1}\|, \tag{6.2.19}
\]
6.2. \(H(\cdot, \cdot)\)-mixed relaxed co-\(\eta\) monotone mapping with an application

\[
\|u_n - u_{n-1}\| \leq \frac{\theta}{|1 - \sqrt{1 - 2t + \lambda_g^2}|} \|z_n - z_{n-1}\|. \tag{6.2.20}
\]

Combining (6.2.20) with (6.2.16), we obtain

\[
\|z_{n+1} - z_n\| \leq \frac{\theta \theta_1}{|1 - \sqrt{1 - 2t + \lambda_g^2}|} \|z_n - z_{n-1}\|,
\]

\[
\|z_{n+1} - z_n\| \leq P(\theta) \|z_n - z_{n-1}\|, \tag{6.2.21}
\]

where \(P(\theta) = \frac{\theta \theta_1}{|1 - \sqrt{1 - 2t + \lambda_g^2}|}\), \(\theta = [\frac{\sigma}{\lambda - \mu \lambda + \lambda \varepsilon_T + \lambda \varepsilon_F}]\) and \(\theta_1 = [(r_1 + r_2) \lambda_g + \lambda (\lambda_s \delta_T + \lambda_s \delta_F)]\).

From (6.2.12), it follows that \(P(\theta) < 1\). Consequently from (6.2.21) it follows that \(\{z_n\}\) is a cauchy sequence in \(X\) and as \(X\) is complete, \(z_n \to z\) as \(n \to \infty\). From (6.2.20), it follows that \(\{u_n\}\) is also a cauchy sequence in \(X\) such that \(u_n \to u\) as \(n \to \infty\). From \(D\)-Lipschitz continuous of \(T, F\) and (ii) and (iii) of Algorithm (6.2.1), we know that \(\{x_n\}\) and \(\{y_n\}\) are Cauchy sequences in \(X\) such that \(x_n \to x\) and \(y_n \to y\) as \(n \to \infty\). Further, we show that \(x \in T(u)\), we have

\[
d(x, T(u)) \leq \|x - x_n\| + d(x_n, T(u)) \\
\leq \|x - x_n\| + \mathcal{D}(T(u_n), T(u)) \\
\leq \|x - x_n\| + \delta_T \|u_n - u_{n-1}\| \to 0 \text{ as } n \to \infty.
\]

Which implies that \(d(x, T(u)) = 0\), it follows that \(x \in T(u)\). Similarly we can show that \(y \in F(u)\).

Since \(H, A, g, T, F\) and \(S\) all are continuous and by (iv) of Algorithm (6.2.1), it follows that

\[
z_{n+1} = H(A(g(u_n)), B(g(u_n))) - \lambda S(x_n, y_n) \\
\to z = H(A(g(u)), B(g(u))) - \lambda S(x, y). \tag{6.2.22}
\]

Consequently,

\[
R_{\lambda, M}^{H(\cdot, \cdot) - \eta}(z_n) = g(u_n) \to g(u) = R_{\lambda, M}^{H(\cdot, \cdot) - \eta}(z), \text{ as } n \to \infty. \tag{6.2.23}
\]
From (6.2.22), (6.2.23) and by Lemma (6.2.2), the result follows.

6.3 Co-resolvent equation problem

In this section, we study co-variational inequality problem and a co-resolvent equation problem in Banach space. Now we give some important definitions.

Definition 6.3.1. [21] Let $Y$ be a Banach space with its dual space $Y^*$, $\phi : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper subdifferentiable (may not be convex) functional, and $J : Y \rightarrow Y^*$ be a mapping. If for any given point $x^* \in Y^*$ and $\rho > 0$, there is a unique point $x \in Y$ satisfying

$$\langle J(x) - x^*, y - x \rangle + \rho \phi(y) - \rho \phi(x) \geq 0, \quad \forall y \in Y.$$ 

The mapping $x^* \rightarrow x$, denoted by $J_{\rho}^\phi(x^*)$, is said to be $J$-proximal mapping of $\phi$. We have $x^* - J(x) \in \rho \partial \phi(x)$, it follows that

$$J_{\rho}^\phi(x^*) = (J + \rho \partial \phi)^{-1}(x^*).$$

Theorem 6.3.1. [21] Let $Y$ be a reflexive Banach space with its dual space $Y^*$, and $\phi : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, subdifferentiable, proper functional which may not be convex. Let $J : Y \rightarrow Y^*$ be $\alpha$-strongly monotone continuous mapping. Then for any $\rho > 0$, $x^* \in Y^*$, there exists a unique $x \in Y$ such that

$$\langle J(x) - x^*, y - x \rangle + \rho \phi(y) - \rho \phi(x) \geq 0, \quad \forall y \in Y.$$ 

That is $x = J_{\rho}^\phi(x^*)$ and so the $J$-proximal mapping of $\phi$ is well-defined and $\frac{1}{\alpha}$-Lipschitz continuous.

Now, we state our problem.

Let $T, A : Y \rightarrow CB(Y)$ be the multi-valued mappings, $J : Y \rightarrow Y^*$, $N : Y \times Y \rightarrow Y$ and $f, h, g : Y \rightarrow Y$ be single-valued mappings. Let $\phi : Y \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be such that for each fixed $x \in Y$, $\phi(\cdot, x)$ is lower semicontinuous, subdifferentiable functional (which may not be convex) on $E$ satisfying $g(Y) \cap \text{dom}(\partial \phi(\cdot, x)) \neq \emptyset$, where $\partial \phi(\cdot, x)$ is
6.3. Co-resolvent equation problem

the subdifferential of \(\phi(\cdot, x)\) at \(x\).

We consider the following co-variational inclusion problem:

Find \(x \in Y, u \in T(x)\) and \(v \in A(x)\) such that \(g(x) \in \text{dom}(\partial\phi(\cdot, x))\) and

\[
\langle J(N(f(u), h(v)), y - g(x)) \rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in Y.
\] (6.3.1)

Below we mention some special cases of problem (6.3.1).

(I) If \(Y\) is a real Banach space and \(J, f, h\) are identity mappings, then problem (6.3.1) reduces to the following problem:

Find \(x \in Y, u \in T(x)\) and \(v \in A(x)\) such that

\[
\langle N(u, v), y - g(x) \rangle \geq \phi(g(x), x) - \phi(y, x), \quad \forall y \in Y.
\] (6.3.2)

Problem (6.3.2) includes problems studied by Hassouni and Moudafi [37], Kazmi [41], and Ding [24, 25] as special cases.

(II) If \(X\) is a real Hilbert space, \(\phi(x, y) = \phi(x)\), for all \(x, y \in X\), \(J\) is the identity mapping, and \(N(f(u), h(v)) = f(u) - h(v)\), for all \(u, v \in X\), then problem (6.3.1) becomes the problem of finding \(x \in X, u \in T(x)\) and \(v \in A(x)\) such that

\[
\langle f(u) - h(v), y - g(x) \rangle \geq \phi(g(x)) - \phi(y), \quad \forall y \in X.
\] (6.3.3)

Problem (6.3.3) was introduced and studied by Huang [39].

It is clear that the suitable choices of mappings involved in the formulation of problem (6.3.1), one can obtain many previously known problems, see e.g., [16, 22, 23, 83].

In connection with co-variational inequality problem (6.3.1), we consider the following co-resolvent equation problem:

Find \(z \in Y^*, x \in Y, u \in T(x)\) and \(v \in A(x)\) such that

\[
J(N(f(u), h(v))) + \rho^{-1} R^{\partial\phi(\cdot, x)}(z) = 0.
\] (6.3.4)
Chapter 6. Resolvent equations and co-resolvent equations

where \( \rho > 0 \) is a constant, \( R_{\rho}^{\partial \phi} = [I - J(J_{\rho}^{\partial \phi})] \), where \( J(J_{\rho}^{\partial \phi}(z)) = [J(J_{\rho}^{\partial \phi})](z) \) and \( I \) is the identity mapping.

In support of problem (6.3.1), we provide the following example.

**Example 6.3.1.** Let us suppose that \( Y = \mathbb{R} \). Define \( T, A : Y \rightarrow CB(Y) \) by \( T(x) = A(x) = [-x, x] \), \( g \) and \( \phi \) by \( g(x) = x - 1 \) and \( \phi(y, x) = y - x \), for all \( x, y \in Y \).

We define for \( x \in Y, u \in T(x) \) and \( v \in A(x) \)

\[(i) \quad J(x) = 2x,\]
\[(ii) \quad N(x, y) = 2x + 3y,\]
\[(iii) \quad f(u) = \frac{u}{2},\]
\[(iv) \quad h(v) = \frac{v}{3},\]
\[(v) \quad \partial \phi(\cdot, x) = \{\alpha \in \mathbb{R} : \phi(\cdot, y) - \phi(\cdot, x) \geq \alpha(y - x)\}.\]

Then, it is easy to check that \( g(x) \in \text{dom}(\partial \phi(\cdot, x)) \). As \( \partial \phi(\cdot, x) \neq \emptyset \), therefore \( \phi(\cdot, x) \) is lower semicontinuous at \( x \). Thus, co-variational inequality problem (6.3.1) is satisfied.

Now we establish an equivalence between co-variational inequality problem (6.3.1) and a fixed point problem, which can be proved easily by using the definition of resolvent operator and subdifferentiability of \( \phi(\cdot, x) \).

**Lemma 6.3.1.** Let \((x, u, v)\), where \( x \in Y, u \in T(x) \) and \( v \in A(x) \), be a solution of co-variational inequality problem (6.3.1) if and only if it is the solution of the following equation

\[
g(x) = J_{\rho}^{\partial \phi(x)} \{J(g(x)) - \rho J(N(f(u), h(v)))\}. \tag{6.3.5}\]

**Lemma 6.3.2.** If \( J \) is one-one, then co-variational inequality problem (6.3.1) has a solution \((x, u, v)\), where \( x \in Y, u \in T(x) \) and \( v \in A(x) \), if and only if co-resolvent equation problem (6.3.4) has a solution \((z, x, u, v)\), \( z \in Y^*, x \in Y, u \in T(x) \) and \( v \in A(x) \), where

\[
g(x) = J_{\rho}^{\partial \phi(\cdot, x)}(z), \tag{6.3.6}\]
6.3. Co-resolvent equation problem

and

\[ z = J(g(x)) - \rho J(N(f(u), h(v))). \]

**Proof.** Let \((x, u, v)\) be a solution of co-variational inequality problem (6.3.4). Then by Lemma (6.3.1), it is a solution of the following equation:

\[ g(x) = J_{\rho}^{\phi(x)} \{ J(g(x)) - \rho J(N(f(u), h(v))) \}. \]

Using the fact that \(R_{\rho}^{\phi(x)} = \left[ I - J \left( J_{\rho}^{\phi(x)} \right) \right]\) and (6.3.6), we have

\[
\begin{align*}
R_{\rho}^{\phi(x)} \{ J(g(x)) - \rho J(N(f(u), h(v))) \} &= J(g(x)) - \rho J(N(f(u), h(v))) - J \left[ J_{\rho}^{\phi(x)} \{ J(g(x)) - \rho J(N(f(u), h(v))) \} \right] \\
&= J(g(x)) - \rho J(N(f(u), h(v))) - J(g(x)) \\
&= -\rho J(N(f(u), h(v))),
\end{align*}
\]

which implies that

\[ J(N(f(u), h(v))) + \rho^{-1} R_{\rho}^{\phi(x)}(z) = 0, \]

with \(z = J(g(x)) - \rho J(N(f(u), h(v)))\), i.e., \((z, x, u, v)\) is the solution of co-resolvent equation problem (6.3.4).

Conversely, let \((z, x, u, v)\) be the solution of co-resolvent equation problem (6.3.4), then

\[ \rho J(N(f(u), h(v))) = -R_{\rho}^{\phi(x)}(z) = J \left( J_{\rho}^{\phi(x)}(z) \right) - z. \]  \hspace{1cm} (6.3.7)

From (6.3.6) and (6.3.7), we have

\[
\begin{align*}
\rho J(N(f(u), h(v))) &= J \left( J_{\rho}^{\phi(x)} \{ J(g(x)) - \rho J(N(f(u), h(v))) \} \right) - J(g(x)) - \rho J(N(f(u), h(v))), \\
\end{align*}
\]

which implies that

\[ J(g(x)) = J \left( J_{\rho}^{\phi(x)} \{ J(g(x)) - \rho J(N(f(u), h(v))) \} \right). \]
Since $J$ is one-one, we have

$$g(x) = J^\partial \phi(x) \left\{ J(g(x)) - \rho J(N(f(u), h(v))) \right\},$$

i.e., $(x, u, v)$ is the solution of co-variational inequality problem (6.3.1).

**Alternative Proof.** Let

$$z = J(g(x)) - \rho J(N(f(u), h(v))).$$

Then from (6.3.8), we have

$$g(x) = J^\partial \phi(x)(z),$$

and

$$z = J \left( J^\partial \phi(x)(z) \right) - \rho J(N(f(u), h(v))).$$

By using the fact that $J \left( J^\partial \phi(x)(z) \right) = \left[ J \left( J^\partial \phi(x) \right) \right](z)$, it follows that

$$J(N(f(u), h(v))) + \rho^{-1} R^\partial \phi(x)(z) = 0,$$

which is the required co-resolvent equation problem (6.3.4).

We suggest the following iterative algorithm for solving co-resolvent equation problem (6.3.4).

**Iterative Algorithm 6.3.1.** For any $z_0 \in Y^*$, $x_0 \in Y$, $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, let

$$z_1 = J(g(x_0)) - \rho J(N(f(u_0), h(v_0)) \in Y^*,$$

and take $x_1 \in Y$ such that

$$g(x_1) = J^\partial \phi(x_1)(z_1).$$

Since $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, by Nadler’s Theorem [56], there exists $u_1 \in T(x_1)$ and $v_1 \in A(x_1)$ such that

$$\| u_0 - u_1 \| \leq (1 + 1) D(T(x_0), T(x_1)),$$

$$\| v_0 - v_1 \| \leq (1 + 1) D(A(x_0), A(x_1)).$$
6.3. Co-resolvent equation problem

where $\mathcal{D}(\cdot,\cdot)$ is the Hausdorff metric on $CB(Y)$. Let

$$z_2 = J(g(x_1)) - \rho J(N(f(u_1),h(v_1))),$$

and take any $x_2 \in Y$ such that

$$g(x_2) = J^0_{\phi}(x_2)(z_2).$$

Continuing the above process inductively, we can obtain the following scheme:

For any $z_0 \in Y^*$, $x_0 \in Y$, $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$, compute the sequences $\{z_n\}$, $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ by the iterative schemes such that

(i) $g(x_n) = J^0_{\phi}(x_n)(z_n), \quad (6.3.4)$

(ii) $u_n \in T(x_n),\|u_n - u_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(T(x_n),T(x_{n+1})), \quad (6.3.5)$

(iii) $v_n \in A(x_n),\|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) \mathcal{D}(A(x_n),A(x_{n+1})), \quad (6.3.6)$

(iv) $z_{n+1} = J(g(x_n)) - \rho J(N(f(u_n),h(v_n))), \quad (6.3.7)$

for $n = 0,1,2,\cdots$ and $\rho > 0$ is a constant.

**Theorem 6.3.2.** Let $Y$ be a uniformly smooth Banach space with the module of smoothness $\rho_Y(t) \leq Ct^2$, for some $t > 0$. Let $T,A : Y \rightarrow CB(Y)$ be the $\mathcal{D}$-Lipschitz continuous mappings with constants $\delta_T$ and $\delta_A$, respectively. Let $f,h : Y \rightarrow Y$ be the Lipschitz continuous mappings with constants $\lambda_f$ and $\lambda_h$, respectively, and $g : Y \rightarrow Y$ be a Lipschitz continuous mapping with constant $\lambda_g$ and strongly accretive with constant $\gamma > 0$. Let $J : Y \rightarrow Y^*$ be a Lipschitz continuous mapping with constant $\lambda_j$ and strongly monotone with constant $\alpha > 0$, and $N : Y \times Y \rightarrow Y$ be Lipschitz continuous in both the arguments with constants $\lambda_{N_1}$ and $\lambda_{N_2}$, respectively. Let $\phi : Y \times Y \rightarrow \mathbb{R} \cup \{+\infty\}$ be a mapping such that for each fixed $x \in Y$, $\phi(\cdot,x)$ is lower semicontinuous, subdifferentiable, proper functional satisfying $g(x) \in \text{dom} \partial \phi(\cdot,x)$, for all $x \in Y$. Suppose that
there exists a constant \( \rho > 0 \) such that for each \( x, y \in Y, x^* \in Y^* \)

\[
\| J^{\delta(\cdot, x)}_\rho(x^*) - J^{\delta(\cdot, y)}_\rho(x^*) \| \leq \mu \| x - y \|, 
\]  

(6.3.8)

and the following condition is satisfied:

\[
\left| \rho - \frac{\alpha \mu - \lambda_\rho \lambda_j}{\lambda_j (\lambda_{N1} \lambda_f \delta_T + \lambda_{N2} \lambda_h \delta_A)} \right| < \frac{\alpha \sqrt{1 - 2\gamma + 64C\lambda_j^2}}{\lambda_j (\lambda_{N1} \lambda_f \delta_T + \lambda_{N2} \lambda_h \delta_A)}, 
\]  

(6.3.9)

\( \alpha \mu > \lambda_\rho \lambda_j \), then there exists \( z \in Y^*, x \in Y, u \in T(x) \) and \( v \in A(x) \) satisfying co-resolvent equation problem (6.3.4), and the iterative sequences \( \{z_n\}, \{x_n\}, \{u_n\} \) and \( \{v_n\} \) generated by Algorithm 6.3.1 converge strongly to \( z, x, u \) and \( v \), respectively.

Proof. From Algorithm (6.3.1), we have

\[
\| z_{n+1} - z_n \| = \| J(g(x_n)) - J(N(f(u_n), h(v_n))) - J(g(x_{n-1})) - J(N(f(u_{n-1}), h(v_{n-1}))) \| 
\]

\[
\leq \| J(g(x_n)) - J(g(x_{n-1})) \| + \rho \| J(N(f(u_n), h(v_n))) - J(N(f(u_{n-1}), h(v_{n-1}))) \|. 
\]

(6.3.10)

By the Lipschitz continuity of \( J \) and \( g \), we have

\[
\| J(g(x_n)) - J(g(x_{n-1})) \| \leq \lambda_j \| g(x_n) - g(x_{n-1}) \| \leq \lambda_j \lambda_\rho \| x_n - x_{n-1} \|. 
\]

(6.3.11)

By the Lipschitz continuity of \( J, f, h \), \( D \)-Lipschitz continuity of \( T, A \) and Lipschitz continuity of \( N \) in both the arguments, we have

\[
\| J(N(f(u_n), h(v_n))) - J(N(f(u_{n-1}), h(v_{n-1}))) \| 
\]

\[
\leq \lambda_j \lambda_{N1} \| f(u_n) - f(u_{n-1}) \| + \lambda_{N2} \| h(v_n) - h(v_{n-1}) \| 
\]

\[
\leq \lambda_j \lambda_{N1} \lambda_f \| u_n - u_{n-1} \| + \lambda_j \lambda_{N2} \lambda_h \| v_n - v_{n-1} \| 
\]

\[
\leq \lambda_j \lambda_{N1} \lambda_f [D(T(x_n), T(x_{n-1}))] + \lambda_j \lambda_{N2} \lambda_h [D(A(x_n), A(x_{n-1}))] 
\]

\[
\leq \lambda_j \lambda_{N1} \lambda_f \delta_T \| x_n - x_{n-1} \| + \lambda_j \lambda_{N2} \lambda_h \delta_A \| x_n - x_{n-1} \| 
\]

123
6.3. Co-resolvent equation problem

Combining (6.3.11) and (6.3.12) with (6.3.10), we obtain

\[ \|z_{n+1} - z_n\| \leq [\lambda_j \lambda_g + \rho \lambda_j (\lambda_{N_1} \lambda_f \delta_T + \lambda_{N_2} \lambda_h \delta_A)] \|x_n - x_{n-1}\|. \]  

(6.3.13)

Using condition (6.3.8) and Lipschitz continuity of \( J_\rho^{\partial \phi(\cdot, x)} \), we get

\[
\|x_n - x_{n-1}\| \\
= \|J_\rho^{\partial \phi(\cdot, x_n)}(z_n) - J_\rho^{\partial \phi(\cdot, x_{n-1})}(z_{n-1}) - [g(x_n) - x_n - \{g(x_{n-1}) - x_{n-1}\}]\| \\
\leq \|J_\rho^{\partial \phi(\cdot, x_n)}(z_n) - J_\rho^{\partial \phi(\cdot, x_{n-1})}(z_{n-1}) + J_\rho^{\partial \phi(\cdot, x_{n-1})}(z_n) - J_\rho^{\partial \phi(\cdot, x_{n-1})}(z_{n-1})\| \\
+ \|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\| \\
\leq \|J_\rho^{\partial \phi(\cdot, x_n)}(z_n) - J_\rho^{\partial \phi(\cdot, x_{n-1})}(z_{n-1})\| + \|J_\rho^{\partial \phi(\cdot, x_{n-1})}(z_n) - J_\rho^{\partial \phi(\cdot, x_{n-1})}(z_{n-1})\| \\
+ \|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\| \\
\leq \mu \|x_n - x_{n-1}\| + \frac{1}{\alpha} \|z_n - z_{n-1}\| + \|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\|. 
\]  

(6.3.14)

By proposition (1.2.1), we have

\[
\|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\|^2 \\
\leq \|x_n - x_{n-1}\|^2 - 2 \langle g(x_n) - g(x_{n-1}), J(x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}) \rangle \\
= \|x_n - x_{n-1}\|^2 - 2 \langle g(x_n) - g(x_{n-1}), J(x_n - x_{n-1}) \rangle \\
- 2 \langle g(x_n) - g(x_{n-1}), J(x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\} - J(x_n - x_{n-1}) \rangle \\
\leq \|x_n - x_{n-1}\|^2 - 2 \gamma \|x_n - x_{n-1}\|^2 + 4k^2 \rho_E \left( \frac{4 \|g(x_n) - g(x_{n-1})\|}{k} \right) \\
\leq \|x_n - x_{n-1}\|^2 - 2 \gamma \|x_n - x_{n-1}\|^2 + 64C^2 \|g(x_n) - g(x_{n-1})\|^2 \\
\leq \left( 1 - 2 \gamma + 64C\lambda_g^2 \right) \|x_n - x_{n-1}\|^2, 
\]

which implies that

\[
\|x_n - x_{n-1} - \{g(x_n) - g(x_{n-1})\}\| \leq \sqrt{\left( 1 - 2 \gamma + 64C\lambda_g^2 \right) \|x_n - x_{n-1}\|}. 
\]

(6.3.15)
Using (6.3.15), (6.3.14) becomes
\[
\|x_n - x_{n-1}\| \leq \left[ \sqrt{1 - 2\gamma + 64C\lambda^2 g} + \mu \right] \|x_n - x_{n-1}\| + \frac{1}{\alpha} \|z_n - z_{n-1}\|,
\]
which implies that
\[
\|x_n - x_{n-1}\| \leq \frac{1}{\alpha \left[ \sqrt{1 - 2\gamma + 64C\lambda^2 g} + \mu \right]} \|z_n - z_{n-1}\|. \tag{6.3.16}
\]
By the application of (6.3.16), (6.3.13) reduces to
\[
\|z_{n+1} - z_n\| \leq \frac{\lambda_j \lambda_\rho + \rho \lambda_j (\lambda_{N_1} \lambda_j \delta_T + \lambda_{N_2} \lambda h \delta_A)}{\alpha \left[ \sqrt{1 - 2\gamma + 64C\lambda^2 g} + \mu \right]} \|z_n - z_{n-1}\|,
\]
i.e.,
\[
\|z_{n+1} - z_n\| \leq \Theta \|z_n - z_{n-1}\|, \tag{6.3.17}
\]
where \(\Theta = \frac{\lambda_j \lambda_\rho + \rho \lambda_j (\lambda_{N_1} \lambda_j \delta_T + \lambda_{N_2} \lambda h \delta_A)}{\alpha \left[ \sqrt{1 - 2\gamma + 64C\lambda^2 g} + \mu \right]}\).

From (6.3.9), we have \(\Theta < 1\), and consequently \(\{z_n\}\) is a cauchy sequence in \(Y^*\). Since \(Y^*\) is a Banach space, there exists \(z \in Y^*\) such that \(z_n \to z\) as \(n \to \infty\). From (6.3.14), we can see that \(\{x_n\}\) is also a cauchy sequence in \(Y\). Therefore, there exists \(x \in Y\) such that \(x_n \to x\) as \(n \to \infty\). Since the mappings \(T\) and \(A\) are \(D\)-Lipschitz continuous, it follows from (6.3.5) and (6.3.6) of Algorithm 6.3.1 that \(\{u_n\}\) and \(\{v_n\}\) are also cauchy sequences, we may assume that \(u_n \to u\) and \(v_n \to v\).

Since \(J, N, g, f\) and \(h\) are continuous mappings, and by (6.3.7) of Algorithm 6.3.1, it follows that
\[
J(g(x_n)) - \rho J(N(f(u_n), h(v_n))) = z_{n+1} \to z = J(g(x)) - \rho J(N(f(u), h(v))), n \to \infty, \tag{6.3.18}
\]
and
\[
J_\rho^{\partial \phi(\cdot, x_n)}(z_n) = g(x_n) \to g(x) = J_\rho^{\partial \phi(\cdot, x)}(z), n \to \infty. \tag{6.3.19}
\]
6.3. Co-resolvent equation problem

By (6.3.18), (6.3.19) and Lemma 6.3.1, we have

\[ J(N(f(u), h(v))) + \rho^{-1} R^\rho_{\mathcal{B}}(x)(z) = 0. \]

Finally, we prove that \( u \in T(x) \) and \( v \in A(x) \). In fact, since \( u_n \in T(x_n) \) and

\[
d(u_n, T(x)) \leq \max \left\{ d(u_n, T(x)), \sup_{q \in T(x)} d(T(x), q) \right\}
\]

\[
= \mathcal{D}(T(x_n), T(x)).
\]

Therefore, we have

\[
d(u, T(x)) \leq \|u - u_n\| + d(u_n, T(x))
\]

\[
\leq \|u - u_n\| + \mathcal{D}(T(x_n), T(x))
\]

\[
\leq \|u - u_n\| + \delta_T \|x_n - x\| \to 0, \text{ as } n \to \infty,
\]

which implies that \( d(u, T(x)) = 0 \). Since \( T(x) \in CB(E) \), it follows that \( u \in T(x) \). Similarly, we can prove that \( v \in A(x) \). By Lemma 6.3.1, the required result follows. This completes the proof. \( \square \)