APPENDIX-A

SOME MATHEMATICAL CONCEPTS AND DEFINITIONS

This appendix provides some important terms and definitions of mathematical concepts used throughout the thesis. Proofs are also given wherever necessary.

1 TRANSITION PROBABILITIES

Let $p_{ij}(m,n)$ denote the probability that the process will go from state $S_i$ to state $S_j$ in $(n-m)$ steps, given that it was in state $S_i$ at step $m$.

Mathematically,

$$p_{ij}(m,n) = P(X_n = j | X_m = i), \quad 0 \leq m \leq n$$

(i) One-step transition probability

It is defined as the probability that the process will go from state ‘i’ to ‘j’ in exactly one step given that it was in state ‘i’ previously.

It is given by

$$p_{ij} = P(X_m = j | X_{m-1} = i)$$

(ii) The n-step transition probability

It is defined as follows:

$$p_{ij}^{(n)} = P(X_{t+n} = j | X_t = i); \quad t, n \in T$$

with $\sum p_{ij}^{(n)} = 1$

(iii) Transition probability matrix (Stochastic matrix)

A square matrix with non negative elements and unit row sum is called transition probability matrix (t.p.m.) or stochastic matrix. It is doubly stochastic, if in addition, it has unit column sum.

The one step t.p.m. of order $n$ is defined as

$$P = (p_{ij})_{n \times n} \quad s.t \quad p_{ij} \geq 0 \forall i, j$$

$$\sum_j p_{ij} = 1$$
The m-step t.p.m. of a homogeneous Markov chain can also be defined in similar way.

2 CHAPMAN – KOLMOGOROV EQUATIONS

The Chapman-Kolmogorov (C-K) equation provides a method to relate the transition probabilities between successive steps.

Let

\[ P_{uv}^{(s)} = \text{probability that the Markov chain will go from state } S_u \text{ to state } S_v \text{ in exactly } s \text{-steps given that it was in state } S_u \text{ previously.} \]

Then the C-K equations are defined as

\[ p_{ij}^{(m+n)} = \sum_k p_{ik}^{(m)} p_{kj}^{(n)} \quad \forall i, j, k \in \text{State space.} \]

3 CONVOLUTION

Let \( f(t) \) and \( g(t) \) be two non-negative functions defined for \( t > 0 \), then the function defined by

\[ f(t) \circ g(t) = \int_0^t f(t - u) \ g(u) \ du \]

is called the convolution (or ordinary convolution) of function \( f(t) \) and \( g(t) \).

The Stieltjes convolution of two non-negative real valued function \( f(t) \) and \( g(t) \) is defined as

\[ f(t) \circ g(t) = \int_0^t f(t - u) \ dg(u) \]

4 LAPLACE TRANSFORM

A transform is merely a mapping of a function from one space to another. While it may be very difficult to solve certain equations directly for a particular function of interest, it is often easier to solve a corresponding equation in terms of transform of the function and then invert the transform to obtain the function. One particular transform, which is very useful for solving some type of differential as well as certain integral equations, is the Laplace transform.
4.1. Definition: Let \( f(t) \) be a function of positive real variables \( t \), then the Laplace transform (L. T.) of \( f(t) \) is defined as

\[
f^*(s) = \int e^{-st}f(t) \, dt
\]

for the range of values of ‘\( s \)’ for which the integral exists.

4.2. Properties of L.T.

Some important properties of the L.T. of a function are illustrated below:

(i) Linearity property

The L.T. of the sum of ‘\( n \)’ functions is equal to the sum of their Laplace transforms.

Mathematically, if

\[ f(t) = \sum_{i=1}^{n} f_i(t) \]

then,

\[ f^*(s) = \sum_{i=1}^{n} f_i^*(s) \]

more generally, if \( C_1, C_2, \ldots, C_n \) are constants then

\[ L[\sum_{i=1}^{n} C_i f_i(t)] = \sum_{i=1}^{n} C_i f_i^*(s) \]

(ii) L.T. of derivatives of a function

The L.T. of the first derivative of a function \( f(t) \) is obtained by multiplying the transition of \( f(t) \) by the argument \( s \) and subtracting the value of the function at \( t = 0 \) from this product i.e.

\[ L[f'(t)] = \int e^{-st} f'(t) \, dt = sf^*(s) - f(0) \]

Similarly the L.T. of second derivative is

\[ L[f''(t)] = s^2f^*(s) - sf(0) - f'(0) \]

In general, the L.T. of the \( n \)th derivative of \( f(t) \) is given by

\[ L[f^{(n)}(t)] = s^n f^*(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0) \]
(iii) L.T. of the integral of a function

The L.T of the integral of a function \( f(t) \) is equal to the L.T. of the function multiplied by the inverse of the arguments, i.e.

\[
L \left[ \int_0^t f(u) \, du \right] = s^{-1} f^*(s)
\]

Proof: We have by definition

\[
L \left[ \int_0^t f(u) \, du \right] = \int e^{-st} \left( \int_0^t f(u) \, du \right) \, dt
\]

Integrating by parts,

\[
L \left[ \int_0^t f(u) \, du \right] = -s^{-1} e^{-st} \int_0^t f(u) \, du \bigg|_0^\infty + s^{-1} \int_0^\infty e^{-st} f(t) \, dt
\]

\[
= e^{-st} f^*(s)
\]

(iv) Limit Properties

(a) Final Value Theorem

\[\lim_{t \to \infty} f(t) = \lim_{s \to 0} s f^*(s)\]

Proof: To prove the result, we proceed as

\[L[f'(t)] = s f^*(s) - f(0)\]

The L.T of \( f'(t) \) can be written as \( \int e^{-st} f(t) \, dt \), taking limit \( s \to 0 \) on both sides, we get

\[\lim_{s \to 0} \int e^{-st} f'(t) \, dt = \lim_{s \to 0} [s f^*(s) - f(0)]\]

\[\Rightarrow \int f'(t) \, dt = \lim_{s \to 0} [s f^*(s) - f(0)]\]

\[\Rightarrow f(\infty) - f(0) = \lim_{s \to 0} s f^*(s) - f(0)\]

\[\Rightarrow \lim_{t \to \infty} f(t) = \lim_{s \to 0} s f^*(s)\]

(b) Initial Value Theorem

\[\lim_{t \to 0} f(t) = \lim_{s \to \infty} s f^*(s)\]

Proof: To prove the result we again use property (ii) and take the limit \( s \to \infty \) on both sides. Thus
\[ \lim_{s \to \infty} \left[ \int e^{-st} f'(t) \, dt \right] = \lim_{s \to \infty} \left[ s f^*(s) - f(0) \right] \]

\[ 0 = \lim_{s \to \infty} s f^*(s) - f(0) \]

\[ \Rightarrow \lim_{t \to 0} f(t) = \lim_{s \to \infty} s f^*(s) \]

(v) L. T. of Convolution

The L.T. of an ordinary convolution of two functions \( f(t) \) and \( g(t) \) is equal to the product of their Laplace transform i.e.

\[
\mathcal{L}[g(t) \ast f(t)] = \int e^{-st} \left[ \int_0^t f(t - u) g(u) \, du \right] \, dt = \int g(u) \, du \int_0^\infty e^{-st} f(t - u) \, dt
\]

(on changing the order of integration)

\[
= \int e^{-st} g(u) \, du \int e^{-sv} f(v) \, dv
\]

\[ = g^*(s) f^*(s) \]

5 LAPLACE STIELTJES TRANSFORM

The Laplace Stieltjes transform of a positive real valued function \( f(t) \) is defined as

\[
\tilde{f}(s) = \int e^{-st} \, df(t) = e^{-st} f(t) \bigg|_0^\infty + \int s e^{-st} f(t) \, dt \]

(on integrating by parts)

\[ = s f^*(s) - f(0) \quad \text{provided } \lim_{t \to \infty} e^{-st} f(t) \, dt = 0 \]

If \( f(t) \) and \( F(t) \) are the p.d.f and c.d.f of a random variable \( T \), then

\[ F(0) = 0, \quad F(\infty) = 1 \quad \text{and} \quad F'(t) = f(t) \]

It follows then

\[
\tilde{F}(s) = s F^*(s) = s \int e^{-st} F(t) \, dt
\]

\[ = s \left[ s^{-1} e^{-st} F(t) \bigg|_0^\infty + s^{-1} \int e^{-st} F'(t) \, dt \right]
\]

\[ = f^*(s) \]
Thus the L.S.T of c.d.f is equal to L.T. of p.d.f.

Thus if \( q_{ij}(t) \) and \( Q_{ij}(t) \) represent the p.d.f and c.d.f respectively of the transition time \( T_{ij} \) from state \( S_i \) to \( S_j \) of the process \( \{X(t), t > 0\} \), then

\[
\tilde{Q}_{ij}(s) = \int e^{-st} dQ_{ij}(t) = \int e^{-st} q_{ij}(t) dt = q_{ij}^*(s)
\]

Further

\[
\tilde{Q}_{ij}(0) = \int dQ_{ij}(t) = Q_{ij}(\infty) - Q_{ij}(0)
\]

\[
= p_{ij} \quad \text{for } Q_{ij}(0) = 0
\]

\[
= q_{ij}^*(s)|_{s=0}
\]

The L.S.T of the Stieltjes convolution of two functions \( A(t) \) and \( B(t) \) is equal to the product of their L.S.T. i.e.

\[
\text{L.S.T. } [A(t) \mathbin{\circledcirc} B(t)] = \tilde{A}(s) \tilde{B}(s)
\]