RELIABILITY MODELING AND COST BENEFIT ANALYSIS OF PAPER RECYCLING SYSTEM WITH PREVENTIVE MAINTENANCE

2.1 INTRODUCTION

The effectiveness of the system is understood to mean the suitability of the system for the fulfillment of the intended task and the efficiency of utilizing the means put into it. The suitability of performing the definite task is primarily determined by the reliability of the system. In other words we can say that the theory of reliability studies the problem of occurrence of failures in equipment/systems. The incorporation of preventive maintenance is one of the important techniques to improve the reliability of the system. Preventive maintenance is a sort of repair, given to the system to improve its working capability and to delay its failures so that we can enhance the benefits made by the system.

The system models with preventive maintenance have been frequently analysed by various authors including L R Goel et. al. [26-27, 30, 34], Gupta [46-47], Kapur et. al. [71], Nakagawa et. al. [94], Osaki [97] and Rander et. al. [106] with different assumptions due their vital existence in modern industries and business.

For the purpose of analyzing real existing system model, a paper recycling system, located at J&K S.I.D.C.O. (J&K State Industrial Development Corporation) Jammu, Jammu and Kashmir is selected for study. J&K S.I.D.C.O. has been incorporated as a completely Govt. owned company. The deeds of the company are to make development in infrastructure for establishing large sized Industrial complexes and Estates.

The given system consists of two units of varying nature. The first unit consists of three subunits namely Paper Pulper, Deinker and Bleaching Machine, all the three are revolving machines and are connected with same conveyer belt and the second unit is Paper Making Machine. The working of different units and subunits of the system is described as follows:
Paper Pulper: The Paper Pulper chops the collected waste paper into small pieces and heating effect of the subunit breaks the paper into tiny strands of cellulose (organic plant material) called fibers or paper pulp.

Deinker: Deinker performs Paper laundering operation to remove printing ink and stickies (sticky material like glue residues and adhesives).

Bleaching Machine: Bleaching Machine enhances the brightness of the paper by using chemicals like hydrogen peroxide, oxygen or chlorine dioxide.

Paper Making Machine: The bleached pulp is mixed with chemicals and water and the water content is around 99.5 percent. The watery mixture is then poured into the Paper Making Machine, during the screening process, the water begins to drain and the recycled pulp starts forming watery sheets. More water is drained out when this sheet moves through press rollers of the Paper making Machine a number of times. The paper sheet thus formed is dried by passing it through heated metal rollers.

Using regenerative point technique the following important reliability characteristics of interest are obtained:

1. Transition probabilities and mean sojourn times.
2. Reliability and mean time to system failure (MTSF).
3. Point wise and steady-state availabilities of the system.
4. Expected up-time of the system.
5. Expected busy period of the repairman during (0, t] and in the steady state.
6. Expected number of repairs during (0, t] and in the steady state.
7. Net expected profit incurred by the system during (0, t] and in steady state.

2.2 ASSUMPTIONS

The system model is analyzed under following practical assumptions:

1. Initially the first unit is operative and the second unit is kept idle it will work only after the first unit produces some required material.
2. If any of the subunit of first unit stops working the whole unit stops working.
3. Single repair facility is always available with the system.
4. Repair is done on the basis of first come first serve.
5. First unit requires preventive maintenance after a random operation of time.
6. System keeps on working even if the first unit is under preventive maintenance.
7. All the failure rates are taken to be negative exponential with different parameters.
8. Repair rate of first unit follows negative exponential distribution and repair rate of second unit is taken to be general.
9. The rate with which first unit comes under preventive maintenance follows negative exponential distribution and the rate with which the first unit transits from preventive maintenance state to operative state is taken to be general.
10. Repairs are perfect, that is repair facility never does any damage to the system.

### 2.3 NOTATIONS AND STATES OF THE SYSTEM

- \(\alpha_1\) : Failure rate of the first unit i.e. \([P, DI & B]\).
- \(\alpha_2\) : Failure rate of the second unit i.e. PM.
- \(\beta_1\) : Rate of transition of second unit from idle to operative.
- \(\beta_2\) : Rate of transition of second unit from operative to idle.
- \(\Lambda\) : Repair rate of first unit i.e.\([P, DI & B]\).
- \(F(.)\) : Repair rate of second unit i.e. PM.
- \(G(.)\) : Rate of performance of Preventive Maintenance.
- \(\gamma\) : Rate with which a unit comes under preventive maintenance.
Symbols for the states of the system

\[
\begin{align*}
[P,DI & B]_o & : \text{ Pulper, Deinker and bleaching machine is operative.} \\
[P,DI & B]_{pm} & : \text{ Pulper, Deinker and bleaching machine is under preventive maintenance.} \\
[P,DI & B]_r & : \text{ Pulper, Deinker and bleaching machine is under repair.} \\
[P,DI & B]_{wr} & : \text{ Pulper, Deinker and bleaching machine is waiting for repair} \\
PM_1 & : \text{ Paper machine is under idle condition.} \\
PM_o & : \text{ Paper machine is operative.} \\
PM_r & : \text{ Paper machine is under repair.} \\
PM_{wr} & : \text{ Paper machine is waiting for repair.}
\end{align*}
\]

With the help of the above symbols the possible states of the system are:

\[
\begin{align*}
S_0 &= [(P,DI & B)_o, PM_1] & S_1 &= [(P,DI & B)_r, PM_o] \\
S_4 &= [(P,DI & B)_r, PM_{wr}] & S_5 &= [(P,DI & B)_{pm}, PM_o] \\
S_8 &= [(P,DI & B)_r, PM_i]
\end{align*}
\]

The transition diagram along with all the transitions is shown in fig 4.1
Fig. 2.1
2.4 TRANSITION PROBABILITIES AND SOJOURN TIMES

Let $T_0(\equiv 0), \ T_1, T_2, \ldots$ denote the regenerative epochs and $X_n$ denotes the state visited at epoch $T_n$, i.e. just after the transition at $T_n$ then $\{X_n, T_n\}$ constitute a Markov-Renewal process with state space $E$, set of regenerative states and

$$Q_{ij}(t) = P[X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i].$$

is the semi Markov kernel over $E$.

Then the transition probability matrix of the embedded Markov chain is

$$p = p_{ij} = [Q_{ij}(\infty)] = [Q(\infty)]$$

The various transitions probabilities may be obtained as follows:

$$Q_{01}(t) = \alpha_1 \int_0^t \exp[-(\alpha_1 + \beta_1)u] \, du$$

$$Q_{02}(t) = \beta_1 \int_0^t \exp[-(\alpha_1 + \beta_1)u] \, du$$

$$Q_{12}(t) = \lambda \int_0^t \exp[-(\lambda + \alpha_2 + \beta_2)u] \, du$$

$$Q_{14}(t) = \alpha_2 \int_0^t \exp[-(\lambda + \alpha_2 + \beta_2)u] \, du$$

$$Q_{18}(t) = \beta_2 \int_0^t \exp[-(\lambda + \alpha_2 + \beta_2)u] \, du$$

$$Q_{21}(t) = \alpha_1 \int_0^t \exp[-(\alpha_1 + \alpha_2 + \gamma)u] \, du$$

$$Q_{23}(t) = \alpha_2 \int_0^t \exp[-(\alpha_1 + \alpha_2 + \gamma)u] \, du$$

$$Q_{25}(t) = \gamma \int_0^t \exp[-(\alpha_1 + \alpha_2 + \gamma)u] \, du$$

$$Q_{31}^{(7)}(t) = \alpha_1 \int_0^t \exp[-(\alpha_1)u] \, du \int_u^t dF(v)$$
\[ Q_{32}(t) = \int_0^t dF(u) \exp[-(\alpha_1)u] \]
\[ Q_{42}^{(3)}(t) = \lambda \int_0^t \exp[-(\lambda)u] \, du \int_u^t dF(v) \]
\[ Q_{51}(t) = \alpha_1 \int_0^t \exp[-(\alpha_1 + \alpha_2)u] \tilde{G}(u) \, du \]
\[ Q_{52}(t) = \int_0^t \exp[-(\alpha_1 + \alpha_2)u] dG(u) \, du \]
\[ Q_{61}^{(7)}(t) = \alpha_1 \int_0^t \exp[-(\alpha_1)u] \, du \int_u^t dF(v) \]
\[ Q_{62}^{(3)}(t) = \int_0^t dG(u) \int_u^t dF(v) \]
\[ Q_{65}(t) = \int_0^t \exp[-(\alpha_1)u] dG(u) \tilde{G}(u) \, du \]
\[ Q_{80}(t) = \lambda \int_0^t \exp[-(\lambda)u] \, du \quad (1) \]

**Steady state transition probabilities**

By taking the limit as \( t \) tends to \( \infty \) in equation (1), we obtain the following steady state transition probabilities:

\[ p_{01} = \lim_{t \to \infty} Q_{01}(t) = \alpha_1 \int_0^\infty \exp[-(\alpha_1 + \beta_1)t] \, dt = \frac{\alpha_1}{\alpha_1 + \beta_1} \]
\[ p_{02} = \frac{\beta_1}{\alpha_1 + \beta_1} \]
\[ p_{14} = \frac{\alpha_2}{\lambda + \alpha_2 + \beta_2} \]
\[ p_{18} = \frac{\beta_2}{\lambda + \alpha_2 + \beta_2} \]
\[ p_{21} = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \gamma} \]
\[ p_{23} = \frac{\alpha_2}{\alpha_1 + \alpha_2 + \gamma} \]
\[ p_{25} = \frac{\gamma}{\alpha_1 + \alpha_2 + \gamma} \]
\[ p_{31}^{(7)} = 1 - f^*(\alpha_1) \]

\[ \text{37} \]
From the obtained steady state probabilities, it can be easily seen that the following results hold good:

\[ p_{01} + p_{02} = 1 \]
\[ p_{21} + p_{23} + p_{25} = 1 \]
\[ p_{41}^{(3,7)} + p_{45}^{(3)} = 1 \]
\[ p_{61}^{(7)} + p_{62}^{(3)} + p_{65} = 1 \]
\[ p_{80} = 1 \]

\[ p_{12} + p_{14} + p_{18} = 1 \]
\[ p_{31}^{(7)} + p_{32} = 1 \]
\[ p_{51} + p_{51}^{(6,7)} + p_{52} = 1 \]
\[ p_{71} = 1 \]

(3)

**Mean sojourn time**

The mean sojourn time in state \( S_i \) denoted by \( \mu_i \) is defined as the expected time taken by the system in state \( S_i \) before transiting to any other state. To obtain mean sojourn time \( \mu_i \) in state \( S_i \), we observe that as long as the system is in state \( S_i \), there is no transition from \( S_i \) to any other state. If \( T_i \) denotes the sojourn time in state \( S_i \) then mean sojourn time in state \( S_i \) is

\[
\mu_i = \mathbb{E}[T_i] = \int_0^\infty P[T_i > t] \, dt
\]

Thus
\[
\mu_0 = \int_0^\infty \exp[-(\alpha_1 + \beta_1)t] \, dt = \frac{1}{\alpha_1 + \beta_1} \\
\mu_1 = \int_0^\infty \exp[-(\alpha_1 + \beta_2 + \lambda)t] \, dt = \frac{1}{\alpha_2 + \beta_2 + \lambda} \\
\mu_2 = \int_0^\infty \exp[-(\alpha_1 + \alpha_2 + \gamma)t] \, dt = \frac{1}{\alpha_1 + \alpha_2 + \gamma} \\
\mu_3 = \int_0^\infty \exp[-(\alpha_1) t] \bar{F}(t) \, dt = \frac{1}{\alpha_1} \left[ 1 - f^*(\alpha_1) \right] \\
\mu_4 = \int_0^\infty \exp[-(\lambda)t] \, dt = \frac{1}{\lambda} \\
\mu_5 = \int_0^\infty \exp[-(\alpha_1 + \alpha_2)t] \bar{G}(t) \, dt = \frac{1}{\alpha_1 + \alpha_2} \left[ 1 - g^*(\alpha_1 + \alpha_2) \right] \\
\mu_6 = \int_0^\infty \exp[-(\alpha_1)t] \bar{F}(t) \bar{G}(t) \, dt \\
\mu_7 = \int_0^\infty \bar{F}(t) \, dt \\
\mu_8 = \int_0^\infty \exp[-(\lambda)t] \, dt = \frac{1}{\lambda}
\]

(4)

### 2.5 MEAN TIME TO SYSTEM FAILURE

Let the random variable $T_i$ denotes the time to system failure when $E_0 = E_i \in E$ and $\phi_1(t)$ is the c.d.f. of the time to system failure for the first time when the system starts operation from state $S_i$. On the basis of arguments used for regenerative processes we obtain following relations for $\phi_1(t)$.

\[
\begin{align*}
\phi_0(t) &= Q_{01}(t) \phi_1(t) + Q_{02}(t) \phi_2(t) \\
\phi_1(t) &= Q_{12}(t) \phi_2(t) + Q_{14}(t) + Q_{18}(t) \phi_6(t) \\
\phi_2(t) &= Q_{21}(t) \phi_1(t) + Q_{23}(t) \phi_3(t) + Q_{25}(t) \phi_5(t) \\
\phi_3(t) &= Q_{31}(t) \phi_1(t) + Q_{32}(t) \phi_2(t) \\
\phi_5(t) &= [Q_{51}(t) + Q_{51}^{(6,7)}(t)] \phi_1(t) + Q_{52}(t) \phi_2(t) \\
\phi_6(t) &= Q_{60}(t) \phi_0(t)
\end{align*}
\]
Taking the Laplace Transform of above equations we get:

\[
\tilde{\Phi}_0(s) = \tilde{Q}_{01}(s)\tilde{\Phi}_1(s) + \tilde{Q}_{02}(s)\tilde{\Phi}_2(s)
\]

\[
\tilde{\Phi}_1(s) = \tilde{Q}_{12}(s)\tilde{\Phi}_2(s) + \tilde{Q}_{14}(s) + \tilde{Q}_{16}(s)\tilde{\Phi}_8(s)
\]

\[
\tilde{\Phi}_2(s) = \tilde{Q}_{21}(s)\tilde{\Phi}_1(s) + \tilde{Q}_{23}(s)\tilde{\Phi}_3(s) + \tilde{Q}_{25}(s)\tilde{\Phi}_5(s)
\]

\[
\tilde{\Phi}_3(s) = \tilde{Q}_{31}^{(7)}(s)\tilde{\Phi}_1(s) + \tilde{Q}_{32}(s)\tilde{\Phi}_2(s)
\]

\[
\tilde{\Phi}_5(s) = \left[\tilde{Q}_{51}(s) + \tilde{Q}_{51}^{(6,7)}(s)\right]\tilde{\Phi}_1(s) + \tilde{Q}_{52}(s)\tilde{\Phi}_2(s)
\]

\[
\tilde{\Phi}_0(s) = \tilde{Q}_{00}(s)\tilde{\Phi}_0(s)
\]

on solving above equations for \(\tilde{\Phi}_0(s)\), we have

\[
\tilde{\Phi}_0(s) = \frac{N_1(s)}{D_1(s)}
\]

where

\[
N_1(s) = \tilde{Q}_{14}(s)[\tilde{Q}_{01}(s) + \tilde{Q}_{02}(s)\tilde{Q}_{01}(s) - \tilde{Q}_{23}(s)\{\tilde{Q}_{01}(s)\tilde{Q}_{32}(s) - \tilde{Q}_{02}(s)\tilde{Q}_{31}^{(7)}(s)\}] - \tilde{Q}_{14}(s)\tilde{Q}_{25}(s)[\tilde{Q}_{01}(s)\tilde{Q}_{52}(s) - \tilde{Q}_{02}(s)\{\tilde{Q}_{51}(s) + \tilde{Q}_{51}^{(6,7)}\}]
\]

and

\[
D_1(s) = 1 - \tilde{Q}_{12}(s)\tilde{Q}_{21}(s) - \tilde{Q}_{23}(s)\{\tilde{Q}_{31}^{(7)}(s)\tilde{Q}_{12}(s)\} - \tilde{Q}_{25}(s)[\tilde{Q}_{52}(s) + \tilde{Q}_{12}(s) - \tilde{Q}_{51}(s) + \tilde{Q}_{51}^{(6,7)}\}] + \tilde{Q}_{18}(s)\tilde{Q}_{80}(s)[\tilde{Q}_{23}(s)\{\tilde{Q}_{01}(s)\tilde{Q}_{32}(s) - \tilde{Q}_{02}(s)\tilde{Q}_{31}^{(7)}(s)\} - \tilde{Q}_{01}(s) - \tilde{Q}_{02}(s)\tilde{Q}_{21}(s)] + \tilde{Q}_{18}(s)\tilde{Q}_{80}(s)\tilde{Q}_{25}(s)[\tilde{Q}_{01}(s)\tilde{Q}_{52}(s) - \tilde{Q}_{02}(s)\{\tilde{Q}_{51}(s) + \tilde{Q}_{51}^{(6,7)}\}]
\]

on taking limit \(s \to 0\) in (6) and (7) and using the relation \(\lim_{s \to 0} \tilde{Q}_{ij}(s) \to p_{ij}\), we get

\[
N_1(0) = p_{14}[p_{01} + p_{02}p_{21} - p_{23}(p_{01}p_{32} - p_{02}p_{31}^{(7)})] - p_{14}p_{25}[p_{01}p_{52} - p_{02}(p_{51} + p_{51}^{(6,7)})]
\]

\[
= (1 - p_{12} - p_{10})[p_{01} + p_{02}p_{21} - p_{01}p_{23}p_{32} + p_{02}p_{23}p_{31}^{(7)} - p_{01}p_{25}p_{52} + p_{02}p_{25}(p_{51} + p_{51}^{(6,7)})]
\]
\[ p_{01} + p_{02}p_{21} - p_{01}p_{23}p_{32} + p_{02}p_{23}p_{31}^{(7)} - p_{01}p_{25}p_{52} + p_{02}p_{25}(p_{51} + p_{51}^{(6,7)}) - p_{01}p_{12} - p_{02}p_{21}p_{12} + p_{01}p_{12}p_{23}p_{32} - p_{02}p_{23}p_{31}^{(7)}p_{12} + p_{01}p_{12}p_{25}p_{52} - p_{02}p_{25}(p_{51} + p_{51}^{(6,7)}) - p_{01}p_{18} - p_{02}p_{21}p_{18} + p_{01}p_{18}p_{23}p_{32} - p_{02}p_{23}p_{31}^{(7)}p_{18} + p_{01}p_{18}p_{25}p_{52} - p_{02}p_{25}p_{18}(p_{51} + p_{51}^{(6,7)}) \]

\[ = p_{18}p_{25}[p_{01}p_{52} - p_{02}(p_{51} + p_{51}^{(6,7)})] + p_{18}[p_{23}(p_{01}p_{32} - p_{02}p_{31}^{(7)}) - p_{02}p_{21} - p_{01}] - p_{25}[p_{01}p_{52} - p_{02} + p_{02}p_{52} - p_{01}p_{12}p_{52} + p_{02}p_{12} - p_{02}p_{12}p_{52}] + p_{23}[p_{02} - p_{032}p_{12} - p_{02}p_{12}p_{31}^{(7)}] + p_{02}p_{21} + p_{01}p_{12} - p_{02}p_{21}p_{12} + p_{01} \]

\[ = p_{18}p_{25}[p_{01}p_{52} - p_{02}(p_{51} + p_{51}^{(6,7)})] + p_{18}[p_{23}(p_{01}p_{32} - p_{02}p_{31}^{(7)}) - p_{02}p_{21} - p_{01}] - p_{25}[p_{52} - p_{12} + p_{12}(p_{51} + p_{51}^{(6,7)}) - p_{02} + p_{02}p_{12}] + p_{23}[p_{02} - p_{32} + p_{01}p_{12} - p_{12}p_{31}^{(7)}] + 1 - p_{02}p_{25} - p_{02}p_{23} - p_{12} + p_{02}p_{12}(1 - p_{21}) \]

\[ = p_{18}p_{25}[p_{01}p_{52} - p_{02}(p_{51} + p_{51}^{(6,7)})] + p_{18}[p_{23}(p_{01}p_{32} - p_{02}p_{31}^{(7)}) - p_{02}p_{21} - p_{01}] - p_{25}[p_{52} + p_{12}(p_{51} + p_{51}^{(6,7)})] - p_{23}[p_{32} + p_{12}p_{31}^{(7)}] + p_{12}p_{25} + p_{02}p_{25} - p_{02}p_{12}p_{25} + p_{02}p_{23} + p_{01}p_{12}p_{23} + 1 - p_{02}p_{25} - p_{02}p_{23} - p_{12} + p_{12}p_{02}p_{23} - p_{02}p_{12}p_{25} \]

\[ = p_{18}p_{25}[p_{01}p_{52} - p_{02}(p_{51} + p_{51}^{(6,7)})] + p_{18}[p_{23}(p_{01}p_{32} - p_{02}p_{31}^{(7)}) - p_{02}p_{21} - p_{01}] - p_{25}[p_{52} + p_{12}(p_{51} + p_{51}^{(6,7)})] - p_{23}[p_{32} + p_{12}p_{31}^{(7)}] + 1 - p_{12}p_{21} \]

\[ = 1 - p_{12}p_{21} - p_{23}p_{32} + p_{18}p_{25}[p_{01}p_{52} - p_{02} + p_{02}p_{52}] + p_{18}[p_{01} + p_{02}p_{21}] - p_{25}[p_{52} + p_{12} - p_{12}p_{52}] - p_{18}p_{23}[p_{01}p_{32} - p_{02}p_{31}^{(7)}] + p_{12}p_{23}p_{31}^{(7)} \]

\[ = 1 - p_{12}p_{21} - p_{23}p_{32} - p_{25}p_{52}(1 - p_{12} - p_{18}) - p_{25}p_{18}(p_{21} + p_{25}) - p_{12}p_{23} + p_{12}p_{23}p_{32} + p_{18}p_{23}p_{32} - p_{02}p_{18}p_{23} + p_{02}p_{18}p_{23}p_{32} - p_{01}p_{18} - p_{12}p_{25} \]

\[ = 1 - p_{12}p_{21} - p_{23}p_{32} - p_{14}p_{25}p_{52} - p_{18} - p_{12}(p_{23} + p_{25}) + p_{23}p_{32}(p_{12} + p_{18}) \]

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\[= 1 - p_{12} - p_{18} - p_{14}(p_{25}p_{52} + p_{23}p_{32})\]

\[= p_{14}(1 - p_{25}p_{52} + p_{23}p_{32})\]

and

\[D_1(0) = p_{14}(1 - p_{25}p_{52} + p_{23}p_{32})\]  

(8)

which is equal to \(N_1(0)\)

\[\tilde{\Phi}_0(s) = \frac{N_1(s)}{D_1(s)} = 1\]

This shows that \(\tilde{\Phi}_0(s)\) is a proper cdf. Therefore, mean time to system failure will be

\[MTSF = E(T) = -\frac{\partial}{\partial s} \tilde{\Phi}_0(s) \bigg|_{s=0} = \frac{D_1'(0) - N_1'(0)}{D_1(0)}\]  

(9)

To obtain numerator of (9), we collect the coefficients of the relevant \(m_{ij}\)'s, where \(m_{ij}\)'s is the mean elapsed time of the system in state \(S_i\) before transiting to state \(S_j\).

In notation

\[m_{ij} = \tilde{Q}_{ij}'(s) \bigg|_{s=0} = \frac{\partial}{\partial s} \int_0^\infty \exp[-(st)] \partial Q_{ij}(t) \bigg|_{s=0}\]

also we know that \(\sum_i m_{ij} = \mu_i\)

Thus the coefficient of various \(m_{ij}\)'s in \(D_1'(0) - N_1'(0)\)

The coefficient of \(m_{01} = p_{18} + p_{14} - p_{18}p_{25}p_{52} - p_{14}p_{23}p_{32} - p_{14}p_{25}p_{52} - p_{18}p_{23}p_{32}\)

\[= (p_{18} + p_{14})(1 - p_{25}p_{52} - p_{23}p_{32})\]

The coefficient of \(m_{02} = p_{18}p_{23}p_{31}^{(7)} + p_{18}p_{21} + p_{18}p_{25}(p_{51} + p_{51}^{(6,7)}) + p_{14}p_{23}p_{31}^{(7)} + p_{14}p_{21} + p_{48}p_{25}(p_{51} + p_{51}^{(6,7)})\)

\[= p_{18}p_{23} - p_{18}p_{23}p_{32} + p_{18}p_{21} + p_{18}p_{25} - p_{18}p_{25}p_{52} + p_{14}p_{21}\]

\[+ p_{14}p_{25} + p_{14}p_{23} - p_{14}p_{23}p_{32} - p_{14}p_{25}p_{52}\]
\[= (p_{18} + p_{14})(1 - p_{25}p_{52} - p_{23}p_{32})\]

The coefficient of \(m_{12} = p_{21} + p_{23}p_{31}^{(7)} + p_{25}(p_{51} + p_{51}^{(6,7)})\)
\[= 1 - p_{23}p_{32} - p_{25}p_{52}\]

The coefficient of \(m_{14} = p_{01} - p_{23}(p_{01}p_{32} - p_{02}p_{31}^{(7)}) - p_{25}[p_{01}p_{52} - p_{02}(p_{51} + p_{51}^{(6,7)})]\]
\[+ p_{02}p_{21}\]
\[= p_{01} + p_{02}(1 - p_{23}) - p_{01}p_{23} + p_{01}p_{23}p_{31}^{(7)} + p_{02}p_{23}p_{31}^{(7)} - p_{25}p_{52}\]
\[= p_{01} + p_{02} - p_{02}p_{23} - p_{01}p_{23} + p_{23}p_{31}^{(7)} - p_{25}p_{52}\]
\[= 1 - p_{23} + p_{23}p_{31}^{(7)} - p_{25}p_{52}\]
\[= 1 - p_{23}p_{32} - p_{25}p_{52}\]

The coefficient of \(m_{18} = p_{01} - p_{23}(p_{01}p_{32} - p_{02}p_{31}^{(7)}) - p_{25}[p_{01}p_{52} - p_{02}(p_{51} + p_{51}^{(6,7)})]\]
\[+ p_{02}p_{21}\]
\[= p_{01} + p_{02}(1 - p_{23}) - p_{01}p_{23} + p_{01}p_{23}p_{31}^{(7)} + p_{02}p_{23}p_{31}^{(7)} - p_{25}p_{52}\]
\[= p_{01} + p_{02} - p_{02}p_{23} - p_{01}p_{23} + p_{23}p_{31}^{(7)} - p_{25}p_{52}\]
\[= 1 - p_{23} + p_{23}p_{31}^{(7)} - p_{25}p_{52}\]
\[= 1 - p_{23}p_{32} - p_{25}p_{52}\]

The coefficient of \(m_{21} = p_{12} + p_{18}p_{02} + p_{02}p_{14}\)
\[= p_{12} + p_{02} - p_{02}p_{12}\]
\[= p_{02} + p_{01}p_{12}\]

The coefficient of \(m_{23} = p_{32} + p_{12}p_{31}^{(7)} - (p_{18} + p_{14})(p_{01}p_{32} - p_{02}p_{31})\)
\[= 1 - p_{31}^{(7)} + p_{31}^{(7)} - p_{18} p_{31}^{(7)} - p_{14} p_{31}^{(7)} + p_{02} (p_{18} + p_{14}) + p_{19} p_{31}^{(7)} + p_{14} p_{31}^{(7)} \]
\[= 1 - p_{18} - p_{14} + p_{02} (p_{18} + p_{14}) \]
\[= p_{02} + p_{01} p_{12} \]

The coefficient of \( m_{25} \) is \( p_{52} - p_{12} (p_{51} + p_{51}^{(6,7)}) - p_{18} [p_{01} p_{52} - p_{02} (p_{51} + p_{51}^{(6,7)})] - p_{18} [p_{01} p_{52} - p_{02} (p_{51} + p_{51}^{(6,7)})] \)
\[= p_{12} - p_{18} p_{52} - p_{14} p_{52} + p_{02} (p_{14} + p_{18}) + p_{52} - p_{12} p_{52} \]
\[= p_{12} - p_{18} p_{52} (p_{14} + p_{18}) + p_{02} (p_{14} + p_{18}) + p_{52} \]
\[= p_{14} + p_{02} (p_{18} + p_{14}) \]
\[= p_{02} + p_{01} p_{12} \]

The coefficient of \( m_{31}^{(7)} \) is \( p_{12} p_{23} + p_{02} p_{18} p_{23} + p_{02} p_{14} p_{23} \)
\[= p_{02} p_{23} + p_{01} p_{12} p_{23} \]

The coefficient of \( m_{32} \) is \( p_{23} + p_{01} p_{18} p_{23} + p_{01} p_{14} p_{23} \)
\[= p_{23} + p_{02} p_{23} (p_{14} + p_{18}) - p_{18} p_{23} - p_{14} p_{23} \]
\[= p_{14} p_{23} + p_{02} p_{23} (p_{14} + p_{18}) \]
\[= p_{02} p_{23} + p_{12} p_{23} (1 - p_{02}) \]
\[= p_{02} p_{23} + p_{01} p_{12} p_{23} \]

The coefficient of \( m_{51} \) is \( p_{12} p_{25} + p_{02} p_{18} p_{25} + p_{02} p_{14} p_{25} \)
\[= p_{12} p_{25} + p_{02} p_{25} - p_{02} p_{12} p_{25} \]
\[= p_{02} p_{25} + p_{02} p_{12} p_{25} \]

The coefficient of \( m_{51}^{(6,7)} \) is \( p_{12} p_{25} + p_{02} p_{18} p_{25} + p_{02} p_{14} p_{25} \)
\[= p_{12} p_{25} + p_{02} p_{25} - p_{02} p_{12} p_{25} \]
The coefficient of \( m_{52} \) = \( p_{25} + p_{01}p_{18}p_{25} + p_{01}p_{14}p_{25} \)
\[ = p_{25} + p_{02}p_{25}(p_{14} + p_{18}) - p_{14}p_{25} - p_{18}p_{25} \]
\[ = p_{02}p_{25} + p_{02}p_{25}(p_{14} + p_{18}) \]
\[ = p_{02}p_{25} + p_{02}p_{12}p_{25} \]

The coefficient of \( m_{80} \) = \( p_{18}[(p_{01} + p_{02}p_{21}) - p_{23}(p_{01}p_{32} - p_{02}p_{31}^{(7)})] - p_{18}p_{25} \)
\[ [p_{01}p_{52} - p_{02}(p_{51} + p_{51}^{(6,7)})] \]
\[ = p_{18}[p_{01}(1 - p_{23}p_{32}) - p_{02}(p_{21} + p_{23} - p_{23}p_{32})] - p_{18}p_{25}p_{52} \]
\[ + p_{02}p_{18}p_{25} \]
\[ = p_{18}[(p_{01} + p_{02})(1 - p_{23}p_{32})] - p_{18}p_{25}p_{52} + p_{18}p_{25}p_{52} - \]
\[ p_{18}p_{25}p_{52} \]
\[ = p_{18}(1 - p_{23}p_{32} - p_{25}p_{52}) \]

Therefore, on using the relations (3) and above coefficients, the numerator of (9) becomes
\[
D_{1}^{\prime}(0) - N_{1}^{\prime}(0) = \mu_{0}[(p_{18} + p_{14})(1 - p_{25}p_{52} - p_{23}p_{32})] + \mu_{1}(1 - p_{23}p_{32} - p_{25}p_{52}) + \\
\mu_{2}(p_{02} + p_{01}p_{12}) + \mu_{3}(p_{02}p_{23} + p_{01}p_{12}p_{23}) + \mu_{5}(p_{02}p_{25} + p_{02}p_{12}p_{25}) + \\
\mu_{6}[p_{18}(1 - p_{23}p_{32} - p_{25}p_{52})] \\
= [\mu_{0}(1 - p_{12}) + \mu_{1} + \mu_{8}p_{18}][1 - p_{23}p_{32} - p_{25}p_{52}] + [\mu_{2} + \mu_{3}p_{23} + \\
\mu_{5}p_{25}][(p_{02} + p_{01}p_{12})] \\
\]

MTSF = \( \frac{D_{1}^{\prime}(0) - N_{1}^{\prime}(0)}{D_{1}(0)} \) \hspace{1cm} (10)

where \( D_{1}(0) \) is given by (8)

Now by substituting the values of \( \mu_{i} \)'s and \( p_{ij} \)'s in (10), we get the expressions for numerator and denominator of MTSF as
\[ D_1(0) - N_1(0) = [\alpha_1 + \alpha_2]\alpha_1(\alpha_2 + \beta_2) + \alpha_1(\alpha_1 + \beta_1) + (\alpha_1 + \beta_1)(\alpha_2 + \beta_2)(1 - f^*(\alpha_1))][(\alpha_1 + \alpha_2 + \gamma) - \alpha_2f^*(\alpha_1) - \gamma g^*(\alpha_1 + \alpha_2)] + [\alpha_1(\alpha_1 + \alpha_2) + \alpha_2(\alpha_1 + \alpha_2)(1 - f^*(\alpha_1)) + \gamma [g^*(\alpha_1 + \alpha_2)][(\alpha_1 + \alpha_2 + \gamma) - \gamma g^*(\alpha_1 + \alpha_2)] + \alpha_1(\alpha_1 + \alpha_2) + \alpha_2f^*(\alpha_1)] \]

and

\[ D_1(0) = \alpha_1\beta_1[\alpha_1 + \alpha_2][\alpha_2 + \beta_2 + \lambda][(\alpha_1 + \alpha_2 + \gamma) - \gamma g^*(\alpha_1 + \alpha_2) + \alpha_2f^*(\alpha_1)] \]

### 2.6 AVAILABILITY ANALYSIS

Let \( Z_i(t) \) be the probability that the system is up at epoch \( t \) when it initially started operation from regenerative state \( S_i \) and remains up continuously till time \( t \) without passing through any other regenerative state. Therefore we have

\[
Z_0(t) = \exp[-(\alpha_1 + \beta_1)t] \quad Z_1(t) = \exp[-(\alpha_1 + \beta_2 + \lambda)t]
\]

\[
Z_2(t) = \exp[-(\alpha_1 + \alpha_2 + \gamma)t] \quad Z_3(t) = \exp[-(\alpha_1)t]\bar{F}(t)
\]

\[
Z_4(t) = \exp[-(\lambda)t] \quad Z_5(t) = \exp[-(\alpha_1 + \alpha_2)t]\bar{G}(t)
\]

\[
Z_6(t) = \exp[-(\alpha_1)t]\bar{F}(t)\bar{G}(t) \quad Z_7(t) = \bar{F}(t)
\]

\[
Z_8(t) = \exp[-(\lambda)t]
\]

Also we define \( A_i(t) \) as the probability that the system is in up-state at epoch \( t \)’ when it initially started from regenerative state \( S_i \). To obtain recurrence relations among different point wise availabilities we use the simple probabilistic arguments.

As an illustration \( A_0(t) \) is the sum of the following probabilities:

1. The system remains up in state \( S_0 \) without making any transition to any other regenerative state up to time ‘\( t \)’, the probability of this event equals

\[
Z_0(t) = \exp[-(\alpha_1 + \beta_1)t]
\]

2. The system transits from state \( S_0 \) to state \( S_1 \) during \((u, u + du), u \leq t\) and then starting at epoch \( u \) from \( S_1 \) it is available for remaining time \((t - u)\) the probability of this event is:
\[ \int_0^t q_{01}(u)A_1(t-u)du = q_{01}(t)\oplus A_1(t). \]

3. The system transits from state \( S_0 \) to state \( S_2 \) during \( (u, u + du), u \leq t \) and then starting at epoch \( u \) from \( S_3 \) it is available for remaining time \( (t - u) \) the probability of this event is:

\[ \int_0^t q_{02}(u)A_2(t-u)du = q_{02}(t)\oplus A_2(t). \]

Therefore \( A_0(t) \) becomes

\[ A_0(t) = Z_0(t) + q_{01}(t)\oplus A_1(t) + q_{02}(t)\oplus A_2(t) \]

By similar arguments, we have

\[ A_1(t) = Z_1(t) + q_{12}(t)\oplus A_2(t) + q_{14}(t)\oplus A_4(t) + q_{18}(t)\oplus A_8(t) \]
\[ A_2(t) = Z_2(t) + q_{21}(t)\oplus A_1(t) + q_{23}(t)\oplus A_3(t) + q_{25}(t)\oplus A_5(t) \]
\[ A_3(t) = Z_3(t) + q_{31}^{(7)}(t)\oplus A_1(t) + q_{32}(t)\oplus A_2(t) \]
\[ A_4(t) = q_{41}^{(6,7)}(t)\oplus A_1(t) + q_{42}^{(3)}(t)\oplus A_2(t) \]
\[ A_5(t) = Z_5(t) + [q_{51}(t) + q_{51}^{(6,7)}(t)]\oplus A_1(t) + q_{52}(t)\oplus A_2(t) \]
\[ A_8(t) = Z_8(t) + q_{80}(t)\oplus A_0(t) \]

Where \( Z_i(t)'s \) are same as defined above

Taking the Laplace transform of above equations, we get a set of linear equations in \( A_i^*(s) \) as

\[ A_0^*(s) = Z_0^*(s) + q_{01}^*(s)A_1^*(s) + q_{02}^*(s)A_2^*(s) \]
\[ A_1^*(s) = Z_1^*(s) + q_{12}^*(s)A_2^*(s) + q_{14}^*(s)A_4^*(s) + q_{18}^*(s)A_8^*(s) \]
\[ A_2^*(s) = Z_2^*(s) + q_{21}^*(s)A_1^*(s) + q_{23}^*(s)A_3^*(s) + q_{25}^*(s)A_5^*(s) \]
\[ A_3^*(s) = Z_3^*(s) + q_{31}^{(3,7)}(s)A_1^*(s) + q_{32}^*(s)A_2^*(s) \]
\[ A_4^*(s) = q_{41}^{(3,7)}(s)A_1^*(s) + q_{42}^{(3)}(s)A_2^*(s) \]
\[ A_5^*(s) = Z_5^*(s) + [q_{51}^*(s) + q_{51}^{(6,7)}(s)]A_1^*(s) + q_{52}^*(s)A_2^*(s) \]
\[ A_8^*(s) = Z_8^*(s) + q_{80}^*(s)A_0^*(s) \]

On solving above equations, the Laplace transformation of the point wise availability is
\[ A_0'(s) = \frac{N_2(s)}{D_2(s)} \]  

where

\[ N_2(s) = Z_0' \{ q_{14} q_{25} q_{52} q_{41}(s) - q_{42}(s) (q_{51} + q_{51}^{(6,7)}) - q_{25} (q_{52} + q_{12} (q_{51} + q_{51}^{(6,7)}) \} + 
(1 - q_{23} q_{32}) - q_{12}(q_{21} + q_{23} q_{31}) - q_{14} q_{23} (q_{31} q_{42} - q_{41} q_{32}) - q_{14} (q_{21} q_{42} + q_{41} q_{32}) + Z_1[q_{01} + q_{02} q_{21} - q_{23} (q_{01} q_{32} - q_{31} q_{02}) - q_{01} q_{25} q_{52} + q_{02} q_{25} q_{51}^{(6,7)}] + Z_2[q_{01} (q_{12} + q_{14} q_{42}) + q_{02} (1 - q_{41} q_{41}^{(6,7)})] + Z_3 q_{23} [q_{01} q_{12} + q_{02} + q_{14} (q_{01} q_{42} - q_{41} q_{02})] + Z_5 q_{25} [q_{01} q_{12} + q_{02} + q_{14} (q_{01} q_{42} - q_{41} q_{02})] + Z_6[q_{01} q_{18} (1 - q_{23} q_{32})] - q_{01} q_{18} q_{25} q_{52} + q_{18} q_{02} q_{25} (q_{51} + q_{51}^{(6,7)}) + q_{02} q_{18} (q_{21} + q_{23} + q_{31})] \]

and

\[ D_2(s) = q_{14} q_{25} [q_{52} q_{41}(s) - q_{42}(s) (q_{51} + q_{51}^{(6,7)})] - q_{25} [q_{52} + q_{12} (q_{51} + q_{51}^{(6,7)})] + 1 - q_{23} q_{32} - q_{12}(q_{21} + q_{23} q_{31}) - q_{14} q_{23} (q_{31} q_{42} - q_{41} q_{32}) - q_{14} (q_{21} q_{42} + q_{41} q_{32}) + q_{80} q_{01} q_{18} q_{25} q_{52} - q_{80} q_{01} q_{18} (1 - q_{23} q_{32}) - q_{80} q_{01} q_{02} q_{25} (q_{51} + q_{51}^{(6,7)}) - q_{80} q_{02} q_{18} (q_{21} + q_{23} + q_{31})] \]

The steady state availability of the system will be given by

\[ A_0 = \lim_{t \to \infty} A_0(t) = \lim_{s \to 0} s A_0'(s) = N_2(0)/D_2(0) \]

As we know that, \( q_{ij}(t) \) is the pdf of the time of transition from state \( S_i \) to \( S_j \) and \( q_{ij}(t)dt \) is the probability of transition from state \( S_i \) to \( S_j \) during the interval \( (t, t + dt) \), thus

\[ q_{ij}(s)|_{s=0} = q_{ij}^{(0)} = p_{ij} \] and

\[ -q_{ij}'(s)|_{s=0} = m_{ij} \]

Also we know that

\[ \lim_{s \to 0} Z_i(s) = \int_0^\infty Z_i(t)dt = \mu_i \]
Therefore, we have

\[ Z_0^+(s) = \frac{1}{\alpha_1 + \beta_1} = \mu_0 \]

\[ Z_1^+(s) = \frac{1}{\alpha_2 + \beta_2 + \lambda} = \mu_1 \]

\[ Z_2^+(s) = \frac{1}{\alpha_1 + \alpha_2 + \gamma} = \mu_2 \]

\[ Z_3^+(s) = \frac{1}{\alpha_1} \left[ 1 - g^* (\alpha_1) \right] = \mu_3 \]

\[ Z_4^+(s) = \frac{1}{\lambda} = \mu_4 \]

\[ Z_5^+(s) = \frac{1}{\alpha_1 + \alpha_2} \left[ 1 - g^* (\alpha_1 + \alpha_2) \right] = \mu_5 \]

\[ Z_6^+(s) = \int_0^\infty \exp \left[ - (\alpha_1)t \right] \tilde{F}(t) \tilde{G}(t) \, dt = \mu_6 \]

\[ Z_7^+(s) = \int_0^\infty \tilde{F}(t) \, dt = \mu_7 \]

\[ Z_8^+(s) = \frac{1}{\lambda} = \mu_8 \]  

Using relations in (3) and (14) in (11), the expressions of \( N_2(0) \) and \( D_2(0) \) becomes

\[ N_2(0) = (1 - p_{23}p_{32} - p_{25}p_{52}) \left[ (\mu_0 + \mu_8)p_{18} + \mu_1 \right] + \left[ p_{02} + p_{01}p_{12} + p_{14} (p_{01}p_{42}^{(3,7)} + p_{02}p_{41}^{(3,7)}) \right] \left[ \mu_2 + \mu_3p_{23} + \mu_5p_{25} \right] \]  

(14)

\[ D_2(0) = p_{14}p_{25} [p_{52}p_{41}^{(3,7)} - p_{42}^{(3)} (p_{51} + p_{51}^{(6,7)})] - p_{25} [p_{52}p_{12} (p_{51} + p_{51}^{(6,7)})] + \]

\[ (1 - p_{23}p_{32}) - p_{12} (p_{21} + p_{23}p_{31}^{(7)}) - p_{14}p_{23} (p_{31}p_{42}^{(3)} - p_{41}^{(3,7)} p_{32}) - \]

\[ p_{14} (p_{21}p_{42}^{(3)} - p_{41}^{(3,7)}) - p_{01}p_{18}p_{80} (1 - p_{23}p_{32} - p_{25}p_{52}) - p_{18}p_{80}p_{02} \]

\[ (p_{21} + p_{23}p_{31}^{(7)}) - p_{18}p_{80}p_{02}p_{25} (p_{51} + p_{51}^{(6,7)}) \]

\[ = p_{14}p_{25}p_{52} - p_{14}p_{25}p_{42}^{(3)} - p_{12}p_{25} - p_{14}p_{25}p_{52} - p_{18}p_{25}p_{52} + (1 - p_{23}p_{32}) \]

\[ (1 - p_{01}p_{18}) - (p_{21} + p_{23}p_{31}^{(7)}) (p_{12} + p_{02}p_{18}) - p_{14}p_{23} (p_{31}p_{42}^{(3)} - p_{41}^{(3,7)} p_{32}) \]

\[ - p_{14} (p_{21}p_{42}^{(3)} - p_{41}^{(3,7)}) + p_{01}p_{18}p_{25}p_{52} - p_{02}p_{18}p_{25} + p_{02}p_{18}p_{25}p_{52} \]

\[ = p_{12}p_{23} - p_{14}p_{42}^{(3)} (p_{21} + p_{25}) - p_{12} + 1 - p_{01}p_{18} - p_{23}p_{32} + p_{01}p_{18}p_{23}p_{32} - \]

\[ p_{02}p_{18} (p_{21} + p_{25}) - p_{12}p_{23}p_{31}^{(7)} - p_{02}p_{18}p_{23}p_{31}^{(7)} - p_{14}p_{23} (p_{31}p_{42}^{(3)} - p_{41}^{(3,7)} p_{32}) \]

\[ + p_{14}p_{23}p_{32}p_{41}^{(3,7)} \]

\[ = p_{14}p_{23}p_{42}^{(3)} (1 - p_{31}^{(7)}) - p_{14}p_{42}^{(3)} + p_{41}^{(3,7)} + 1 - p_{12} - p_{23}p_{32} - p_{01}p_{18} + \]

\[ p_{12}p_{23} (1 - p_{31}^{(7)}) + p_{01}p_{18}p_{23}p_{32} - p_{02}p_{18} + p_{02}p_{18}p_{23} - p_{02}p_{18}p_{23}p_{31}^{(7)} + \]

\[ p_{14}p_{41}^{(3,7)} p_{23}p_{32} \]
The steady state probability that the system will be up in the long run is given by

\[ A_0 = \lim_{t \to \infty} A_0(t) = \lim_{s \to 0} s A_0^*(s) \]

\[ A_0 = \lim_{s \to 0} \frac{sN_2(s)}{D_2(s)} \]

As \( s \to 0 \), \( D_2(s) \) becomes zero. Thus above equation becomes indeterminate form.

Hence on using L’Hospital’s rule, \( A_0 \) becomes

\[ A_0 = \frac{N_2(0)}{D_2(0)} \]  \hspace{1cm} (17)

Where \( N_2(0) \) is given by equation (15)

To obtain \( D_2'(0) \), we first collect the coefficients of \( m_{ij} \)'s as

The coefficient of \( m_{01} = p_{18}(1 - p_{23}p_{32}) - p_{18}p_{25}p_{52} \)

The coefficient of \( m_{02} = p_{18}p_{25}(p_{51} + p_{51}^{(6,7)}) + p_{18}(p_{21} + p_{23}p_{31})^{(7)} \)

\[ = p_{18}p_{25}(1 - p_{52}) + p_{18}p_{21} + p_{18}p_{23}p_{31}^{(7)} \]

\[ = p_{18} - p_{18}p_{21} - p_{18}p_{25}p_{52} + p_{18}p_{23}p_{31}^{(7)} + p_{18}p_{23} + p_{18}p_{21} \]

\[ = p_{18} - p_{18}p_{23} - p_{18}p_{25}p_{52} + p_{18}p_{23} - p_{18}p_{23}p_{32} \]
= p_{18}(1 - p_{25}p_{52} - p_{23}p_{32})

The coefficient of $m_{12} = p_{25}(p_{51} + p_{51}^{(6,7)}) + (p_{21} + p_{23}p_{31}^{(7)})$

= p_{25} - p_{25}p_{52} + p_{21} - p_{23}p_{31}^{(7)}

= 1 - p_{25}p_{52} - p_{23}(1 - p_{31}^{(7)})

= 1 - p_{25}p_{52} - p_{23}p_{32}

The coefficient of $m_{14} = p_{23}(p_{31}^{(7)}p_{42}^{(3)} - p_{41}^{(3,7)}p_{32}) - p_{25}[p_{52}p_{41}^{(3,7)} - p_{42}^{(3)}(p_{51} + p_{51}^{(6,7)})]$

+ p_{21}p_{42}^{(3)} + p_{41}^{(3,7)}

= p_{23}p_{31}^{(7)}p_{42}^{(3)} - p_{23}p_{41}^{(3,7)}p_{32} - p_{25}[p_{52}p_{41}^{(3,7)} - p_{42}^{(3)} + p_{42}^{(3)}p_{52}] +

p_{21}p_{42}^{(3)} + p_{41}^{(3,7)}

= p_{25}p_{42}^{(3)} - p_{25}p_{52} + p_{23}p_{51}^{(7)}p_{42}^{(3)} - p_{23}p_{41}^{(3,7)}p_{32} + p_{42}^{(3)}p_{21} + p_{41}^{(3,7)}

= 1 - p_{25}p_{52} + p_{23}p_{32}p_{42}^{(3)} - p_{23}p_{41}^{(3,7)}p_{32}

= 1 - p_{25}p_{52} - p_{23}p_{32}(p_{42}^{(3)} + p_{41}^{(3,7)})

= 1 - p_{25}p_{52} - p_{23}p_{32}

The coefficient of $m_{18} = p_{80}p_{02}p_{25}(p_{51} + p_{51}^{(6,7)}) + p_{80}p_{01}(1 - p_{23}p_{32}) - p_{18}p_{80}p_{25}p_{52}$

+ p_{80}p_{02}(p_{21} + p_{23}p_{31}^{(7)})

= p_{01} - p_{01}p_{25}p_{52} - p_{01}p_{23}p_{32} + p_{02}p_{25} - p_{02}p_{25}p_{52} + p_{02}p_{21}

+ p_{02}p_{23}p_{31}^{(7)}

= 1 - p_{25}p_{52} - p_{02}p_{23}(1 - p_{31}^{(7)}) - p_{01}p_{23}p_{32}

= 1 - p_{25}p_{52} - p_{01}p_{23}p_{32} - p_{02}p_{23}p_{32}

= 1 - p_{25}p_{52} - p_{23}p_{32}

The coefficient of $m_{21} = p_{12} + p_{14}p_{42}^{(3)} + p_{18}p_{02}$
The coefficient of \( m_{23} \) = \( p_{32} + p_{14}(p_{31}^{(7)}p_{42}^{(3)} - p_{41}^{(3,7)}p_{32}) - p_{01}p_{18}p_{32} + p_{02}p_{18}p_{31}^{(7)} + p_{12}p_{31}^{(7)} \)
= \( p_{32} + p_{14}p_{42}^{(3)} + p_{02}p_{18} - p_{04}p_{32} - p_{18}p_{32}p_{31}^{(7)} + p_{12}p_{31}^{(7)} \)
= \( p_{32} + p_{14}p_{42}^{(3)} + p_{02}p_{18} - p_{32}(1 - p_{12}) + p_{12}p_{31}^{(7)} \)
= \( p_{32} + p_{14}p_{42}^{(3)} + p_{02}p_{18} - p_{32}p_{12} + p_{12}p_{31}^{(7)} \)
= \( p_{12} + p_{14}p_{42}^{(3)} + p_{18}p_{02} \)

The coefficient of \( m_{25} \) = \( p_{52} + p_{12}(p_{51} + p_{51}^{(6,7)}) - p_{14}[p_{52}p_{41}^{(3)} - p_{42}^{(3)}(p_{51} + p_{51}^{(6,7)})] - p_{01}p_{18}p_{52} + p_{02}p_{18}(p_{51} + p_{51}^{(6,7)}) \)
= \( (1 - p_{52})(p_{14}p_{42}^{(3)} + p_{12}p_{02}p_{18}) - p_{14}p_{52}p_{41}^{(3,7)} + p_{52} - p_{01}p_{18}p_{52} \)
= \( p_{12} + p_{14}p_{42}^{(3)} + p_{18}p_{02} - p_{14}p_{42}^{(3)}p_{52} - p_{12}p_{52} - p_{02}p_{18}p_{52} - p_{14}p_{52}p_{41}^{(3,7)} + p_{52} - p_{01}p_{18}p_{52} \)
= \( p_{12} + p_{14}p_{42}^{(3)} + p_{18}p_{02} - p_{14}p_{52} + p_{14}p_{52} - p_{18}p_{52} + p_{18}p_{52} \)
= \( p_{12} + p_{14}p_{42}^{(3)} + p_{18}p_{02} \)

The coefficient of \( m_{31}^{(7)} \) = \( p_{12}p_{23} + p_{14}p_{42}^{(3)}p_{23} + p_{18}p_{02}p_{23} \)

The coefficient of \( m_{32} \) = \( p_{23} + p_{14}p_{41}^{(3,7)}p_{23} + p_{01}p_{18}p_{23} \)
= \( p_{23}(1 - p_{14}) + p_{14}p_{42}^{(3)}p_{23} + p_{02}p_{18}p_{23} - p_{18}p_{23} \)
= \( p_{23}(1 - p_{14} - p_{18}) + p_{14}p_{42}^{(3)}p_{23} + p_{02}p_{18}p_{23} \)
= \( p_{12}p_{23} + p_{14}p_{42}^{(3)}p_{23} + p_{18}p_{02}p_{23} \)

The coefficient of \( m_{41}^{(3,7)} \) = \( p_{14} - p_{14}p_{25}p_{52} - p_{14}p_{23}p_{32} \)
The coefficient of \( m_{42}^{(3)} = p_{14}p_{25}(p_{51} + p_{51}^{(6,7)}) - p_{14}p_{23}p_{31}^{(7)} + p_{21}p_{14} \)

\[= p_{14} - p_{14}p_{25}p_{52} - p_{14}p_{23}p_{32} \]

\[= p_{14}(1 - p_{25}p_{52} - p_{23}p_{32}) \]

The coefficient of \( m_{51} = p_{14}p_{42}p_{25} + p_{12}p_{25} + p_{02}p_{18}p_{23} \)

The coefficient of \( m_{51}^{(6,7)} = p_{14}p_{42}p_{25} + p_{12}p_{25} + p_{02}p_{18}p_{23} \)

The coefficient of \( m_{52} = p_{25} - p_{14}p_{42}p_{25} - p_{01}p_{18}p_{25} \)

\[= p_{25} - p_{14}p_{25} + p_{14}p_{42}p_{25} - p_{18}p_{25} + p_{02}p_{18}p_{23} \]

\[= p_{14}p_{42}p_{25} + p_{12}p_{25} + p_{02}p_{18}p_{23} \]

The coefficient of \( m_{80} = p_{01}p_{18}(1 - p_{23}p_{32}) + p_{02}p_{18}p_{25}(p_{51} + p_{51}^{(6,7)}) - \)

\[p_{01}p_{18}p_{25}p_{52} + p_{02}p_{18}(p_{21} + p_{23}p_{31}^{(7)}) \]

\[= p_{01}p_{18} - p_{01}p_{18}p_{23}p_{32} + p_{02}p_{18}p_{25} - p_{18}p_{25}p_{52} - \]

\[p_{02}p_{18}p_{21} + p_{02}p_{18}p_{23}p_{31}^{(7)} \]

\[= p_{18}(p_{01} + p_{02}) - p_{18}p_{23}p_{32}(p_{01} + p_{02}) - p_{18}p_{25}p_{52} \]

\[= p_{18} - p_{18}p_{23}p_{32} - p_{18}p_{25}p_{52} \]

Therefore, on collecting the above coefficients and using the relations (3), denominator of (17) becomes

Therefore \( D_2'(0) = \mu_0p_{18}p_{80}[1 - p_{25}p_{52} - p_{23}p_{32}] + \mu_1[1 - p_{25}p_{52} - p_{23}p_{32}] + \)

\[\mu_2[p_{12} + p_{14}p_{42}^{(3)} + p_{18}p_{02}] + \mu_3p_{23}[p_{12} + p_{14}p_{42}^{(3)} + p_{18}p_{02}] + \]

\[\mu_4p_{14}[1 - p_{25}p_{52} - p_{23}p_{32}] + \mu_5p_{25}[p_{12} + p_{14}p_{42}^{(3)} + p_{18}p_{02}] + \]

\[\mu_8p_{18}[1 - p_{25}p_{52} - p_{23}p_{32}] \]
\[
= [1 - p_{25}p_{52} - p_{23}p_{32}][\mu_0p_{18} + \mu_1 + \mu_4p_{14} + \mu_6p_{18}] + \\
[p_{12} + p_{14}p_{42}^{(3)} + p_{18}p_{02}][\mu_2 + \mu_3p_{23} + \mu_5p_{25}]
\]

(18)

Now by substituting the values of \(\mu_i\)'s and \(p_{ij}\)'s in (17), we get the expressions for numerator and denominator of steady state availability as

\[
N_2(0) = [\alpha_1(\alpha_1 + \alpha_2)][(\alpha_1 + \alpha_2 + \gamma) - \alpha_2f^*(\alpha_1) - \gamma g^*(\alpha_1 + \alpha_2)][\beta_2(1 + \lambda) + \lambda (\alpha_1 + \beta_1)] + \lambda[\beta_1(\alpha_2 + \beta_2 + \lambda) + \alpha_1\lambda + \alpha_1\alpha_2\{1 - f^*(\lambda)\} + \alpha_2\beta_1f^*(\lambda)]
\]
\[
[\alpha_1(\alpha_1 + \alpha_2) + \alpha_2(\alpha_1 + \alpha_2)\{1 - f^*(\alpha_1)\} + \alpha_1\gamma\{1 - g^*(\alpha_1 + \alpha_2)\}]
\]

and

\[
D_2(0) = [\alpha_1(\alpha_1 + \alpha_2)][(\alpha_1 + \alpha_2 + \gamma) - \alpha_2f^*(\alpha_1) - \gamma g^*(\alpha_1 + \alpha_2)][\beta_2\lambda + \alpha_1\alpha_2\beta_1 + \lambda(\alpha_1 + \beta_1) + \alpha_1\beta_1\beta_2] + \lambda[\beta_1(\alpha_1 + \beta_1) + \alpha_2(\alpha_1 + \beta_1)\{1 - f^*(\lambda)\} + \beta_1\beta_2]\n\]
\[
[\alpha_1(\alpha_1 + \alpha_2) + \alpha_2(\alpha_1 + \alpha_2)\{1 - f^*(\alpha_1)\} + \alpha_1\gamma\{1 - g^*(\alpha_1 + \alpha_2)\}]
\]

2.7 BUSY PERIOD ANALYSIS

Let \(B_i(t)\) be the probability that the system is under repair at epoch \(t\) given that the system entered regenerative state \(S_i\) at \(t = 0\). Now we will determine these probabilities. To illustrate the calculations we consider \(B_0(t)\) and similar arguments may be employed for other probabilities.

\(B_0(t)\) consists of the sum of the following independent contingencies and their respective probabilities:

1. The system transits to state \(S_1\) from \(S_0\) during \((u, u + du), u \leq t\) and then repairman may be found busy at epoch \((t - u)\), starting from \(S_1\), the probability of this event is

\[
\int_0^t q_{01}(t)B_1(t - u)du = q_{01}(t)\circ B_1(t)
\]

2. The system transits to state \(S_2\) from \(S_0\) during \((u, u + du), u \leq t\) and then repairman may be found busy at epoch \((t - u)\), starting from \(S_2\), the probability of this event is

\[
\int_0^t q_{02}(t)B_2(t - u)du = q_{02}(t)\circ B_2(t)
\]
the expression for $B_0(t)$ becomes

$$B_0(t) = q_{01}(t)\oplus B_1(t) + q_{02}(t)\oplus B_2(t)$$

By similar arguments we have

$$B_1(t) = Z_1(t) + q_{12}(t)\oplus B_2(t) + q_{14}(t)\oplus B_4(t) + q_{18}(t)\oplus B_8(t)$$

$$B_2(t) = q_{21}(t)\oplus B_1(t) + q_{23}(t)\oplus B_3(t) + q_{25}(t)\oplus B_5(t)$$

$$B_3(t) = Z_3(t) + q_{31}^{(7)}(t)\oplus B_1(t) + q_{32}(t)\oplus B_2(t)$$

$$B_4(t) = Z_4(t) + q_{41}^{(3,7)}(t)\oplus B_1(t) + q_{42}(t)\oplus B_2(t)$$

$$B_5(t) = [q_{51}(t) + q_{51}^{(6,7)}(t)]\oplus B_3(t) + q_{52}(t)\oplus B_2(t)$$

$$B_8(t) = Z_8(t) + q_{80}(t)\oplus B_0(t)$$

(19)

Taking Laplace transform of these relations, we get

$$B_0^*(s) = q_{01}^*(s)\oplus B_1^*(s) + q_{02}^*(s)\oplus B_2^*(s)$$

$$B_1^*(s) = Z_1^*(s) + q_{12}^{*}\oplus B_2^*(s) + q_{14}(s)\oplus B_4^*(s) + q_{18}(s)\oplus B_8^*(t)$$

$$B_2^*(s) = q_{21}(s)\oplus B_1^*(s) + q_{23}(s)\oplus B_3^*(s) + q_{25}(s)\oplus B_5^*(s)$$

$$B_3^*(s) = Z_3^*(s) + q_{31}^{*(7)}(s)\oplus B_0^*(s) + q_{32}(s)\oplus B_2^*(s)$$

$$B_4^*(s) = Z_4^*(s) + q_{41}^{*(3,7)}(s)\oplus B_1^*(s) + q_{42}(s)\oplus B_2^*(s)$$

$$B_5^*(s) = [q_{51}^*(s) + q_{51}^{*(6,7)}(s)]\oplus B_3^*(s) + q_{52}(s)\oplus B_2^*(s)$$

$$B_8^*(s) = Z_8^*(s) + q_{80}(s)\oplus B_0^*(s)$$

on solving above equations for $B_0^*(s)$ we have

$$B_0^*(s) = \frac{N_3(s)}{D_3(s)}$$

(20)

where

$$N_3(s) = Z_1^*\{q_{01}^* + q_{02}^*q_{21} - q_{23}^*(q_{01}^*q_{32} - q_{31}^*q_{02}^*) - q_{01}^*q_{25}q_{52} + q_{02}^*q_{25}
\]

$$(q_{51}^* + q_{51}^{*}\{q_{01}^*q_{23} + q_{02}^*q_{12} + q_{02}^*q_{14}(q_{25}^*q_{42}^* - q_{25}^*q_{41}^{*})
\} + Z_4^*q_{14}^*q_{01}^* + q_{02}^*q_{21} - q_{23}^*(q_{01}^*q_{32} - q_{02}^*q_{31}^{*}) - q_{01}^*q_{52} + q_{02}^*(q_{51}^*
\} + q_{51}^{*}\{q_{01}^*(1 - q_{23}^*q_{32} - q_{25}^*q_{52}^*) + q_{02}^*q_{25}^*(q_{51}^* + q_{51}^{*}
\) + q_{02}^*q_{25}^*(q_{51}^* + q_{51}^{*}\{q_{21}^* + q_{23}^*q_{31}^{*}\})$$
and

\( D_3(s) \) is same as \( D_2(s) \) as obtained in availability is given by (13)

Thus in the long run, the fraction of time for which system is under repair is given by

\[
B_0 = \lim_{t \to \infty} B_0(t) = \lim_{s \to 0} sB'_0(s) = \frac{N_3(0)}{D'_3(0)} \tag{21}
\]

where

\[
N_3(0) = [1 - p_{25}p_{52} - p_{23}p_{32}][\mu_1 + \mu_8p_{18}] + \mu_3p_{23}[p_{01}p_{12} + p_{02} + p_{14}(p_{01}p_{42}^{(3)}) - p_{02}p_{41}^{(3,7)}] + \mu_4p_{14}[1 - p_{23}p_{32} - p_{02}p_{25} + p_{02} - p_{52}] \tag{22}
\]

and

\( D'_3(0) = D'_2(0) \) is same as obtained in availability analysis which is given by (18)

Now by substituting the values of \( \mu_i \)'s and \( p_{ij} \)'s in (21), we get the expressions for numerator and denominator of \( B_0 \) as

\[
N_3(0) = [\alpha_1 + \alpha_2][\alpha_1 + \beta_1][\alpha_1(\lambda + \beta_2)][(\alpha_1 + \alpha_2 + \gamma) - \alpha_2f^*(\alpha_1) - \gamma g^*(\alpha_1 + \alpha_2)] + [\alpha_1 + \alpha_2][\alpha_2\lambda[1 - f^*(\alpha_1)]][\beta_1(\alpha_2 + \beta_2 + \lambda) + \alpha_1\lambda + \alpha_1\alpha_2[1 - f^*(\lambda)] + \alpha_2\beta_1f^*(\lambda)] + [\alpha_1 + \alpha_2][\alpha_1\alpha_2(\alpha_1 + \alpha_2 + \gamma)(\alpha_1 + \beta_1) - \alpha_2(\alpha_1 + \beta_1)f^*(\alpha_1) - \beta_1\gamma + \beta_1(\alpha_1 + \alpha_2 + \gamma) - (\alpha_1 + \beta_1)(\alpha_1 + \alpha_2 + \gamma)g^*(\alpha_1 + \alpha_2)]
\]

and

\[
D'_3(0) = [\alpha_1(\alpha_1 + \alpha_2)][(\alpha_1 + \alpha_2 + \gamma) - \alpha_2f^*(\alpha_1) - \gamma g^*(\alpha_1 + \alpha_2)][\beta_2\lambda + \alpha_1\alpha_2\beta_1 + \lambda(\alpha_1 + \beta_1) + \alpha_1\beta_1\beta_2] + \lambda[\lambda(\alpha_1 + \beta_1) + \alpha_2(\alpha_1 + \beta_1)[1 - f^*(\lambda) + \beta_1\beta_2]]
\]

\[
[\alpha_1(\alpha_1 + \alpha_2) + \alpha_2(\alpha_1 + \alpha_2)[1 - f^*(\alpha_1)] + \alpha_1\gamma[1 - g^*(\alpha_1 + \alpha_2)]]
\]

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2.8 EXPECTED NUMBER OF VISITS BY THE REPAIR FACILITIES

Let us define \( V_i(t) \) as the expected no. of repairs of the unit during the time interval \((0, t]\) when the system initially starts from regenerative state \( S_i \). Using the definition of \( V_i(t) \), the recursive relation among \( V_i(t) \)'s can be easily developed.

\[
V_0(t) = Q_{01}(t)\bar{V}_1(t) + Q_{02}(t)\bar{V}_2(t)
\]
\[
V_1(t) = Q_{12}(t)[1 + V_2(t)] + Q_{14}(t)\bar{V}_4(t) + Q_{10}(t)\bar{V}_8(t)
\]
\[
V_2(t) = Q_{21}(t)\bar{V}_1(t) + Q_{23}(t)\bar{V}_3(t) + Q_{25}(t)\bar{V}_5(t)
\]
\[
V_3(t) = Q_{31}^{(7)}(t)[1 + V_1(t)] + Q_{32}(t)[1 + V_2(t)]
\]
\[
V_4(t) = Q_{41}^{(3,7)}(t)[1 + V_1(t)] + Q_{42}^{(3)}(t)[1 + V_2(t)]
\]
\[
V_5(t) = \left[ Q_{51}(t) + Q_{51}^{(6,7)}(t) \right] \bar{V}_1(t) + Q_{52}(t)\bar{V}_2(t).
\]
\[
V_8(t) = Q_{80}(t)[1 + V_0(t)]
\]

Taking Laplace Stieltjes transformation of these relations, we get

\[
\bar{V}_0(s) = \tilde{Q}_{01}(s)\bar{V}_1(s) + \tilde{Q}_{02}(s)\bar{V}_2(s)
\]
\[
\bar{V}_1(s) = \tilde{Q}_{12}(s)[1 + \bar{V}_1(s)] + \tilde{Q}_{14}(s)\bar{V}_4(s)
\]
\[
\bar{V}_2(s) = \tilde{Q}_{21}(s)[1 + \bar{V}_1(s)] + \tilde{Q}_{23}(s)\bar{V}_3(s) + \tilde{Q}_{25}(s)\bar{V}_5(s)
\]
\[
\bar{V}_3(s) = \tilde{Q}_{31}^{(7)}(s)[1 + \bar{V}_1(s)] + \tilde{Q}_{32}(s)[1 + \bar{V}_2(s)]
\]
\[
\bar{V}_4(s) = \tilde{Q}_{41}^{(3,7)}(s)[1 + \bar{V}_1(s)] + \tilde{Q}_{42}^{(3)}(s)[1 + \bar{V}_2(s)]
\]
\[
\bar{V}_5(s) = \left[ \tilde{Q}_{51}(s) + \tilde{Q}_{51}^{(6,7)}(s) \right] \bar{V}_1(s) + \tilde{Q}_{52}(s)\bar{V}_2(s)
\]
\[
\bar{V}_8(s) = \tilde{Q}_{80}(s)[1 + \bar{V}_0(s)]
\]

On solving the above equations \( \bar{V}_0(s) \) the Laplace Stieltjes transformation of the expected number of visits is given by we get

\[
\bar{V}_0(s) = \frac{N_4(s)}{D_4(s)}
\]

Where

\[
N_4(s) = \tilde{Q}_{01}\tilde{Q}_{01} + \tilde{Q}_{02}\tilde{Q}_{21} - \tilde{Q}_{23}(\tilde{Q}_{01}\tilde{Q}_{32} - \tilde{Q}_{02}\tilde{Q}_{31}) - \tilde{Q}_{01}\tilde{Q}_{25}\tilde{Q}_{52} + \tilde{Q}_{02}\tilde{Q}_{25}
\]
\[
(\tilde{Q}_{51} + \tilde{Q}_{51}^{(6,7)}) + (\tilde{Q}_{31}^{(7)} + \tilde{Q}_{32})\tilde{Q}_{23}[\tilde{Q}_{01}\tilde{Q}_{12} + \tilde{Q}_{02} + \tilde{Q}_{14}(\tilde{Q}_{01}\tilde{Q}_{42} -
\]

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and $D_4(s)$ can be obtained by replacing $q_{ij}$'s by $\tilde{q}_{ij}$'s equation (13)

In steady state the number of visits per unit time is given by

$$V_0 = \lim_{t \to \infty} V_0(t) = \lim_{s \to 0} s \tilde{V}_0(s) = \frac{N_4(0)}{D_4'(0)} \quad (25)$$

Where

$$N_4(0) = [1 - p_{25}p_{52} - p_{23}p_{32}] [p_{12} + p_{18}] + p_{23} [p_{01}p_{12} + p_{02} + p_{14} (p_{01}p_{42}^{(3)} - p_{02}p_{41}^{(3,7)})] + p_{14} [1 - p_{23}p_{32} - p_{02}p_{25} + p_{02} - p_{52}] \quad (26)$$

and $D_4'(0) = D_2'(0)$ is already specified in equation (18)

Now by substituting the values of $\mu_i$'s and $p_{ij}$'s in (25), we get the expressions for numerator and denominator of $V_0$ as

$$N_4(0) = [\alpha_1 \lambda (\alpha_1 + \alpha_2)] [\alpha_1 + \beta_1] (\lambda + \beta_2) [(\alpha_1 + \alpha_2 + \gamma) - \gamma g^*(\alpha_1 + \alpha_2) - \alpha_2 f^*(\alpha_1)] + [\alpha_1 \lambda (\alpha_1 + \alpha_2)] \alpha_2 [\beta_1 (\alpha_2 + \beta_2 + \lambda) + \alpha_1 \lambda + \alpha_2 \beta_1 f^*(\lambda) + \alpha_1 \alpha_2 (1 - f^*(\lambda))] + [\alpha_1 \lambda (\alpha_1 + \alpha_2)] \alpha_2 [(\alpha_1 + \alpha_2 + \gamma) (\alpha_1 + \beta_1) + \beta_1 \gamma - \alpha_2 (\alpha_1 + \beta_1) f^*(\alpha_1) + \beta_1 (\alpha_1 + \alpha_2 + \gamma) - (\alpha_1 + \alpha_2 + \gamma) g^*(\alpha_1 + \alpha_2)] (\alpha_1 + \beta_1)$$

and

$$D_4'(0) = [\alpha_1 (\alpha_1 + \alpha_2)] [(\alpha_1 + \alpha_2 + \gamma) - \alpha_2 f^*(\alpha_1) - \gamma g^*(\alpha_1 + \alpha_2)] [\beta_2 \lambda + \alpha_1 \alpha_2 \beta_1 + \lambda (\alpha_1 + \beta_1) + \alpha_1 \beta_1 \beta_2] + \lambda [\lambda (\alpha_1 + \beta_1) + \alpha_2 (\alpha_1 + \beta_1) (1 - f^*(\lambda) + \beta_1 \beta_2)] [\alpha_1 (\alpha_1 + \alpha_2) + \alpha_2 (\alpha_1 + \alpha_2) (1 - f^*(\alpha_1)) + \alpha \gamma (1 - g^*(\alpha_1 + \alpha_2))]$$

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2.9 Preventive Maintenance

Let us define $V_i(t)$ as the expected no. of repairs of the unit during the time interval $(0, t]$ when the system initially starts from regenerative state $S_i$. Using the definition of $V_i(t)$, the recursive relation among $V_i(t)$’s can be easily developed

\[
\begin{align*}
  P_0(t) &= q_{01}(t)P_1(t) + q_{02}(t)P_2(t) \\
  P_1(t) &= q_{12}(t)P_2(t) + q_{14}(t)P_4(t) + q_{18}(t)P_8(t) \\
  P_2(t) &= q_{21}(t)P_1(t) + q_{23}(t)P_3(t) + q_{25}(t)P_5(t) \\
  P_3(t) &= q_{31}^{(7)}(t)P_1(t) + q_{32}(t)P_2(t) \\
  P_4(t) &= q_{41}^{(3,7)}(t)P_1(t) + q_{42}^{(3)}(t)P_2(t) \\
  P_5(t) &= Z_5 + [q_{51}(t) + q_{51}^{(6,7)}(t)]P_1(t) + q_{52}(t)P_2(t) \\
  P_8(t) &= q_{80}(t)P_0(t)
\end{align*}
\]

Taking Laplace Stieltjes transformation of these relations, we get

\[
\begin{align*}
  P_0^*(s) &= q_{01}^*(s)P_1^*(s) + q_{02}^*(s)P_2^*(s) \\
  P_1^*(s) &= q_{12}^*(s)P_2^*(s) + q_{14}^*(s)P_4^*(s) + q_{18}^*(s)P_8^*(t) \\
  P_2^*(s) &= q_{21}^*(s)P_1^*(s) + q_{23}^*(s)P_3^*(s) + q_{25}^*(s)P_5^*(s) \\
  P_3^*(s) &= q_{31}^{(7)}(s)P_1^*(s) + q_{32}^*(s)P_2^*(s) \\
  P_4^*(s) &= q_{41}^{*(3,7)}(s)P_1^*(s) + q_{42}^{*(3)}(s)P_2^*(s) \\
  P_5^*(s) &= Z_5^*(s) + [q_{51}^*(s) + q_{51}^{*(6,7)}(s)]P_1^*(s) + q_{52}^*(s)P_2^*(s) \\
  P_8^*(s) &= q_{80}^*(s)P_0^*(s)
\end{align*}
\]

on solving the equation (27) we get

\[
\bar{P}_0(s) = \frac{N_5(s)}{D_5(s)}
\]

where

\[
N_5(s) = Z_5^*q_{21}^{*}[q_{01}^*q_{12}^* + q_{02}^* + q_{14}^*(q_{01}^*q_{42}^{(3)} - q_{41}^{*(3,7)}q_{02}^*)] \\
\text{and } D_5(s) = D_2(s) \text{ is already specified in eq (13)}
\]
In steady state the number of visits per unit time is given by

\[ P_0 = \lim_{t \to \infty} P_0(t) = \lim_{s \to 0} s \tilde{P}_0(s) = \frac{N_5(0)}{D'_5(0)} \tag{30} \]

\[ N_5(0) = \mu_5 P_{25} [p_{01}p_{12} + p_{02} + p_{14} (p_{01}p_{42}^{(3)} - p_{02}p_{41}^{(3,7)})] \tag{31} \]

and \( D'_5(0) = D_2(0) \) is already specified in eq (18)

Now by substituting the values of \( \mu_i 's \) and \( p_{ij} 's \) in (30), we get the expressions for numerator and denominator of \( P_0 \) as

\[ N_5(0) = \alpha_1 \lambda \gamma [1 - g^*(\alpha_1 + \alpha_2)] [\beta_1 (\alpha_2 + \beta_2 + \lambda) + \alpha_1 \lambda + \alpha_1 \alpha_2 \{1 - f^*(\lambda)\} + \alpha_2 \beta_1 f^*(\lambda) \]

and

\[ D'_5(0) = [\alpha_1 (\alpha_1 + \alpha_2)] [(\alpha_1 + \alpha_2 + \gamma) - \alpha_2 f^*(\alpha_1) - \gamma g^*(\alpha_1 + \alpha_2)] [\beta_2 \lambda + \alpha_1 \alpha_2 \beta_1 + \lambda (\alpha_1 + \beta_1) + \alpha_1 \beta_1 \beta_2] + \lambda [\lambda (\alpha_1 + \beta_1) + \alpha_2 (\alpha_1 + \alpha_2) \{1 - f^*(\lambda) + \beta_1 \beta_2\}] \]

\[ [\alpha_1 (\alpha_1 + \alpha_2) + \alpha_2 (\alpha_1 + \alpha_2) \{1 - f^*(\alpha_1)\} + \alpha_1 \gamma (1 - g^*(\alpha_1 + \alpha_2))] \]

### 2.10 COST BENEFIT ANALYSIS

The expected uptime and down time of the system and busy period of the repair man in \((0, t]\) are

\[ \mu_{up}(t) = \int_0^t A_0(u)du \]

and

\[ \mu_{dn}(t) = t - \mu_{up}(t) \]

and

\[ \mu_b = \int_0^t B_0(u)du \]

So that
\[ \mu_{up}^*(s) = A_0^*(s)/s \]
\[ \mu_{dn}^*(s) = 1/s^2 - \mu_{up}^*(s) \]

and
\[ \mu_h^*(s) = B_0^*(s)/s \]

The expected profits incurred in \((0, t]\) = expected total revenue in \((0, t]\) – expected total repair in \((0, t]\) – expected cost of visits by repairman in \((0, t]\).

we have profit functions as
\[ P_1 = K_0 A_0 - K_1 B_0 - K_2 V_0 \]  \hfill (32)
\[ P_2 = K_0 A_0 - K_1 B_0 - K_3 P_0 \]  \hfill (33)

where
\[ K_0 = \text{Revenue per unit up time of the system}, \]
\[ K_1 = \text{Cost per unit time for which the repair is busy}, \]
\[ K_2 = \text{Cost per unit visits by the repairman}. \]
\[ K_3 = \text{Cost per unit preventive maintenance by the repairman}. \]

2.11 GRAPHICAL STUDY OF SYSTEM BEHAVIOUR

The behavior of MTSF and availability of the system is studied graphically in this section and to plot their graphs, the replacement and repair time distributions are also assumed to be distributed exponentially. The graphs of MTSF and that of availability are depicted with respect to the different parameters. It is observed that the MTSF decreases uniformly as the failure rates of the system increases irrespective of the other fixed parameters. However, we note that MTSF increases with increasing repair rates. Thus, we can conclude that the expected life of the system can be increased by increasing repair rate of the unit. The availability of the system gradually decreases with increasing failure rates irrespective of type of failure and increases with increasing repair rate of the unit. Further, the profit function decreases with the increase in the failure rate of the system and it increases with the increase in the value of repair rate of the unit.
For fixed values of the parameters $\alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1, \beta_2$ and changing $\alpha_1$, TABLE-1 is obtained.

**TABLE-1:** Effect of $\alpha_1$ and fixed parameters $\alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1$ and $\beta_2$ on MTSF for two different values of $\lambda$.

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>MTSF</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\alpha_2 = 0.25, \alpha_3 = 0.12$ , $\gamma = 0.08, \gamma_2 = 0.15$ , $\lambda = 0.50, \beta_1 = 0.15, \beta_2 = 0.55$</td>
</tr>
<tr>
<td>0.1</td>
<td>27.1368</td>
</tr>
<tr>
<td>0.2</td>
<td>22.7861</td>
</tr>
<tr>
<td>0.3</td>
<td>19.9342</td>
</tr>
<tr>
<td>0.4</td>
<td>17.9720</td>
</tr>
<tr>
<td>0.5</td>
<td>16.5561</td>
</tr>
<tr>
<td>0.6</td>
<td>15.4930</td>
</tr>
<tr>
<td>0.7</td>
<td>14.6684</td>
</tr>
<tr>
<td>0.8</td>
<td>14.0116</td>
</tr>
<tr>
<td>0.9</td>
<td>13.4771</td>
</tr>
<tr>
<td>1.0</td>
<td>13.0340</td>
</tr>
</tbody>
</table>

In fig. 2.2, we plot MTSF w.r.t. $\alpha_1$ and fixed values of parameters $\alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1$ and $\beta_2$ for two different values of $\lambda$. It is observed that MTSF of the system decreases w.r.t. $\alpha_1$ irrespective of the other parameters so that we conclude that expected life of the system increases with decreasing failure rate.
Behaviour of MTSF w. r. t. $\alpha_1$ for two different values of $\lambda$

Fig. 2.2
For fixed values of the parameters $\alpha_1, \alpha_2, \alpha_3, \gamma, \gamma_2, \beta_1, \beta_2$ and changing $\lambda$, TABLE-2 is obtained.

**TABLE-2**: Effect of $\lambda$ and fixed parameters $\alpha_1, \alpha_2, \alpha_3, \gamma, \gamma_2, \beta_1$ and $\beta_2$ on MTSF two different values of $\alpha_1$.

| $\lambda$ | MTSF \[\begin{align*}
\alpha_1 &= 0.02, \alpha_2 = 0.25, \\
\alpha_3 &= 0.12, \gamma = 0.08, \\
\gamma_2 &= 0.15, \beta_1 = 0.15 \\
\beta_2 &= 0.55
\end{align*}\
| $\alpha_1 = 0.06, \alpha_2 = 0.25, \\
\alpha_3 &= 0.12, \gamma = 0.08, \\
\gamma_2 &= 0.15, \beta_1 = 0.15 \\
\beta_2 &= 0.55
| 0.1 | 26.8732 | 28.3030 \\
| 0.2 | 26.8954 | 26.1458 \\
| 0.3 | 28.3843 | 26.7148 \\
| 0.4 | 30.2399 | 27.9653 \\
| 0.5 | 32.2421 | 29.4884 \\
| 0.6 | 34.3176 | 31.1479 \\
| 0.7 | 36.4351 | 32.8852 \\
| 0.8 | 38.5788 | 34.6713 \\
| 0.9 | 40.7399 | 36.4898 \\
| 1.0 | 42.9132 | 38.3310 |

In fig. 2.3, we plot MTSF w.r.t. $\lambda$ and fixed values of parameter $\alpha_1, \alpha_2, \alpha_3, \gamma, \gamma_2, \beta_1$ and $\beta_2$ for two different values of $\alpha_1$. It is quite clear that MTSF of the system increases w.r.t. $\lambda$ irrespective of the other parameters so that we conclude that expected life of the system increases with increasing repair rate.
Behaviour of MTSF w. r. t. $\lambda$ for two different values of $\alpha_1$

![Graph showing the behaviour of MTSF w.r.t. $\lambda$ for two different values of $\alpha_1$.]

Fig. 2.3
For fixed values of the parameters $\alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1, \beta_2$ and changing $\alpha_1$, TABLE-3 is obtained.

TABLE-3: Effect of $\alpha_1$ and fixed parameters $\alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1$ and $\beta_2$ on Availability for two different values of $\lambda$.

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_2 = 0.25, \alpha_3 = 0.12, \gamma = 0.08, \gamma_2 = 0.15, \lambda = 0.50, \beta_1 = 0.15, \beta_2 = 0.55$</th>
<th>$\alpha_2 = 0.25, \alpha_3 = 0.12, \gamma = 0.08, \gamma_2 = 0.15, \lambda = 0.95, \beta_1 = 0.15, \beta_2 = 0.55$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.99365</td>
<td>0.999970</td>
</tr>
<tr>
<td>0.2</td>
<td>0.98162</td>
<td>0.989820</td>
</tr>
<tr>
<td>0.3</td>
<td>0.97999</td>
<td>0.979064</td>
</tr>
<tr>
<td>0.4</td>
<td>0.95327</td>
<td>0.969360</td>
</tr>
<tr>
<td>0.5</td>
<td>0.93289</td>
<td>0.960086</td>
</tr>
<tr>
<td>0.6</td>
<td>0.91710</td>
<td>0.950836</td>
</tr>
<tr>
<td>0.7</td>
<td>0.90463</td>
<td>0.940698</td>
</tr>
<tr>
<td>0.8</td>
<td>0.89458</td>
<td>0.932196</td>
</tr>
<tr>
<td>0.9</td>
<td>0.88633</td>
<td>0.924975</td>
</tr>
<tr>
<td>1.0</td>
<td>0.87945</td>
<td>0.918769</td>
</tr>
</tbody>
</table>

In fig. 2.4, we plot Availability w.r.t. $\alpha_1$ and fixed values of parameter $\alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1$ and $\beta_2$ for two different values of $\lambda$. It is observed that Availability of the system decreases w.r.t. $\alpha_1$ irrespective of the other parameters. Therefore, we conclude that expected life of the system increases with decreasing failure rate.
Behaviour of Availability w.r.t. $\alpha_1$ for different values of $\lambda$

Fig. 2.4
For fixed values of the parameters $\alpha_1, \alpha_2, \alpha_3, \gamma, \gamma_2, \beta_1, \beta_2$ and changing $\alpha_1$, TABLE-4 is obtained.

TABLE-4: Effect of $\lambda$ and fixed parameters $\alpha_1, \alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1$ and on Availability for two different values of $\alpha_1$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Availability $\alpha_1 = 0.02, \alpha_2 = 0.25, \alpha_3 = 0.12, \gamma = 0.08, \gamma_2 = 0.15, \beta_1 = 0.15, \beta_2 = 0.55$</th>
<th>Availability $\alpha_1 = 0.06, \alpha_2 = 0.25, \alpha_3 = 0.12, \gamma = 0.08, \gamma_2 = 0.15, \beta_1 = 0.15, \beta_2 = 0.55$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.3876</td>
<td>0.2933</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4321</td>
<td>0.3772</td>
</tr>
<tr>
<td>0.3</td>
<td>0.4450</td>
<td>0.3975</td>
</tr>
<tr>
<td>0.4</td>
<td>0.4466</td>
<td>0.4060</td>
</tr>
<tr>
<td>0.5</td>
<td>0.4477</td>
<td>0.4103</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4490</td>
<td>0.4126</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4502</td>
<td>0.4141</td>
</tr>
<tr>
<td>0.8</td>
<td>0.4508</td>
<td>0.4149</td>
</tr>
<tr>
<td>0.9</td>
<td>0.4510</td>
<td>0.4155</td>
</tr>
<tr>
<td>1.0</td>
<td>0.4511</td>
<td>0.4159</td>
</tr>
</tbody>
</table>

In fig. 2.5, we plot Availability w.r.t. $\lambda$ and fixed values of parameter $\alpha_1, \alpha_2, \alpha_3, \gamma, \gamma_2, \beta_1$ and $\beta_2$ for two different values of $\alpha_1$. It is observed that Availability of the system increases w.r.t. $\lambda$ irrespective of the other parameters. Therefore, we conclude that expected life of the system increases with increasing repair rate.
Behaviour of Availability w.r.t. $\lambda$ for two different values of $\alpha_1$

Fig. 2.5
For fixed values of the parameters $\alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1, \beta_2, K_0, K_1, K_2, K_3$ and changing $\alpha_1$, Table-5 is obtained

TABLE-5: Effect of $\alpha_1$ and fixed parameters $\alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1$ and $\beta_2$ on Profit function for two different values of $\lambda$.

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>Profit function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_2 = 0.25, \alpha_3 = 0.12, \gamma = 0.08,$</td>
<td></td>
</tr>
<tr>
<td>$\gamma_2 = 0.15, \lambda = 0.50, \beta_1 = 0.15,$</td>
<td></td>
</tr>
<tr>
<td>$\beta_2 = 0.55, K_0 = 1000, K_1 = 500$</td>
<td></td>
</tr>
<tr>
<td>$K_2 = 400, K_3 = 300$</td>
<td></td>
</tr>
<tr>
<td>$\alpha_2 = 0.25, \alpha_3 = 0.12, \gamma = 0.08,$</td>
<td></td>
</tr>
<tr>
<td>$\gamma_2 = 0.15, \lambda = 0.95, \beta_1 = 0.15,$</td>
<td></td>
</tr>
<tr>
<td>$\beta_2 = 0.55, K_0 = 1000, K_1 = 500$</td>
<td></td>
</tr>
<tr>
<td>$K_2 = 400, K_3 = 300$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_1$</th>
<th>$P_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>175.543</td>
<td>201.596</td>
<td>195.096</td>
<td>224.198</td>
</tr>
<tr>
<td>0.2</td>
<td>146.459</td>
<td>179.725</td>
<td>173.047</td>
<td>212.668</td>
</tr>
<tr>
<td>0.3</td>
<td>120.941</td>
<td>159.779</td>
<td>151.486</td>
<td>200.070</td>
</tr>
<tr>
<td>0.4</td>
<td>99.5339</td>
<td>142.742</td>
<td>131.728</td>
<td>187.968</td>
</tr>
<tr>
<td>0.5</td>
<td>81.6784</td>
<td>128.380</td>
<td>113.976</td>
<td>176.800</td>
</tr>
<tr>
<td>0.6</td>
<td>66.7075</td>
<td>116.251</td>
<td>98.1153</td>
<td>166.645</td>
</tr>
<tr>
<td>0.7</td>
<td>54.0474</td>
<td>105.941</td>
<td>83.9483</td>
<td>157.459</td>
</tr>
<tr>
<td>0.8</td>
<td>43.2412</td>
<td>97.1062</td>
<td>71.2656</td>
<td>149.157</td>
</tr>
<tr>
<td>0.9</td>
<td>33.9328</td>
<td>89.4718</td>
<td>59.8747</td>
<td>141.644</td>
</tr>
<tr>
<td>1.0</td>
<td>25.8458</td>
<td>82.8222</td>
<td>49.6059</td>
<td>134.829</td>
</tr>
</tbody>
</table>

In fig. 2.6, we plot Profit Function w.r.t. $\alpha_1$ and fixed values of parameter $\alpha_2, \alpha_3, \lambda, \gamma, \gamma_2, \beta_1$ and $\beta_2$ for two different values of $\lambda$. It is observed that Profit Function of the system decreases w.r.t. $\alpha_1$ irrespective of the other parameters. Therefore, we conclude Profit increases with decreasing failure rate.
Behaviour of Profit function w. r. t. $\alpha_1$ for two different values of $\lambda$

Fig. 2.6
For fixed values of the parameters $\alpha_1, \alpha_2, \alpha_3, \gamma, \gamma_2, \beta_1, \beta_2$ and changing $\lambda$, Table-6 is obtained.

TABLE-6: Effect of $\lambda$ and fixed parameters $\alpha_1, \alpha_2, \alpha_3, \gamma, \gamma_2, \beta_1$ and $\beta_2$ on Profit Function for two different values of $\alpha_1$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>Profit function</th>
<th>$\alpha_2 = 0.25, \alpha_3 = 0.12, \gamma = 0.08, \gamma_2 = 0.15, \beta_1 = 0.15, \beta_2 = 0.55, K_0 = 1000, K_1 = 500$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$P_1$</td>
</tr>
<tr>
<td>0.1</td>
<td>105.136</td>
<td>119.339</td>
</tr>
<tr>
<td>0.2</td>
<td>157.943</td>
<td>174.754</td>
</tr>
<tr>
<td>0.3</td>
<td>179.315</td>
<td>197.203</td>
</tr>
<tr>
<td>0.4</td>
<td>190.561</td>
<td>209.028</td>
</tr>
<tr>
<td>0.5</td>
<td>197.341</td>
<td>216.165</td>
</tr>
<tr>
<td>0.6</td>
<td>201.786</td>
<td>220.851</td>
</tr>
<tr>
<td>0.7</td>
<td>204.874</td>
<td>224.111</td>
</tr>
<tr>
<td>0.8</td>
<td>207.112</td>
<td>226.477</td>
</tr>
<tr>
<td>0.9</td>
<td>208.788</td>
<td>228.251</td>
</tr>
<tr>
<td>1.0</td>
<td>210.074</td>
<td>229.616</td>
</tr>
</tbody>
</table>

In fig 2.7, we plot Profit Function w.r.t $\lambda$ and fixed values of parameter $\alpha_1, \alpha_2, \alpha_3, \gamma, \gamma_2, \beta_1$ and $\beta_2$ for two different values of $\alpha_1$. It is observed that Profit Function of the system increases w.r.t $\lambda$ irrespective of the other parameters. Therefore, we conclude Profit increases with increasing repair rate.
Behaviour of Profit function w.r.t. $\lambda$ for two different values of $\alpha_1$

Fig. 2.7
**Recommendations to the entrepreneur**:- On the basis of above analysis, it is recommended to the entrepreneur to adopt the tool of preventive maintenance and to increase the repair rates by enhancing the repair facilities which in turn ultimately lead to decrease in failure rate and increase in the profit of the firm.