Chapter 3

Bold Signed Total Domination For Complete Multi-partite Graphs

In the chapter we have established three lemmas and one theorem determining r—the number of vertices assigned with -2. Using the lemmas and the theorem, we have determined bold signed total domination number for complete multi-partite graph in five cases.

Definition 3.1 A k-partite graph is one whose vertex set can be partitioned into k subsets so that no edge has both ends in any one subset; a complete k-partite graph is one that is simple and in which each vertex is joined to every vertex that is not in the same subset [4].

3.1 Some Basic Results

Theorem 3.2 Let $G$ be a complete multi-partite graph. Let $n$ be the number of partitions of the graph $G$, $V_i$ (i = 1, 2, ..., n) be the partitions of $V(G)$ and $n_i$ be the number of vertices in the vertex set $V_i$ and $r_i$ be the number of -2's in $V_i$. Then

$$r_i \leq \begin{cases} \frac{(n-1)n_i-1}{3(n-1)} + \frac{1}{3(n-1)} & \text{if } n_i \equiv 0 \mod 3 \\ \frac{(n-1)n_i-1}{3(n-1)} + \frac{2n_i-1}{3(n-1)} & \text{if } n_i \equiv 1 \mod 3 \\ \frac{(n-1)n_i-1}{3(n-1)} + \frac{n_i}{3(n-1)} & \text{if } n_i \equiv 2 \mod 3, \end{cases}$$

for all $i = 1, 2, ..., n$. 41
**Proof:** $G$ is a complete $n$-partite graph. $V_i$ is the vertex set containing $n_i$ vertices and $r_i$ is the number of vertices assigned with -2 in $V_i$.

We know that $\sum_{v \in V(i)} f(v) \geq 1$.

We prove the result by induction on $n$.

For $n=3$.

$$f(N(v)) = \begin{cases} 
  n_1 - r_1 + n_2 - r_2 - 2r_1 - 2r_2 \geq 1 & \text{if } v \in V_3 \\
  n_2 - r_2 + n_3 - r_3 - 2r_2 - 2r_3 \geq 1 & \text{if } v \in V_1 \\
  n_1 - r_1 + n_3 - r_3 - 2r_1 - 2r_3 \geq 1 & \text{if } v \in V_2.
\end{cases}$$

(i.e.) $n_1 + n_2 - 3r_1 - 3r_2 \geq 1$ implies $n_1 + n_2 \geq 3r_1 + 3r_2 + 1$  \hspace{1cm} (3.1)

$n_2 + n_3 - 3r_2 - 3r_3 \geq 1$ implies $n_2 + n_3 \geq 3r_2 + 3r_3 + 1$  \hspace{1cm} (3.2)

$n_1 + n_3 - 3r_1 - 3r_3 \geq 1$ implies $n_1 + n_3 \geq 3r_1 + 3r_3 + 1$.  \hspace{1cm} (3.3)

(3.1)+(3.2)+(3.3) implies $2n_1 + 2n_2 + 2n_3 \geq 6r_1 + 6r_2 + 6r_3 + 3$.

(i.e.) $n_1 + n_2 + n_3 \geq 3r_1 + 3r_2 + 3r_3 + \frac{3}{2}$.  \hspace{1cm} (3.4)

(3.4)-(3.1) implies $n_3 \geq 3r_3 + \frac{1}{2}$.

(i.e.) $3r_3 \leq n_3 - \frac{1}{2}$ implies $r_3 \leq \frac{2n_3 - 1}{(2)(3)}$.

(3.4)-(3.2) implies $n_1 \geq 3r_1 + \frac{1}{2}$.

(i.e.) $3r_1 \leq n_1 - \frac{1}{2}$ implies $r_1 \leq \frac{2n_1 - 1}{(2)(3)}$.

(3.4)-(3.3) implies $n_2 \geq 3r_2 + \frac{1}{2}$.

(i.e.) $3r_2 \leq n_2 - \frac{1}{2}$ implies $r_2 \leq \frac{2n_2 - 1}{(2)(3)}$.

In general $r_i \leq \frac{2n_i - 1}{(2)(3)}$, for all $i = 1, 2, 3$.\hspace{1cm} (42)
Since \( r_i \) is an integer, \( r_i \leq \begin{cases} \frac{2n-1}{(2k)^2} + \frac{1}{(2k^3)} & \text{if } n_i \equiv 0 \mod 3 \\ \frac{2n-1}{(2k)^2} + \frac{5}{(2k^3)} & \text{if } n_i \equiv 1 \mod 3 \\ \frac{2n-1}{(2k)^2} + \frac{3}{(2k^3)} & \text{if } n_i \equiv 2 \mod 3 \end{cases} \), for all \( i = 1, 2, 3 \).

For \( n=4 \),

\[
f(N(v)) = \begin{cases} 
  n_1 + n_2 + n_3 - 3r_1 - 3r_2 - 3r_3 \geq 1 & \text{if } v \in V_4 \\
  n_2 + n_3 + n_4 - 3r_2 - 3r_3 - 3r_4 \geq 1 & \text{if } v \in V_1 \\
  n_1 + n_3 + n_4 - 3r_1 - 3r_3 - 3r_4 \geq 1 & \text{if } v \in V_2 \\
  n_1 + n_2 + n_4 - 3r_1 - 3r_2 - 3r_4 \geq 1 & \text{if } v \in V_3. 
\end{cases}
\]

(i.e.) \( n_1 + n_2 + n_3 \geq 3r_1 + 3r_2 + 3r_3 + 1 \) \hspace{1cm} (3.5)

\( n_2 + n_3 + n_4 \geq 3r_2 + 3r_3 + 3r_4 + 1 \) \hspace{1cm} (3.6)

\( n_1 + n_3 + n_4 \geq 3r_1 + 3r_3 + 3r_4 + 1 \) \hspace{1cm} (3.7)

\( n_1 + n_2 + n_4 \geq 3r_1 + 3r_2 + 3r_4 + 1 \) \hspace{1cm} (3.8)

(3.5) + (3.6) + (3.7) + (3.8) implies \( 3n_1 + 3n_2 + 3n_3 + 3n_4 \geq 9r_1 + 9r_2 + 9r_3 + 9r_4 + 4. \)

(i.e.) \( n_1 + n_2 + n_3 + n_4 \geq 3r_1 + 3r_2 + 3r_3 + 3r_4 + \frac{4}{3} \). \hspace{1cm} (3.9)

(3.9)-(3.5) implies \( n_4 \geq 3r_4 + \frac{1}{3} \).

(i.e.) \( 3r_4 \leq n_4 - \frac{1}{3} \) implies \( r_4 \leq \frac{3n_4 - 1}{(3k)^3} \).

(3.9)-(3.6) implies \( n_1 \geq 3r_1 + \frac{1}{3} \).

(i.e.) \( 3r_1 \leq n_1 - \frac{1}{3} \) implies \( r_1 \leq \frac{3n_1 - 1}{(3k)^3} \).

(3.9)-(3.7) implies \( n_2 \geq 3r_2 + \frac{1}{3} \).

(i.e.) \( 3r_2 \leq n_2 - \frac{1}{3} \) implies \( r_2 \leq \frac{3n_2 - 1}{(3k)^3} \).

(3.9)-(3.8) implies \( n_3 \geq 3r_3 + \frac{1}{3} \).

(i.e.) \( 3r_3 \leq n_3 - \frac{1}{3} \) implies \( r_3 \leq \frac{3n_3 - 1}{(3k)^3} \).

In general \( r_i \leq \frac{3n_i - 1}{(3k)^3} \), for all \( i = 1, 2, 3, 4 \).
Since \( r_i \) is an integer, \( r_i \leq \begin{cases} \frac{n_i}{3(n-3)} + \frac{1}{3(n-3)} & \text{if } n_i \equiv 0 \mod 3 \\ \frac{n_i}{3(n-3)} + \frac{7}{3(n-3)} & \text{if } n_i \equiv 1 \mod 3 \\ \frac{n_i}{3(n-3)} + \frac{4}{3(n-3)} & \text{if } n_i \equiv 2 \mod 3 \end{cases} \), for all \( i = 1, 2, 3 \).

Now we prove the result for \( n \)-partite graph.

Let \( V_i \) be the \( i^{th} \) vertex set having \( n_i \) vertices. Let \( r_i \) vertices assign \(-2\) in \( V_i \) for \( i = 1, 2, 3, \ldots, n \). Clearly

\[
\begin{align*}
n_2 + n_3 + \ldots + n_n & \geq 3r_2 + 3r_3 + \ldots + 3r_n + 1. \quad (3.10) \\
n_1 + n_3 + \ldots + n_n & \geq 3r_1 + 3r_3 + \ldots + 3r_n + 1. \\
\cdots \cdots \cdots \cdots \\
n_1 + n_2 + \ldots + n_{n-1} & \geq 3r_1 + 3r_2 + \ldots + 3r_{n-1} + 1.
\end{align*}
\]

Adding the above, we get

\[
(n - 1)(n_1 + n_2 + \ldots + n_n) \geq 3(n - 1)(r_1 + r_2 + \ldots + r_n) + n.
\]

\[
(i.e.) \quad n_1 + n_2 + \ldots + n_n \geq 3(r_1 + r_2 + \ldots + r_n) + \frac{n}{n - 1}. \quad (3.11)
\]

(3.11)-(3.10) implies \( n_1 \geq 3r_1 + \frac{n}{n-1} - 1 = 3r_1 + \frac{1}{n-1} \)

(i.e.) \( 3r_1 \leq n_1 - \frac{1}{n-1} \).

Therefore \( r_1 \leq \frac{(n-1)n_1-1}{3(n-1)} \).

Similarly, \( r_2 \leq \frac{(n-1)n_2-1}{3(n-1)} \), \( r_3 \leq \frac{(n-1)n_3-1}{3(n-1)} \), \ldots, \( r_n \leq \frac{(n-1)n_n-1}{3(n-1)} \).

In general, \( r_i \leq \frac{(n-1)n_i-1}{3(n-1)} \), for all \( i = 1, 2, \ldots, n \).

Since \( r_i \) is an integer, \( r_i \leq \begin{cases} \frac{(n-1)n_i-1}{3(n-1)} + \frac{1}{3(n-1)} & \text{if } n_i \equiv 0 \mod 3 \\ \frac{(n-1)n_i-1}{3(n-1)} + \frac{2n-1}{3(n-1)} & \text{if } n_i \equiv 1 \mod 3 \\ \frac{(n-1)n_i-1}{3(n-1)} + \frac{n}{3(n-1)} & \text{if } n_i \equiv 2 \mod 3 \end{cases} \), for all \( i = 1, 2, \ldots, n \).

Thus the result is proved.

**Lemma 3.3** Let \( G \) be a complete multi-partite graph. Let \( n \) be the number of partitions of the graph \( G \), \( V_i (i = 1, 2, \ldots, n) \) be the partitions of \( V(G) \) and \( n_i \) be the number
of vertices in the vertex set $V_i$ and $r_i$ be the number of -2's in $V_i$. If $\sum_{i=1}^{n} n_i = 3k$ where 
k is any positive integer and $k \geq 1, n \geq 3$, then $\sum_{i=1}^{n} r_i \leq \sum_{i=1}^{n} s_i - 2 = k_1 - 2$ for all $n_i = 3s_i$, $s_i$ is any positive integer and $\sum_{i=1}^{n} r_i \leq k_1 + \left(\frac{k_2}{3}\right) - 1$ otherwise, where $k_1$ is the sum of quotients of $n_i$ when it is divided by 3 and $k_2$ is the sum of remainders of $n_i$ when it is divided by 3, for all $i = 1, 2, \ldots, n$.

**Proof:** $G$ is a complete multi-partite graph. $n$ is the number of partitions of the graph $G$, $V_i (i = 1, 2, \ldots, n)$ is the $n$-partitions of $V(G)$ and $n_i$ is the number of vertices in the vertex set $V_i$ and $r_i$ is the number of -2's in $V_i$.

$\sum_{i=1}^{n} n_i = 3k$ where $k$ is any positive integer and $k \geq 1, n \geq 3$.

$k_1$ is the sum of quotients of $n_i$ when it is divided by 3 and $k_2$ be the sum of remainders of $n_i$ when it is divided by 3, for all $i = 1, 2, \ldots, n$.

**Case (i): All $n_i$’s are multiple of 3**

Let $n_i = 3s_i$, where $s_i$ is any positive integer, $i = 1, 2, \ldots, n$.

we prove this by induction on $n$.

For $n = 3$.

Then

\[
 n_1 + n_2 \geq 3r_1 + 3r_2 + 1 \tag{3.12}
\]

\[
 n_2 + n_3 \geq 3r_2 + 3r_3 + 1 \tag{3.13}
\]

\[
 n_1 + n_3 \geq 3r_1 + 3r_3 + 1 \tag{since \ \sum_{v\in N(u)} f(v) \geq 1}. \tag{3.14}
\]

\[i.e.\] 3$s_1$ + 3$s_2$ $\geq$ 3$r_1 + 3r_2 + 1$

\[3s_2 + 3s_3 \geq 3r_2 + 3r_3 + 1\]

\[3s_1 + 3s_3 \geq 3r_1 + 3r_3 + 1.\]
\[(i.e.)\ r_1 + r_2 \leq s_1 + s_2 - \frac{1}{3}\]
\[r_2 + r_3 \leq s_2 + s_3 - \frac{1}{3}\]
\[r_1 + r_3 \leq s_1 + s_3 - \frac{1}{3}\]

Since \(r_1 + r_2 \leq s_1 + s_2 - \frac{1}{3}\), either \(r_1 < s_1\) or \(r_2 < s_2\).

Since \(r_2 + r_3 \leq s_2 + s_3 - \frac{1}{3}\), either \(r_2 < s_2\) or \(r_3 < s_3\).

Since \(r_1 + r_3 \leq s_1 + s_3 - \frac{1}{3}\), either \(r_1 < s_1\) or \(r_3 < s_3\).

On consolidation, we get either \(r_1 < s_1\) and \(r_2 < s_2\). \(\text{(i.e.) } r_1 \leq s_1 - 1 \text{ and } r_2 \leq s_2 - 1\)

(or) \(r_2 < s_2\) and \(r_3 < s_3\). \(\text{(i.e.) } r_2 \leq s_2 - 1 \text{ and } r_3 \leq s_3 - 1\)

(or) \(r_1 < s_1\) and \(r_3 < s_3\). \(\text{(i.e.) } r_1 \leq s_1 - 1 \text{ and } r_3 \leq s_3 - 1\),

as \(r_1, r_2, r_3, s_1, s_2, s_3\) are integers.

Hence \(r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 - 2 = k_1 - 2\).

For \(n=4\).

Then \(n_1 + n_2 + n_3 \geq 3r_1 + 3r_2 + 3r_3 + 1\) \hspace{1cm} (3.15)

\[n_2 + n_3 + n_4 \geq 3r_2 + 3r_3 + 3r_4 + 1\] \hspace{1cm} (3.16)

\[n_1 + n_3 + n_4 \geq 3r_1 + 3r_3 + 3r_4 + 1\] \hspace{1cm} (3.17)

\[n_1 + n_2 + n_4 \geq 3r_1 + 3r_2 + 3r_4 + 1 \text{ (since } \sum_{v \in N(u)} f(v) \geq 1).\] \hspace{1cm} (3.18)

\(i.e.) \ 3s_1 + 3s_2 + 3s_3 \geq 3r_1 + 3r_2 + 3r_3 + 1\)

\[3s_2 + 3s_3 + 3s_4 \geq 3r_2 + 3r_3 + 3r_4 + 1\]

\[3s_1 + 3s_3 + 3s_4 \geq 3r_1 + 3r_3 + 3r_4 + 1\]

\[3s_1 + 3s_2 + 3s_4 \geq 3r_1 + 3r_2 + 3r_4 + 1\].

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\[(i.e.) \quad r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 - \frac{1}{3},
\quad r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 - \frac{1}{3},
\quad r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 - \frac{1}{3},
\quad r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 - \frac{1}{3}.
\]

Since \(r_1, r_2, r_3, r_4\) are integers and \(r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 - \frac{1}{3}\), let \(r_1 \leq s_1, r_2 \leq s_2, r_3 \leq s_3 - 1\). If \(r_4 \leq s_4\), then \(r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 - \frac{1}{3}; r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 - \frac{1}{3}\) but \(r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 - \frac{1}{3}\).

This is a contradiction.

Therefore \(r_4 \leq s_4 - 1\).

Hence \(r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 - 2 = k_1 - 2\).

Assume that this result is true for all complete \((n - 1)\)-partite graph.

Then \(r_1 + r_2 + r_3 + \ldots + r_{n-1} \leq k_1 - 2\), where \(k_1\) is the sum of quotients of \(n_i\), when it is divided by 3, \(i = 1, 2, \ldots, n - 1\).

Now we prove it for complete \(n\)-partite graph.

Let \(n_n = 3s_n\) (since \(\sum_{i=1}^{n-1} n_i = 3k_1\) and \(\sum_{i=1}^{n} n_i = 3k\)).

Clearly \(r_n \leq s_n\) (since by theorem (3.2)).

Therefore \(\sum_{i=1}^{n} r_i = \sum_{i=1}^{n-1} r_i + r_n \leq k_1 - 2 + s_n = k_1 - 2\) where \(k_1 = k_1' + s_n\) is the sum of quotients of \(n_i\) when it is divided by 3, for all \(i = 1, 2, \ldots, n\).

Hence \(\sum_{i=1}^{n} r_i \leq k_1 - 2\), where \(k_1\) is the sum of quotients of \(n_i\) when it is divided by 3, for all \(i = 1, 2, \ldots, n\).

**Case (ii): Some \(n_i\)'s are not multiple of 3**

we prove this by induction on \(n\).
For $n = 3$.

Let $V_1, V_2, V_3$ are tripartitions of $V$. The cardinality of $V_i$ are as follows:

| $|V_1|$  | $|V_2|$  | $|V_3|$  |
|--------|--------|--------|
| $3s_1$ | $3s_2 + 1$ | $3s_3 + 2$ |
| $3s_1 + 1$ | $3s_2 + 1$ | $3s_3 + 1$ |
| $3s_1 + 2$ | $3s_2 + 2$ | $3s_3 + 2$ |

If $|V_1| = 3s_1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 2$, then by (3.12), (3.13) and (3.14),

$$3s_1 + 3s_2 + 1 \geq 3r_1 + 3r_2 + 1$$
$$3s_2 + 3s_3 + 3 \geq 3r_2 + 3r_3 + 1$$
$$3s_1 + 3s_3 + 2 \geq 3r_1 + 3r_3 + 1.$$

(i.e.) $r_1 + r_2 \leq s_1 + s_2$

$$r_2 + r_3 \leq s_2 + s_3 + \frac{2}{3}$$
$$r_1 + r_3 \leq s_1 + s_3 + \frac{1}{3}.$$

Therefore $r_1 \leq s_1, r_2 \leq s_2$ and $r_3 \leq s_3$.

Thus $r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 = k_1 + \left(\frac{k_2}{3}\right) - 1$. \hspace{1cm} (3.19)

If $|V_1| = 3s_1 + 1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 1$, then by (3.12), (3.13) and (3.14),

$$3s_1 + 1 + 3s_2 + 1 \geq 3r_1 + 3r_2 + 1$$
$$3s_2 + 1 + 3s_3 + 1 \geq 3r_2 + 3r_3 + 1$$
$$3s_1 + 1 + 3s_3 + 1 \geq 3r_1 + 3r_3 + 1.$$

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\[(i.e.) \quad r_1 + r_2 \leq s_1 + s_2 + \frac{1}{3} \]
\[r_2 + r_3 \leq s_2 + s_3 + \frac{1}{3} \]
\[r_1 + r_3 \leq s_1 + s_3 + \frac{1}{3}\]

Therefore \(r_1 \leq s_1, r_2 \leq s_2\) and \(r_3 \leq s_3\). Thus
\[r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 = k_1 + \left(\frac{k_2}{3}\right) - 1. \quad (3.20)\]

If \(|V_1| = 3s_1 + 2, \quad |V_2| = 3s_2 + 2, \quad |V_3| = 3s_3 + 2\), then by (3.12), (3.13) and (3.14),
\[3s_1 + 2 + 3s_2 + 2 \geq 3r_1 + 3r_2 + 1 \]
\[3s_2 + 2 + 3s_3 + 2 \geq 3r_2 + 3r_3 + 1 \]
\[3s_1 + 2 + 3s_3 + 2 \geq 3r_1 + 3r_3 + 1. \]

\[(i.e.) \quad r_1 + r_2 \leq s_1 + s_2 + 1 \]
\[r_2 + r_3 \leq s_2 + s_3 + 1 \]
\[r_1 + r_3 \leq s_1 + s_3 + 1.\]

Since \(r_1 + r_2 \leq s_1 + s_2 + 1\), let \(r_1 \leq s_1, r_2 \leq s_2 + 1\).

If \(r_3 \leq s_3 + 1\), then \(r_1 + r_3 \leq s_1 + s_3 + 1\) but \(r_2 + r_3 \not< s_2 + s_3 + 1\). This is a contradiction.

Therefore \(r_3 \leq s_3\).

**Thus** \(r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + 1 = k_1 + \left(\frac{k_2}{3}\right) - 1. \quad (3.21)\)

By inequations (3.19), (3.20) and (3.21), \(r_1 + r_2 + r_3 \leq k_1 + \left(\frac{k_2}{3}\right) - 1.\)

**For \(n = 4\).**

Let \(V_1, V_2, V_3, V_4\) are partitions of \(V\). The cardinality of \(V_i\) are as follows:
\[
\begin{array}{|c|c|c|c|}
\hline
| V_1 | & | V_2 | & | V_3 | & | V_4 | \\
\hline
3s_1 & 3s_2 + 1 & 3s_3 + 1 & 3s_4 + 1 \\
3s_1 & 3s_2 & 3s_3 + 1 & 3s_4 + 2 \\
3s_1 & 3s_2 + 2 & 3s_3 + 2 & 3s_4 + 2 \\
3s_1 + 1 & 3s_2 + 1 & 3s_3 + 2 & 3s_4 + 2 \\
\hline
\end{array}
\]

If \(|V_1| = 3s_1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 1, |V_4| = 3s_4 + 1\), then by (3.15), (3.16), (3.17) and (3.18),

\[
\begin{aligned}
3s_1 + 3s_2 + 1 + 3s_3 + 1 & \geq 3r_1 + 3r_2 + 3r_3 + 1 \\
3s_2 + 1 + 3s_3 + 1 + 3s_4 + 1 & \geq 3r_2 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_3 + 1 + 3s_4 + 1 & \geq 3r_1 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_2 + 1 + 3s_4 + 1 & \geq 3r_1 + 3r_2 + 3r_4 + 1
\end{aligned}
\]

(i.e.)   \[
\begin{aligned}
r_1 + r_2 + r_3 & \leq s_1 + s_2 + s_3 + \frac{1}{3} \\
r_2 + r_3 + r_4 & \leq s_2 + s_3 + s_4 + \frac{2}{3} \\
r_1 + r_3 + r_4 & \leq s_1 + s_3 + s_4 + \frac{1}{3} \\
r_1 + r_2 + r_4 & \leq s_1 + s_2 + s_4 + \frac{1}{3}
\end{aligned}
\]

Since \(r_1, r_2, r_3, r_4\) are integers, \(r_1 \leq s_1, r_2 \leq s_2, r_3 \leq s_3, \) and \(r_4 \leq s_4\).

Thus \[
\begin{aligned}
r_1 + r_2 + r_3 + r_4 & \leq s_1 + s_2 + s_3 + s_4 = k_1 + \left(\frac{k_2}{3}\right) - 1.
\end{aligned}
\] (3.22)

If \(|V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3 + 1, |V_4| = 3s_4 + 2\), then by (3.15), (3.16), (3.17) and (3.18),

\[
\begin{aligned}
3s_1 + 3s_2 + 3s_3 + 1 & \geq 3r_1 + 3r_2 + 3r_3 + 1 \\
3s_2 + 3s_3 + 1 + 3s_4 + 2 & \geq 3r_2 + 3r_3 + 3r_4 + 1
\end{aligned}
\]
\[3s_1 + 3s_3 + 1 + 3s_4 + 2 \geq 3r_1 + 3r_3 + 3r_4 + 1\]
\[3s_1 + 3s_2 + 3s_4 + 2 \geq 3r_1 + 3r_2 + 3r_4 + 1.\]

\[(i.e.) \ r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3\]
\[r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 + \frac{2}{3}\]
\[r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 + \frac{2}{3}\]
\[r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 + \frac{1}{3}\.\]

Since \(r_1, r_2, r_3, r_4\) are integers, \(r_1 \leq s_1, r_2 \leq s_2, r_3 \leq s_3,\) and \(r_4 \leq s_4.\)

Thus \(r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 = k_1 + \left(\frac{k_2}{3}\right) - 1.\) \quad (3.23)

If \(|V_1| = 3s_1, |V_2| = 3s_2 + 2, |V_3| = 3s_3 + 2, |V_4| = 3s_4 + 2,\) then by (3.15), (3.16), (3.17) and (3.18),

\[3s_1 + 3s_2 + 2 + 3s_3 + 2 \geq 3r_1 + 3r_2 + 3r_3 + 1\]
\[3s_2 + 2 + 3s_3 + 2 + 3s_4 + 2 \geq 3r_2 + 3r_3 + 3r_4 + 1\]
\[3s_1 + 3s_3 + 2 + 3s_4 + 2 \geq 3r_1 + 3r_3 + 3r_4 + 1\]
\[3s_1 + 3s_2 + 2 + 3s_4 + 2 \geq 3r_1 + 3r_2 + 3r_4 + 1.\]

\[(i.e.) \ r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + 1\]
\[r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 + \frac{5}{3}\]
\[r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 + 1\]
\[r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 + 1.\]

Since \(r_1, r_2, r_3, r_4\) are integers and \(r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + 1,\) let \(r_1 \leq s_1, r_2 \leq s_2, r_3 \leq s_3 + 1.\) If \(r_4 \leq s_4 + 1,\) then \(r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 + 1,\) but \(r_2 + r_3 + r_4 \notin \)
\[ s_2 + s_3 + s_4 + \frac{5}{3} \] and \( r_1 + r_3 + r_4 \not\leq s_1 + s_3 + s_4 + 1 \). This is a contradiction.

Therefore \( r_4 \leq s_4 \).

Thus \( r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 + 1 = k_1 + \left( \frac{k_2}{3} \right) - 1 \). \hspace{1cm} (3.24)

If \( |V_1| = 3s_1 + 1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 2, |V_4| = 3s_4 + 2 \), then by (3.15), (3.16), (3.17) and (3.18),

\[
\begin{align*}
3s_1 + 3s_2 + 3s_3 + 4 & \geq 3r_1 + 3r_2 + 3r_3 + 1 \\
3s_2 + 3s_3 + 3s_4 + 5 & \geq 3r_2 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_3 + 3s_4 + 5 & \geq 3r_1 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_2 + 3s_4 + 4 & \geq 3r_1 + 3r_2 + 3r_4 + 1.
\end{align*}
\]

\( \text{(i.e.) } r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + 1 \)
\[
\begin{align*}
r_2 + r_3 + r_4 & \leq s_2 + s_3 + s_4 + \frac{4}{3} \\
r_1 + r_3 + r_4 & \leq s_1 + s_3 + s_4 + \frac{4}{3} \\
r_1 + r_2 + r_4 & \leq s_1 + s_2 + s_4 + 1.
\end{align*}
\]

Since \( r_1, r_2, r_3, r_4 \) are integers and \( r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + 1 \), let \( r_1 \leq s_1, r_2 \leq s_2, r_3 \leq s_3 + 1 \). If \( r_4 \leq s_4 + 1 \), then \( r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 + 1 \), but \( r_2 + r_3 + r_4 \not\leq s_2 + s_3 + s_4 + \frac{4}{3} \) and \( r_1 + r_3 + r_4 \not\leq s_1 + s_3 + s_4 + \frac{4}{3} \). This is a contradiction. Therefore \( r_4 \leq s_4 \).

Thus \( r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 + 1 = k_1 + \left( \frac{k_2}{3} \right) - 1 \). \hspace{1cm} (3.25)

By inequations (3.22),(3.23),(3.24) and (3.25), \( r_1 + r_2 + r_3 + r_4 \leq k_1 + \left( \frac{k_2}{3} \right) - 1 \).

Assume that this result is true for all complete \((n - 1)\)-partite graph.

\( \text{(i.e.) } \sum_{i=1}^{n-1} r_i \leq k_i + \left( \frac{k_i'}{n} \right) - 1 \) where \( k_i' \) is the sum of quotients of \( n_i \) when it is divided by

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3, and \( k_2' \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( i = 1, 2, ..., n-1 \) of complete \((n-1)\)-partite graph.

Now we prove it for complete \( n \)-partite graph.

Let \(|V_n| = 3s_n \) (since \( \sum_{i=1}^{n-1} n_i = 3k' \) and \( \sum_{i=1}^{n} n_i = 3k \)).

Clearly \( r_n \leq s_n \) (since by theorem 3.2).

\[
Therefore \sum_{i=1}^{n} r_i = \sum_{i=1}^{n-1} r_i + r_n \\
\leq k'_1 + (\frac{k_2'}{3}) - 1 + s_n \\
= k'_1 + (\frac{k_2'}{3}) - 1,
\]

where \( k_1 = k'_1 + s_n \) is the sum of quotients of \( n_i \) when it is divided by 3 and \( k_2 = k'_2 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( i = 1, 2, ..., n \).

Hence \( \sum_{i=1}^{n} r_i \leq k_1 + (\frac{k_2}{3}) - 1 \), where \( k_1 \) is the sum of quotients of \( n_i \) when it is divided by 3 and \( k_2 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( i = 1, 2, ..., n \).

**Lemma 3.4** Let \( G \) be a complete multi-partite graph. Let \( n \) be the number of partitions of the graph \( G \), \( V_i \) (\( i = 1, 2, ..., n \)) be the partitions of \( V(G) \) and \( n_i \) be the number of vertices in the vertex set \( V_i \) and \( r_i \) be the number of -2's in \( V_i \). If \( \sum_{i=1}^{n} n_i = 3k + 1 \) where \( k \) is any positive integer, \( n \geq 3 \), then \( \sum_{i=1}^{n} r_i \leq k_1 + \frac{k_2}{3} \), where \( k_1 \) is the sum of quotients of \( n_i \) when it is divided by 3 and \( k_2 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( i = 1, 2, ..., n \).

**Proof:** \( G \) is a complete multi-partite graph. \( n \) is the number of partitions of the graph \( G \), \( V_i \) (\( i = 1, 2, ..., n \)) is the \( n \)-partitions of \( V(G) \) and \( n_i \) is the number of vertices in the vertex set \( V_i \) and \( r_i \) is the number of -2's in \( V_i \).
Let \( \sum_{i=1}^{n} n_i = 3k + 1 \) where \( k \) is any positive integer and \( n \geq 3 \).

Let \( k_1 \) be the sum of quotients of \( n_i \) when it is divided by 3 and \( k_2 \) be the sum of remainders of \( n_i \) when it is divided by 3, for all \( i = 1, 2, ..., n \). we prove this by induction on \( n \).

**For \( n = 3 \).**

Let \( V_1, V_2, V_3 \) are tripartitions of \( V \). The cardinality of \( V_i \) are as follows:

| \( |V_1| \) | \( |V_2| \) | \( |V_3| \) |
|---|---|---|
| \( 3s_1 \) | \( 3s_2 + 2 \) | \( 3s_3 + 1 \) |
| \( 3s_1 + 1 \) | \( 3s_2 + 1 \) | \( 3s_3 + 2 \) |

If \( |V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3 + 1 \).

Then
\[
\begin{align*}
    n_1 + n_2 & \geq 3r_1 + 3r_2 + 1 \\
    n_2 + n_3 & \geq 3r_2 + 3r_3 + 1 \\
    n_1 + n_3 & \geq 3r_1 + 3r_3 + 1 \quad (\text{since } \sum_{v \in N(u)} f(v) \geq 1).
\end{align*}
\]

(i.e.)
\[
\begin{align*}
    3s_1 + 3s_2 & \geq 3r_1 + 3r_2 + 1 \\
    3s_2 + 3s_3 + 1 & \geq 3r_2 + 3r_3 + 1 \\
    3s_1 + 3s_3 + 1 & \geq 3r_1 + 3r_3 + 1.
\end{align*}
\]

(i.e.)
\[
\begin{align*}
    r_1 + r_2 & \leq s_1 + s_2 - \frac{1}{3} \\
    r_2 + r_3 & \leq s_2 + s_3 \\
    r_1 + r_3 & \leq s_1 + s_3.
\end{align*}
\]

Therefore \( r_1 \leq s_1 \) and \( r_3 \leq s_3 \) (since \( r_1 + r_3 \leq s_1 + s_3 \)).

If \( r_2 \leq s_2 \), then \( r_2 + r_3 \leq s_2 + s_3 \) but \( r_1 + r_2 \not\leq s_1 + s_2 - \frac{1}{3} \).
This is a contradiction.

Therefore \( r_2 \leq s_2 - 1 \).

\[
\text{Hence } r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 - 1 = k_1 + \frac{k_2 - 4}{3}.
\] (3.29)

If \(|V_1| = 3s_1, |V_2| = 3s_2 + 2, |V_3| = 3s_3 + 2\), then by (3.26), (3.27) and (3.28),

\[
\begin{align*}
3s_1 + 3s_2 + 2 & \geq 3r_1 + 3r_2 + 1 \\
3s_2 + 3s_3 + 4 & \geq 3r_2 + 3r_3 + 1 \\
3s_1 + 3s_3 + 2 & \geq 3r_1 + 3r_3 + 1.
\end{align*}
\]

\[(i.e.)
\begin{align*}
r_1 + r_2 & \leq s_1 + s_2 + \frac{1}{3} \\
r_2 + r_3 & \leq s_2 + s_3 + 1 \\
r_1 + r_3 & \leq s_1 + s_3 + \frac{1}{3}.
\end{align*}
\]

Therefore \( r_1 \leq s_1 \) and \( r_2 \leq s_2 \) (since \( r_1 + r_2 \leq s_1 + s_2 + \frac{1}{3} \)).

If \( r_3 \leq s_3 + 1 \), then \( r_2 + r_3 \leq s_2 + s_3 + 1 \) but \( r_1 + r_3 \leq s_1 + s_3 + \frac{1}{3} \).

This is a contradiction.

Therefore \( r_3 \leq s_3 \).

\[
\text{Hence } r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 = k_1 + \frac{k_2 - 4}{3}.
\] (3.30)

If \(|V_1| = 3s_1 + 1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 2\), then by (3.26), (3.27) and (3.28),

\[
\begin{align*}
3s_1 + 3s_2 + 2 & \geq 3r_1 + 3r_2 + 1 \\
3s_2 + 3s_3 + 3 & \geq 3r_2 + 3r_3 + 1 \\
3s_1 + 3s_3 + 3 & \geq 3r_1 + 3r_3 + 1.
\end{align*}
\]
\[(i.e.) \ r_1 + r_2 \leq s_1 + s_2 + \frac{1}{3} \]
\[r_2 + r_3 \leq s_2 + s_3 + \frac{2}{3} \]
\[r_1 + r_3 \leq s_1 + s_3 + \frac{2}{3}. \]

Therefore \( r_1 \leq s_1, r_2 \leq s_2 \) and \( r_3 \leq s_3. \)

Hence \( r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 = k_1 + \frac{k_2 - 4}{3}. \)\hspace{1cm} (3.31)

By inequations (3.29),(3.30) and (3.31), \( r_1 + r_2 + r_3 \leq k_1 + \frac{k_2 - 4}{3}. \)

For \( n=4. \)

Let \( V_1, V_2, V_3, V_4 \) are partitions of \( V. \) The cardinality of \( V_i \) are as follows:

| \(|V_1|\) | \(|V_2|\) | \(|V_3|\) | \(|V_4|\) |
|---|---|---|---|
| 3s_1 | 3s_2 | 3s_3 | 3s_4 + 1 |
| 3s_1 | 3s_2 + 1 | 3s_3 + 1 | 3s_4 + 2 |
| 3s_1 | 3s_2 | 3s_3 + 2 | 3s_4 + 2 |
| 3s_1 + 1 | 3s_2 + 2 | 3s_3 + 2 | 3s_4 + 2 |

If \( |V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3, |V_4| = 3s_4 + 1. \)

Then \( n_1 + n_2 + n_3 \geq 3r_1 + 3r_2 + 3r_3 + 1 \) \hspace{1cm} (3.32)

\[n_2 + n_3 + n_4 \geq 3r_2 + 3r_3 + 3r_4 + 1 \] \hspace{1cm} (3.33)

\[n_1 + n_3 + n_4 \geq 3r_1 + 3r_3 + 3r_4 + 1 \] \hspace{1cm} (3.34)

\[n_1 + n_2 + n_4 \geq 3r_1 + 3r_2 + 3r_4 + 1 \ (since \ \sum_{v \in N(u)} f(v) \geq 1). \] \hspace{1cm} (3.35)

\[(i.e.) \ 3s_1 + 3s_2 + 3s_3 \geq 3r_1 + 3r_2 + 3r_3 + 1 \]

\[3s_2 + 3s_3 + 3s_4 + 1 \geq 3r_2 + 3r_3 + 3r_4 + 1 \]

\[3s_1 + 3s_3 + 3s_4 + 1 \geq 3r_1 + 3r_3 + 3r_4 + 1 \]

\[3s_1 + 3s_2 + 3s_4 + 1 \geq 3r_1 + 3r_2 + 3r_4 + 1. \]
\begin{align*}
(i.e.) \quad r_1 + r_2 + r_3 & \leq s_1 + s_2 + s_3 - \frac{1}{3} \\
r_2 + r_3 + r_4 & \leq s_2 + s_3 + s_4 \\
r_1 + r_3 + r_4 & \leq s_1 + s_3 + s_4 \\
r_1 + r_2 + r_4 & \leq s_1 + s_2 + s_4.
\end{align*}

Therefore \( r_1 \leq s_1, r_2 \leq s_2 \) and \( r_4 \leq s_4 \) (since \( r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 \)).

If \( r_3 \leq s_3 \), then \( r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 \) and \( r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 \) but \( r_1 + r_2 + r_3 \not\leq s_1 + s_2 + s_3 - \frac{1}{3} \).

This is a contradiction. Therefore \( r_3 \leq s_3 - 1 \).

\[ \text{Hence } r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 - 1 = k_1 + \frac{k_2 - 4}{3}. \quad (3.36) \]

If \( |V_1| = 3s_1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 1, |V_4| = 3s_4 + 2 \), then by (3.32), (3.33), (3.34) and (3.35),

\begin{align*}
3s_1 + 3s_2 + 3s_3 + 2 & \geq 3r_1 + 3r_2 + 3r_3 + 1 \\
3s_2 + 3s_3 + 3s_4 + 4 & \geq 3r_2 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_3 + 3s_4 + 3 & \geq 3r_1 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_2 + 3s_4 + 3 & \geq 3r_1 + 3r_2 + 3r_4 + 1.
\end{align*}

\begin{align*}
(i.e.) \quad r_1 + r_2 + r_3 & \leq s_1 + s_2 + s_3 + \frac{1}{3} \\
r_2 + r_3 + r_4 & \leq s_2 + s_3 + s_4 + 1 \\
r_1 + r_3 + r_4 & \leq s_1 + s_3 + s_4 + \frac{2}{3} \\
r_1 + r_2 + r_4 & \leq s_1 + s_2 + s_4 + \frac{2}{3}.
\end{align*}

Since \( r_1, r_2, r_3, r_4 \) are integers and \( r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + \frac{1}{3}, r_1 \leq s_1, r_2 \leq s_2 \) and \( r_3 \leq s_3 \).
If \( r_4 \leq s_4 + 1 \), then \( r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 + 1 \), but \( r_1 + r_3 + r_4 \not\leq s_1 + s_3 + s_4 + \frac{2}{3} \); \( r_1 + r_2 + r_4 \not\leq s_1 + s_2 + s_4 + \frac{2}{3} \).

This is a contradiction. Therefore \( r_4 \leq s_4 \).

\[
\text{Hence } r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 = k_1 + \frac{k_2 - 4}{3}. \tag{3.37}
\]

If \( |V_1| = 3s_1 + 1, |V_2| = 3s_2 + 2, |V_3| = 3s_3 + 2, |V_4| = 3s_4 + 2 \), then by (3.32), (3.33), (3.34) and (3.35),

\[
3s_1 + 3s_2 + 3s_3 + 5 \geq 3r_1 + 3r_2 + 3r_3 + 1
\]
\[
3s_2 + 3s_3 + 3s_4 + 6 \geq 3r_2 + 3r_3 + 3r_4 + 1
\]
\[
3s_1 + 3s_3 + 3s_4 + 5 \geq 3r_1 + 3r_3 + 3r_4 + 1
\]
\[
3s_1 + 3s_2 + 3s_4 + 5 \geq 3r_1 + 3r_2 + 3r_4 + 1.
\]

(i.e.) \( r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + \frac{4}{3} \)
\[
r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 + \frac{5}{3}
\]
\[
r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 + \frac{4}{3}
\]
\[
r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 + \frac{4}{3}.
\]

Since \( r_1, r_2, r_3, r_4 \) are integers and \( r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + \frac{4}{3} \),

let \( r_1 \leq s_1, r_2 \leq s_2 \) and \( r_3 \leq s_3 + 1 \).

If \( r_4 \leq s_4 + 1 \), then \( r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 + \frac{4}{3} \), but \( r_2 + r_3 + r_4 \not\leq s_2 + s_3 + s_4 + \frac{5}{3} \) and \( r_1 + r_3 + r_4 \not\leq s_1 + s_3 + s_4 + \frac{4}{3} \).

This is a contradiction. Therefore \( r_4 \leq s_4 \).

\[
\text{Hence } r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 + 1 = k_1 + \frac{k_2 - 4}{3}. \tag{3.38}
\]

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If $|V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3 + 2, |V_4| = 3s_4 + 2$, then by (3.32), (3.33), (3.34) and (3.35),

\[
\begin{align*}
3s_1 + 3s_2 + 3s_3 + 2 & \geq 3r_1 + 3r_2 + 3r_3 + 1 \\
3s_2 + 3s_3 + 3s_4 + 4 & \geq 3r_2 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_3 + 3s_4 + 4 & \geq 3r_1 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_2 + 3s_4 + 2 & \geq 3r_1 + 3r_2 + 3r_4 + 1.
\end{align*}
\]

(i.e.) $r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + \frac{1}{3}$

\[
\begin{align*}
r_2 + r_3 + r_4 & \leq s_2 + s_3 + s_4 + 1 \\
r_1 + r_3 + r_4 & \leq s_1 + s_3 + s_4 + 1 \\
r_1 + r_2 + r_4 & \leq s_1 + s_2 + s_4 + \frac{1}{3}.
\end{align*}
\]

Since $r_1, r_2, r_3, r_4$ are integers and $r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + \frac{1}{3}$, let $r_1 \leq s_1, r_2 \leq s_2$ and $r_3 \leq s_3$.

If $r_4 \leq s_4 + 1$, then $r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 + 1$ and $r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 + 1$ but $r_1 + r_2 + r_4 \not\leq s_1 + s_2 + s_4 + \frac{1}{3}$.

This is a contradiction. Therefore $r_4 \leq s_4$.

\[
\text{Hence } r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 = k_1 + \frac{k_2 - 4}{3}. \quad (3.39)
\]

By inequations (3.36),(3.37),(3.38) and (3.39), $r_1 + r_2 + r_3 + r_4 \leq k_1 + \frac{k_2 - 4}{3}$.

Assume that this result is true for all complete $(n-1)$-partite graph.

Then $r_1 + r_2 + r_3 + \ldots + r_{n-1} \leq k_1' + \frac{k_2' - 4}{3}$, where $k_1'$ is the sum of quotients of $n_i$ when it is divided by 3 and $k_2'$ is the sum of remainders of $n_i$ when it is divided by 3, $i = 1, 2, \ldots, n - 1$ of complete $(n-1)$-partite graph.
Now we prove it for complete $n$-partite graph.

Let $|V_n| = 3s_n$ (since $\sum_{i=1}^{n-1} n_i = 3k' + 1$ and $\sum_{i=1}^{n} n_i = 3k + 1$).

Clearly $r_n \leq s_n$ (since by theorem 3.2).

Therefore

$$\sum_{i=1}^{n} r_i = \sum_{i=1}^{n-1} r_i + r_n$$

$$\leq k'_1 + \frac{k'_2 - 4}{3} + s_n$$

$$= k_1 + \frac{k_2 - 4}{3},$$

where $k_1 = k'_1 + s_n$ is the sum of quotients of $n_i$ when it is divided by 3 and $k_2 = k'_2$ is the sum of remainders of $n_i$ when it is divided by 3, for all $i = 1, 2, ..., n$.

Hence $\sum_{i=1}^{n} r_i \leq k_1 + \frac{k_2 - 4}{3}$, where $k_1$ is the sum of quotients of $n_i$ when it is divided by 3 and $k_2$ is the sum of remainders of $n_i$ when it is divided by 3, for all $i = 1, 2, ..., n$.

Thus the lemma is proved.

**Lemma 3.5** Let $G$ be a complete multi-partite graph. Let $n$ be the number of partitions of the graph $G$, $V_i (i = 1, 2, ..., n)$ be the partitions of $V(G)$ and $n_i$ be the number of vertices in the vertex set $V_i$ and $r_i$ be the number of -2’s in $V_i$. If $\sum_{i=1}^{n} n_i = 3k + 2$ where $k$ is any positive integer, $n \geq 3$, then

$$\sum_{i=1}^{n} r_i \leq \begin{cases} k_1 + l_1 + l_2 & \text{if} \ (m_1 \text{ times } n_i = 3s, 2 + 3m_2 \text{ times } n_i = 3s + 1 \\
\text{and } 3m_3 \text{ times } n_i = 3s + 2 \text{ for all } m_1, m_2, m_3 \geq 0) \text{ or} \\
(m_1 \text{ times } n_i = 3s, 3m_2 \text{ times } n_i = 3s + 1 \text{ and } 1 + 3m_3 \text{ times } n_i = 3s + 2 \\
\text{for all } m_1, m_3 \geq 0, m_2 \geq 1), s, m_1, m_2, m_3 \text{ are positive integers .} \\
k_1 + l_1 + l_2 - 1 & \text{otherwise .} \end{cases}$$

where $k_1$ is the sum of quotients of $n_i$ when it is divided by 3, for all $i = 1, 2, ..., n$, $l_1$ is the quotient of $k_2$ when it is divided by 3, where $k_2$ is the sum of remainders of
when it is divided by 3, for all \( n_i = 3s_i + 1 \) and \( l_2 \) is the quotient of \( k_3 \) when it is
divided by 3, where \( k_3 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all
\( n_i = 3s_i + 2 \).

**Proof:** \( G \) is a complete multi-partite graph. \( n \) is the number of partitions of the graph
\( G \), \( V_i (i = 1, 2, ..., n) \) is the \( n \)-partitions of \( V(G) \) and \( n_i \) is the number of vertices in
the vertex set \( V_i \) and \( r_i \) is the number of -2's in \( V_i \).

Let \( \sum_{i=1}^{n} n_i = 3k + 2 \) where \( k \) is any positive integer and \( n \geq 3 \).

**Case (i):** \( (m_1 \) times \( n_i = 3s_i, 2 + 3m_2 \) times \( n_i = 3s_i + 1 \) and \( 3m_3 \) times \( n_i = 3s_i + 2 \)
for all \( m_1, m_2, m_3 \geq 0 \) or \( (m_1 \) times \( n_i = 3s_i, 3m_2 \) times \( n_i = 3s_i + 1 \) and \( 1 + 3m_3 \) times \( n_i = 3s + 2 \)
for all \( m_1, m_2, m_3 \geq 0, m_2 \geq 1 \), \( s, m_1, m_2, m_3 \) are positive integers.

we prove this by induction on \( n \).

**For \( n = 3 \).**

Let \( V_1, V_2, V_3 \) are tripartitions of \( V \). The cardinality of \( V_i \) are as follows:

| \( |V_1| \) | \( |V_2| \) | \( |V_3| \) |
| --- | --- | --- |
| \( 3s_1 \) | \( 3s_2 + 1 \) | \( 3s_3 + 1 \) |

Then \( n_1 + n_2 \geq 3r_1 + 3r_2 + 1 \)  
(3.40)

\( n_2 + n_3 \geq 3r_2 + 3r_3 + 1 \)  
(3.41)

\( n_1 + n_3 \geq 3r_1 + 3r_3 + 1 \) (since \( \sum_{v \in N(u)} f(v) \geq 1 \)).  
(3.42)

(i.e.) \( 3s_1 + 3s_2 + 1 \geq 3r_1 + 3r_2 + 1 \)

\( 3s_2 + 1 + 3s_3 + 1 \geq 3r_2 + 3r_3 + 1 \)

\( 3s_1 + 3s_3 + 1 \geq 3r_1 + 3r_3 + 1 \).

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\[(i.e.) \quad r_1 + r_2 \leq s_1 + s_2 \]
\[r_2 + r_3 \leq s_2 + s_3 + \frac{1}{3} \]
\[r_1 + r_3 \leq s_1 + s_3. \]

Therefore \(r_1 \leq s_1, r_2 \leq s_2\) and \(r_3 \leq s_3\). Thus \(r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 = k_1 + l_1 + l_2\).

For \(n = 4\).

Let \(V_1, V_2, V_3, V_4\) are partitions of \(V\). The cardinality of \(V_i\) are as follows:

| \(|V_1|\) | \(|V_2|\) | \(|V_3|\) | \(|V_4|\) |
|---|---|---|---|
| \(3s_1\) | \(3s_2\) | \(3s_3 + 1\) | \(3s_4 + 1\) |
| \(3s_1 + 1\) | \(3s_2 + 1\) | \(3s_3 + 1\) | \(3s_4 + 2\) |

If \(|V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3 + 1, |V_4| = 3s_4 + 1\).

Then \(n_1 + n_2 + n_3 \geq 3r_1 + 3r_2 + 3r_3 + 1\) \hfill (3.43)
\(n_2 + n_3 + n_4 \geq 3r_2 + 3r_3 + 3r_4 + 1\) \hfill (3.44)
\(n_1 + n_3 + n_4 \geq 3r_1 + 3r_3 + 3r_4 + 1\) \hfill (3.45)
\(n_1 + n_2 + n_4 \geq 3r_1 + 3r_2 + 3r_4 + 1\) (since \(\sum_{v \in H(n)} f(v) \geq 1\)). \hfill (3.46)

\[(i.e.) \quad 3s_1 + 3s_2 + 3s_3 + 1 \geq 3r_1 + 3r_2 + 3r_3 + 1 \]
\[3s_2 + 3s_3 + 3s_4 + 2 \geq 3r_2 + 3r_3 + 3r_4 + 1 \]
\[3s_1 + 3s_3 + 3s_4 + 2 \geq 3r_1 + 3r_3 + 3r_4 + 1 \]
\[3s_1 + 3s_2 + 3s_4 + 1 \geq 3r_1 + 3r_2 + 3r_4 + 1 \]

\[(i.e.) \quad r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 \]
\[r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 + \frac{1}{3} \]

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\[ r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 + \frac{1}{3} \]
\[ r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4. \]

Therefore \( r_1 \leq s_1, r_2 \leq s_2, r_3 \leq s_3, r_4 \leq s_4. \)

Thus \( r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 = k_1 + l_1 + l_2. \)

If \( |V_1| = 3s_1 + 1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 1, |V_4| = 3s_4 + 2 \), then by (3.43), (3.44), (3.45) and (3.46),
\[
3s_1 + 3s_2 + 3s_3 + 3 \geq 3r_1 + 3r_2 + 3r_3 + 1 \\
3s_2 + 3s_3 + 3s_4 + 4 \geq 3r_2 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_3 + 3s_4 + 4 \geq 3r_1 + 3r_3 + 3r_4 + 1 \\
3s_1 + 3s_2 + 3s_4 + 4 \geq 3r_1 + 3r_2 + 3r_4 + 1.
\]

(i.e.) \( r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + \frac{2}{3} \)
\[ r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 + 1 \]
\[ r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 + 1 \]
\[ r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 + 1. \]

Let \( r_2 \leq s_2, r_3 \leq s_3, r_4 \leq s_4 + 1. \)

If \( r_1 \leq s_1 + 1 \), then \( r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 + 1 \) but \( r_1 + r_2 + r_3 \not\leq s_1 + s_2 + s_3 + \frac{2}{3}, r_1 + r_3 + r_4 \not\leq s_1 + s_3 + s_4 + 1 \) and \( r_1 + r_2 + r_4 \not\leq s_1 + s_2 + s_4 + 1. \)

This is a contradiction. Therefore \( r_1 \leq s_1. \)

Thus \( r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 + 1 = k_1 + l_1 + l_2. \)

For \( n = 5. \)

Let \( V_1, V_2, V_3, V_4, V_5 \) are partitions of \( V \). The cardinality of \( V_i \) are as follows:
| $|V_1|$ | $|V_2|$ | $|V_3|$ | $|V_4|$ | $|V_5|$ |
|-------|-------|-------|-------|-------|
| $3s_1$ | $3s_2$ | $3s_3$ | $3s_4 + 1$ | $3s_5 + 1$ |
| $3s_1$ | $3s_2 + 1$ | $3s_3 + 1$ | $3s_4 + 1$ | $3s_5 + 2$ |
| $3s_1 + 1$ | $3s_2 + 1$ | $3s_3 + 1$ | $3s_4 + 1$ | $3s_5 + 1$ |
| $3s_1 + 1$ | $3s_2 + 1$ | $3s_3 + 2$ | $3s_4 + 2$ | $3s_5 + 2$ |

If $|V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3, |V_4| = 3s_4 + 1, |V_5| = 3s_5 + 1.$

We know that $r_2 + r_3 + r_4 + r_5 \leq s_2 + s_3 + s_4 + s_5$ and $r_1 \leq s_1.$

Therefore $r_1 + r_2 + r_3 + r_4 + r_5 \leq s_1 + s_2 + s_3 + s_4 + s_5 = k_1 + l_1 + l_2.$

If $|V_1| = 3s_1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 1, |V_4| = 3s_4 + 1, |V_5| = 3s_5 + 2.$

We know that $r_2 + r_3 + r_4 + r_5 \leq s_2 + s_3 + s_4 + s_5 + 1$ and $r_1 \leq s_1.$

Therefore $r_1 + r_2 + r_3 + r_4 + r_5 \leq s_1 + s_2 + s_3 + s_4 + s_5 + 1 = k_1 + l_1 + l_2.$

If $|V_1| = 3s_1 + 1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 1, |V_4| = 3s_4 + 1, |V_5| = 3s_5 + 1.$

Then

$$n_1 + n_2 + n_3 + n_4 \geq 3r_1 + 3r_2 + 3r_3 + 3r_4 + 1 \quad (3.47)$$

$$n_2 + n_3 + n_4 + n_5 \geq 3r_2 + 3r_3 + 3r_4 + 3r_5 + 1 \quad (3.48)$$

$$n_1 + n_3 + n_4 + n_5 \geq 3r_1 + 3r_3 + 3r_4 + 3r_5 + 1 \quad (3.49)$$

$$n_1 + n_2 + n_4 + n_5 \geq 3r_1 + 3r_2 + 3r_4 + 3r_5 + 1 \quad (3.50)$$

$$n_1 + n_2 + n_3 + n_5 \geq 3r_1 + 3r_2 + 3r_3 + 3r_5 + 1. \quad (3.51)$$

(i.e.) $3s_1 + 3s_2 + 3s_3 + 3s_4 + 4 \geq 3r_1 + 3r_2 + 3r_3 + 3r_4 + 1.$

(i.e.) $r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 + 1$

Similarly $r_2 + r_3 + r_4 + r_5 \leq s_2 + s_3 + s_4 + s_5 + 1$

$$r_1 + r_3 + r_4 + r_5 \leq s_1 + s_3 + s_4 + s_5 + 1$$

$$r_1 + r_2 + r_4 + r_5 \leq s_1 + s_2 + s_4 + s_5 + 1$$

$$r_1 + r_2 + r_3 + r_5 \leq s_1 + s_2 + s_3 + s_5 + 1.$$
Let \( r_1 \leq s_1, r_2 \leq s_2, r_3 \leq s_3, r_4 \leq s_4 + 1. \)

If \( r_5 \leq s_5 + 1, \) then \( r_1 + r_2 + r_3 + r_4 + r_5 \leq s_1 + s_2 + s_3 + s_4 + 1, r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 + 1, r_1 + r_2 + r_3 + r_4 + r_5 \leq s_1 + s_2 + s_3 + s_4 + s_5 + 1 \) but \( r_2 + r_3 + r_4 + r_5 \not\leq s_2 + s_3 + s_4 + s_5 + 1, r_1 + r_3 + r_4 + r_5 \not\leq s_1 + s_3 + s_4 + s_5 + 1, \) and \( r_1 + r_2 + r_4 + r_5 \not\leq s_1 + s_2 + s_4 + s_5 + 1. \)

This is a contradiction. Therefore \( r_5 \leq s_5. \)

Thus \( r_1 + r_2 + r_3 + r_4 + r_5 \leq s_1 + s_2 + s_3 + s_4 + s_5 + 1 = k_1 + l_1 + l_2. \)

If \( |V_1| = 3s_1 + 1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 2, |V_4| = 3s_4 + 2, |V_5| = 3s_5 + 2, \) then by (3.47), (3.48), (3.49), (3.50) and (3.51),

\[
3s_1 + 3s_2 + 3s_3 + 3s_4 + 6 \geq 3r_1 + 3r_2 + 3r_3 + 3r_4 + 1 \\
3s_2 + 3s_3 + 3s_4 + 3s_5 + 7 \geq 3r_2 + 3r_3 + 3r_4 + 3r_5 + 1 \\
3s_1 + 3s_3 + 3s_4 + 3s_5 + 7 \geq 3r_1 + 3r_3 + 3r_4 + 3r_5 + 1 \\
3s_1 + 3s_2 + 3s_4 + 3s_5 + 6 \geq 3r_1 + 3r_2 + 3r_4 + 3r_5 + 1 \\
3s_1 + 3s_2 + 3s_3 + 3s_5 + 6 \geq 3r_1 + 3r_2 + 3r_3 + 3r_5 + 1.
\]

(i.e.) \( r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 + \frac{5}{3} \)

\[
r_2 + r_3 + r_4 + r_5 \leq s_2 + s_3 + s_4 + s_5 + 2 \\
r_1 + r_3 + r_4 + r_5 \leq s_1 + s_3 + s_4 + s_5 + 2 \\
r_1 + r_2 + r_4 + r_5 \leq s_1 + s_2 + s_4 + s_5 + \frac{5}{3} \\
r_1 + r_2 + r_3 + r_5 \leq s_1 + s_2 + s_3 + s_5 + \frac{5}{3}.
\]

Let \( r_2 \leq s_2, r_3 \leq s_3, r_4 \leq s_4 + 1, r_5 \leq s_5 + 1. \)

If \( r_1 \leq s_1 + 1, \) then \( r_2 + r_3 + r_4 + r_5 \leq s_2 + s_3 + s_4 + s_5 + 2 \) but \( r_1 + r_2 + r_3 + r_4 \not\leq s_1 + s_2 + s_3 + s_4 + s_5 + \frac{5}{3}, r_1 + r_3 + r_4 + r_5 \not\leq s_1 + s_3 + s_4 + s_5 + 2, r_1 + r_2 + r_4 + r_5 \not\leq s_1 + s_2 + s_4 + s_5 + 2. \)
\[ s_1 + s_2 + s_4 + s_5 + \frac{5}{3} \text{ and } r_1 + r_2 + r_3 + r_5 \leq s_1 + s_2 + s_3 + s_4 + s_5 + \frac{5}{3}. \]

This is a contradiction. Therefore \( r_1 \leq s_1 \).

Thus \( r_1 + r_2 + r_3 + r_4 + r_5 \leq s_1 + s_2 + s_3 + s_4 + s_5 + 2 = k_1 + l_1 + l_2 \).

Assume that it is true for complete \((n - 1)\)-partite graph.

(i.e.) \( \sum_{i=1}^{n-1} r_i \leq k'_i + l'_1 + l'_2 \) where \( k'_i \) is the sum of quotients of \( n_i \) when it is divided by 3, for all \( i = 1, 2, ..., n - 1 \), \( l'_1 \) is the quotient of \( k'_2 \) when it is divided by 3, where \( k'_2 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( n_i = 3s_i + 1 \) and \( l'_2 \) is the quotient of \( k'_3 \) when it is divided by 3, where \( k'_3 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( n_i = 3s_i + 2 \), for all \( i = 1, 2, ..., n - 1 \).

Now we prove it for complete \( n \)-partite graph.

Let \( |V_n| = 3s_n \), then \( r_n \leq s_n \).

\[
\begin{align*}
\text{Therefore } \sum_{i=1}^{n} r_i &= \sum_{i=1}^{n-1} r_i + r_n \\
&\leq k'_1 + l'_1 + l'_2 + s_n \\
&= k_1 + l_1 + l_2
\end{align*}
\]

where \( k_1 = k'_1 + s_n \) is the sum of quotients of \( n_i \) when it is divided by 3, for all \( i = 1, 2, ..., n \), \( l_1 = l'_1 \) is the quotient of \( k_2 \) when it is divided by 3, where \( k_2 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( n_i = 3s_i + 1 \) and \( l_2 = l'_2 \) is the quotient of \( k_3 \) when it is divided by 3, where \( k_3 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( n_i = 3s_i + 2 \), for all \( i = 1, 2, ..., n \).

Case (ii): 0 (or) 1 + 3m times 3s + 1 in \( n \)-partite complete graph, where \( m \) is any positive integer. we prove this by induction on \( n \).

For \( n = 3 \).
Let $V_1, V_2, V_3$ are tripartitions of $V$. The cardinality of $V_i$ are as follows:

| $|V_1|$ | $|V_2|$ | $|V_3|$ |
|------|------|------|
| $3s_1$ | $3s_2$ | $3s_3 + 2$ |
| $3s_1 + 1$ | $3s_2 + 2$ | $3s_3 + 2$ |

If $|V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3 + 2$, then by (3.40), (3.41) and (3.42)

\[ 3s_1 + 3s_2 \geq 3r_1 + 3r_2 + 1 \]
\[ 3s_2 + 3s_3 + 2 \geq 3r_2 + 3r_3 + 1 \]
\[ 3s_1 + 3s_3 + 2 \geq 3r_1 + 3r_3 + 1. \]

(i.e.)
\[ r_1 + r_2 \leq s_1 + s_2 - \frac{1}{3} \]
\[ r_2 + r_3 \leq s_2 + s_3 + \frac{1}{3} \]
\[ r_1 + r_3 \leq s_1 + s_3 + \frac{1}{3}. \]

Let $r_2 \leq s_2, r_3 \leq s_3$.

If $r_1 \leq s_1$, then $r_2 + r_3 \leq s_2 + s_3 + \frac{1}{3}, r_3 + r_1 \leq s_3 + s_1 + \frac{1}{3}$ but $r_1 + r_2 \notin s_1 + s_2 - \frac{1}{3}$.

Therefore $r_1 \leq s_1 - 1$.

Thus $r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 - 1 = k_1 + l_1 + l_2 - 1$.

If $|V_1| = 3s_1 + 1, |V_2| = 3s_2 + 2, |V_3| = 3s_3 + 2$, then by (3.40), (3.41) and (3.42)

\[ 3s_1 + 3s_2 + 3 \geq 3r_1 + 3r_2 + 1 \]
\[ 3s_2 + 3s_3 + 4 \geq 3r_2 + 3r_3 + 1 \]
\[ 3s_1 + 3s_3 + 3 \geq 3r_1 + 3r_3 + 1. \]
\[(i.e.) \quad r_1 + r_2 \leq s_1 + s_2 + \frac{2}{3} \]
\[r_2 + r_3 \leq s_2 + s_3 + 1 \]
\[r_1 + r_3 \leq s_1 + s_3 + \frac{2}{3} \]

Let \( r_2 \leq s_2, r_1 \leq s_1 \).

If \( r_3 \leq s_3 + 1 \), then \( r_1 + r_2 \leq s_1 + s_2 + \frac{2}{3}, r_2 + r_3 \leq s_2 + s_3 + 1 \) but \( r_1 + r_3 \leq s_1 + s_3 + \frac{2}{3} \).

Therefore \( r_3 \leq s_3 \).

Thus \( r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 = k_1 + l_1 + l_2 - 1 \).

For \( n = 4 \).

Let \( V_1, V_2, V_3, V_4 \) are partitions of \( V \). The cardinality of \( V_i \) are as follows:

| \( |V_1| \) | \( |V_2| \) | \( |V_3| \) | \( |V_4| \) |
|---|---|---|---|
| 3s_1 | 3s_2 | 3s_3 | 3s_4 + 2 |
| 3s_1 | 3s_2 + 1 | 3s_3 + 2 | 3s_4 + 2 |
| 3s_1 + 2 | 3s_2 + 2 | 3s_3 + 2 | 3s_4 + 2 |

If \( |V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3, |V_4| = 3s_4 + 2 \).

We know that \( r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 - 1 \) and \( r_1 \leq s_1 \).

Therefore \( r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 - 1 = k_1 + l_1 + l_2 - 1 \).

If \( |V_1| = 3s_1, |V_2| = 3s_2 + 1, |V_3| = 3s_3 + 2, |V_4| = 3s_4 + 2 \).

We know that \( r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 \) and \( r_1 \leq s_1 \).

Therefore \( r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 = k_1 + l_1 + l_2 - 1 \).

If \( |V_1| = 3s_1 + 2, |V_2| = 3s_2 + 2, |V_3| = 3s_3 + 2, |V_4| = 3s_4 + 2 \), then by (3.43), (3.44), (3.45) and (3.46),

\[3s_1 + 3s_2 + 3s_3 + 6 \geq 3r_1 + 3r_2 + 3r_3 + 1 \]

\[(i.e.) \quad r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + \frac{5}{3} \].

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Similarly $r_2 + r_3 + r_4 \leq s_2 + s_3 + s_4 + \frac{5}{3}$

$r_1 + r_3 + r_4 \leq s_1 + s_3 + s_4 + \frac{5}{3}$

$r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 + \frac{5}{3}$

Let $r_1 \leq s_1, r_2 \leq s_2, r_3 \leq s_3 + 1$.

If $r_4 \leq s_4 + 1$, then $r_1 + r_2 + r_3 \leq s_1 + s_2 + s_3 + s_4 + \frac{5}{3}, r_1 + r_2 + r_4 \leq s_1 + s_2 + s_4 + \frac{5}{3}$ but $r_2 + r_3 + r_4 \not\leq s_2 + s_3 + s_4 + \frac{5}{3}$ and $r_1 + r_3 + r_4 \not\leq s_1 + s_3 + s_4 + \frac{5}{3}$.

This is a contradiction. Therefore $r_4 \leq s_4$.

Thus $r_1 + r_2 + r_3 + r_4 \leq s_1 + s_2 + s_3 + s_4 + 1 = k_1 + l_1 + l_2 - 1$.

For $n = 5$.

Let $V_1, V_2, V_3, V_4, V_5$ are partitions of $V$. The cardinality of $V_i$ are as follows:

| $|V_1|$ | $|V_2|$ | $|V_3|$ | $|V_4|$ | $|V_5|$ |
|--------|--------|--------|--------|--------|
| $3s_1$ | $3s_2$ | $3s_3$ | $3s_4$ | $3s_5 + 2$ |
| $3s_1$ | $3s_2$ | $3s_3 + 1$ | $3s_4 + 2$ | $3s_5 + 2$ |
| $3s_1$ | $3s_2 + 2$ | $3s_3 + 2$ | $3s_4 + 2$ | $3s_5 + 2$ |

If $|V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3, |V_4| = 3s_4, |V_5| = 3s_5 + 2$.

We know that $r_2 + r_3 + r_4 + r_5 \leq s_2 + s_3 + s_4 + s_5 - 1$ and $r_1 \leq s_1$.

Therefore $r_1 + r_2 + r_3 + r_4 + r_5 \leq s_1 + s_2 + s_3 + s_4 + s_5 - 1 = k_1 + l_1 + l_2 - 1$.

If $|V_1| = 3s_1, |V_2| = 3s_2, |V_3| = 3s_3 + 1, |V_4| = 3s_4 + 2, |V_5| = 3s_5 + 2$.

We know that $r_2 + r_3 + r_4 + r_5 \leq s_2 + s_3 + s_4 + s_5$ and $r_1 \leq s_1$.

Therefore $r_1 + r_2 + r_3 + r_4 + r_5 \leq s_1 + s_2 + s_3 + s_4 + s_5 = k_1 + l_1 + l_2 - 1$.

If $|V_1| = 3s_1, |V_2| = 3s_2 + 2, |V_3| = 3s_3 + 2, |V_4| = 3s_4 + 2, |V_5| = 3s_5 + 2$.

We know that $r_2 + r_3 + r_4 + r_5 \leq s_2 + s_3 + s_4 + s_5 + 1$ and $r_1 \leq s_1$.

Therefore $r_1 + r_2 + r_3 + r_4 + r_5 \leq s_1 + s_2 + s_3 + s_4 + s_5 + 1 = k_1 + l_1 + l_2 - 1$.  

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Assume that it is true for complete \((n-1)\)-partite graph.

(i.e.) \( \sum_{i=1}^{n-1} r_i \leq k'_i + l'_1 + l'_2 - 1 \) where \( k'_i \) is the sum of quotients of \( n_i \) when it is divided by 3, for all \( i = 1, 2, ..., n-1 \), \( l'_1 \) is the quotient of \( k'_3 \) when it is divided by 3, where \( k'_2 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( n_i = 3s_i + 1 \) and \( l'_2 \) is the quotient of \( k'_3 \) when it is divided by 3, where \( k'_3 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( n_i = 3s_i + 2 \), for all \( i = 1, 2, ..., n-1 \).

Now we prove it for complete \( n \)-partite graph.

Let \( |V_n| = 3s_n \). Then \( r_n \leq s_n \).

\[
\text{Therefore } \sum_{i=1}^{n} r_i = \sum_{i=1}^{n-1} r_i + r_n \\
\leq k'_1 + l'_1 + l'_2 - 1 + s_n \\
= k_1 + l_1 + l_2 - 1
\]

where \( k_1 = k'_1 + s_n \) is the sum of quotients of \( n_i \) when it is divided by 3, for all \( i = 1, 2, ..., n \), \( l_1 = l'_1 \) is the quotient of \( k_2 \) when it is divided by 3, where \( k_2 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( n_i = 3s_i + 1 \) and \( l_2 = l'_2 \) is the quotient of \( k_3 \) when it is divided by 3, where \( k_3 \) is the sum of remainders of \( n_i \) when it is divided by 3, for all \( n_i = 3s_i + 2 \), for all \( i = 1, 2, ..., n \).

### 3.2 BSTD Number for complete multi-partite graphs

**Theorem 3.6** Let \( G \) be a complete multi-partite graph. Let \( n \) be the number of partitions of the graph \( G \), \( n \geq 3 \), \( V_i (i = 1, 2, ..., n) \) be the partitions of \( V(G) \) and \( n_i \) be the number of vertices in the vertex set \( V_i \) and \( r_i \) vertices are assigned with -2 in \( V_i \).
Then the bold signed total domination number of $G$ is
\[
\gamma_{bs}(G) = \begin{cases} 
2 & \text{if } \sum_{i=1}^{n} n_i = 3k + 2 \text{ and } 3m - 1 \text{ or } 3m \text{ times } n_i = 3s_i + 1, \forall m > 0. \\
3 & \text{if } \sum_{i=1}^{n} n_i = 3k \text{ and at least one } n_i \text{ is not a multiple of } 3. \\
4 & \text{if } \sum_{i=1}^{n} n_i = 3k + 1. \\
5 & \text{if } \sum_{i=1}^{n} n_i = 3k + 2 \text{ and } 0 \text{ or } 3m + 1 \text{ times } n_i = 3s_i + 1, \forall m \geq 0. \\
6 & \text{if } \sum_{i=1}^{n} n_i = 3k \text{ and all } n_i \text{ are multiple of } 3,
\end{cases}
\]
where $k$, $s_i$, $m$ are any positive integers.

**proof:** $G$ is a complete multi-partite graph and $V_1, V_2, ..., V_n$ are the partitions of the vertex set $V$ and $|V_i| = n_i$ for $i = 1, 2, ..., n$ and $r_i$ vertices are assigned with -2 in $V_i$. We prove this result in 3 cases of $\sum_{i=1}^{n} n_i$.

**case (i):** $\sum_{i=1}^{n} n_i = 3k$ where $k$ is any positive integer and $k \geq 1$.

Now we prove $\gamma_{bs}(G)$ in 2 subcases.

(i.e.) $\gamma_{bs}(G) = \begin{cases} 
6 & \text{if all } n_i \text{ are multiple of } 3 \\
3 & \text{if at least one } n_i \text{ is not a multiple of } 3.
\end{cases}$

Let $\sum_{i=1}^{n} n_i = 3k_1 + k_2$, where $k_1$ is the sum of quotients of $n_i$ when it is divided by 3 and $k_2$ is the sum of remainders of $n_i$ when it is divided by 3, for all $i = 1, 2, ..., n$.

**Subcase (i):** All $n_i$s are multiple of 3, we claim that $\gamma_{bs}(G) = 6$.

(i.e.) $n_i = 3s_i$ for $i = 1, 2, ..., n$. Then $k_2 = 0$.

We prove this by induction on $n$.

**For $n = 3$.**

$|V_i| = n_i$ for $i = 1, 2, 3$.

Since $n_i = 3s_i$ for $i = 1, 2, 3$, $r_1 + r_2 + r_3 \leq k_1 - 2$, by Lemma 3.3.

Therefore $w(f) = \sum_{v \in V(G)} f(v) = n_1 + n_2 + n_3 - 3(r_1 + r_2 + r_3) \geq 3k_1 - 3(k_1 - 2) = 6$.
Therefore \( \gamma_{bst}(G) = \min w(f) = 6. \)

For \( n = 4. \)

\( |V_i| = n_i \) for \( i = 1, 2, 3, 4. \)

Since \( n_i = 3s_i \) for \( i = 1, 2, 3, 4, \sum_{i=1}^{4} r_i \leq k_1 - 2, \) by Lemma 3.3.

\[
\therefore w(f) = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{4} n_i - 3 \sum_{i=1}^{4} r_i \\
\geq 3k_1 - 3(k_1 - 2) = 6.
\]

Therefore \( \gamma_{bst}(G) = \min w(f) = 6. \)

Assume that it is true for \( n = m - 1 \) and all \( n_i = 3s_i. \)

Let \( n = m. \) \( |V_i| = n_i \) for \( i = 1, 2, ..., m. \) \( n_i = 3s_i \) for all \( i. \)

Now by Theorem 3.2, we have

\[
\begin{align*}
    r_i & \leq \frac{(n - 1)n_i - 1}{3(n - 1)} + \frac{1}{3(n - 1)} \quad \text{if } n_i \equiv 0 \mod 3 \\
    & = \frac{n_i}{3} \\
    & = s_i \text{ for all } i. \\
\end{align*}
\]

Hence \( r_m \leq s_m. \) \hspace{1cm} (3.52)

\[
\therefore w(f) = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{m} n_i - 3 \sum_{i=1}^{m} r_i \\
= \sum_{i=1}^{m-1} n_i + n_m - 3(\sum_{i=1}^{m-1} r_i + r_m) \\
= (\sum_{i=1}^{m-1} n_i - 3 \sum_{i=1}^{m-1} r_i) + n_m - 3r_m \\
\geq 6 + 3s_m - 3s_m \quad \text{(since by our assumption)} \\
= 6.
\]
Therefore $\gamma_{bst}(G) = \min w(f) = 6$.

Hence $\gamma_{bst}(G) = 6$ if for all $n_i$ are multiple of 3.

**Subcase (ii):** For atleast one $n_i$ is not a multiple of 3. We claim that $\gamma_{bst}(G) = 3$.

(i.e.) $n_i \neq 3s_i$ for atleast one $i$. Then $k_2 = 3s$, where $s$ is any positive integer.

We prove the claim by induction on $n$.

**For $n = 3$.**

Here $r_1 + r_2 + r_3 \leq k_1 + \frac{k_2}{3} - 1$, by Lemma 3.3.

Therefore $w(f) = \sum_{v \in V(G)} f(v) = n_1 + n_2 + n_3 - 3(r_1 + r_2 + r_3) \geq 3k_1 + k_2 - 3(k_1 + \frac{k_2}{3} - 1) = 3$.

Therefore $\gamma_{bst}(G) = \min w(f) = 3$.

**For $n = 4$.**

Here $\sum_{i=1}^{4} r_i \leq k_1 + \frac{k_2}{3} - 1$, by Lemma 3.3.

Therefore $w(f) = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{4} n_i - 3 \sum_{i=1}^{4} r_i \geq 3k_1 + k_2 - 3(k_1 + \frac{k_2}{3} - 1) = 3$.

Therefore $\gamma_{bst}(G) = \min w(f) = 3$.

Assume that it is true for $n = m - 1$.

Let $n = m$. $|V_i| = n_i$ for $i = 1, 2, ..., m$.

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Let $n_m = 3s_m$. Then $r_m \leq s_m$, by equation (3.52)

Therefore $w(f) = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{m} n_i - 3 \sum_{i=1}^{m} r_i$

$= \sum_{i=1}^{m-1} n_i + n_m - 3(\sum_{i=1}^{m-1} r_i + r_m)$

$= \left(\sum_{i=1}^{m-1} n_i - 3 \sum_{i=1}^{m-1} r_i\right) + (n_m - 3r_m)$

$\geq 3 + 3s_m - 3s_m \quad \text{(since by our assumption)}$

$= 3.$

Therefore $\gamma_{bcl}(G) = \min w(f) = 3$.

Hence $\gamma_{bcl}(G) = 3$ if for at least one $n_i$ is not a multiple of 3.

**case (ii):** $\sum_{i=1}^{n} n_i = 3k + 1$, where $k$ is any positive integer and $k \geq 1$.

Now we prove $\gamma_{bcl}(G) = 4$.

Let $\sum_{i=1}^{n} n_i = 3k_1 + k_2$, where $k_1$ is the sum of quotients of $n_i$ when it is divided by 3 and $k_2$ is the sum of remainders of $n_i$ when it is divided by 3, for all $i = 1, 2, ..., n$.

We prove this by induction on $n$.

**For n = 3.**

Here $r_1 + r_2 + r_3 \leq k_1 + \frac{k_2 - 1}{3}$, by Lemma 3.4.

Therefore $w(f) = \sum_{v \in V(G)} f(v) = n_1 + n_2 + n_3 - 3(r_1 + r_2 + r_3)$

$\geq 3k_1 + k_2 - 3\left(k_1 + \frac{k_2 - 4}{3}\right) = 4.$

Therefore $\gamma_{bcl}(G) = \min w(f) = 4$. 

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For $n = 4$.

Here $\sum_{i=1}^{4} r_i \leq k_1 + \frac{k_2 - 1}{3}$, by Lemma 3.4.

Therefore $w(f) = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{4} n_i - 3 \sum_{i=1}^{4} r_i$

\[ \geq 3k_1 + k_2 - 3(k_1 + \frac{k_2 - 4}{3}) = 4. \]

Therefore $\gamma_{\text{bal}}(G) = \min w(f) = 4$.

Assume that it is true for $n = m - 1$.

Let $n = m$. $|V_i| = n_i$ for $i = 1, 2, ..., m$.

Let $n_m = 3s_m$. Then $r_m \leq s_m$, by equation (3.52).

Therefore $w(f) = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{m} n_i - 3 \sum_{i=1}^{m} r_i$

\[ = \sum_{i=1}^{m-1} n_i + n_m - 3(\sum_{i=1}^{m-1} r_i + r_m) \]

\[ = (\sum_{i=1}^{m-1} n_i - 3 \sum_{i=1}^{m-1} r_i) + (n_m - 3r_m) \]

\[ \geq 4 + 3s_m - 3s_m (\text{since by our assumption}) \]

\[ = 4. \]

Therefore $\gamma_{\text{bal}}(G) = \min w(f) = 4$.

Hence $\gamma_{\text{bal}}(G) = 4$ if $\sum_{i=1}^{n} n_i = 3k + 1$.

\textbf{case (iii): $\sum_{i=1}^{n} n_i = 3k + 2$, where $k$ is any positive integer and $k \geq 1$.}

Now we prove $\gamma_{\text{bal}}(G)$ in 2 subcases.

(i.e.) $\gamma_{\text{bal}}(G) = \begin{cases} 2 \text{ if } 3m - 1 \text{ or } 3m \text{ times } n_i = 3s_i + 1 \text{ for all } m > 0 \\ 5 \text{ if } 0 \text{ or } 3m + 1 \text{ times } n_i = 3s_i + 1 \text{ for all } m \geq 0. \end{cases}$

Let $\sum_{i=1}^{n} n_i = 3k_1 + k_2 + k_3$, where $k_1$ is the sum of quotients of $n_i$ when it is divided by

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3, for all \( i = 1, 2, \ldots, n \), \( k_2 \) is the sum of remainders of \( n_i \) when it is divided by 3, for those \( n_i = 3s_i + 1 \) and \( k_3 \) is the sum of remainders of \( n_i \) when it is divided by 3, for those \( n_i = 3s_i + 2 \).

**Subcase (i):** \( 3m - 1 \) or \( 3m \) times \( n_i = 3s_i + 1 \) for all \( m > 0 \).

We claim that \( \gamma_{ba}(G) = 2 \).

Let \( k_2 + k_3 = 3l_1 + 3l_2 + 2 \), where \( l_1 \) is the quotient of \( k_2 \) when it is divided by 3 and \( l_2 \) is the quotient of \( k_3 \) when it is divided by 3.

We prove this by induction on \( n \).

**For \( n = 3 \).**

\( |V_i| = n_i \) for \( i = 1, 2, 3 \).

Then \( r_1 + r_2 + r_3 \leq k_1 + l_1 + l_2 \), where \( l_1 \) is the quotient of \( k_2 \) when it is divided by 3 and \( l_2 \) is the quotient of \( k_3 \) when it is divided by 3, by Lemma 3.5.

\[
\text{Therefore } w(f) = \sum_{v \in V(G)} f(v) = n_1 + n_2 + n_3 - 3(r_1 + r_2 + r_3) \\
\geq (3k_1 + k_2 + k_3) - 3(k_1 + l_1 + l_2) \\
= (3l_1 + 3l_2 + 2) - 3(l_1 + l_2) \\
= 2.
\]

Therefore \( \gamma_{ba}(G) = \min w(f) = 2 \).

**For \( n = 4 \).**

\( |V_i| = n_i \) for \( i = 1, 2, 3, 4 \).

Then \( \sum_{i=1}^{4} r_i \leq k_1 + l_1 + l_2 \), where \( l_1 \) is the quotient of \( k_2 \) when it is divided by 3 and \( l_2 \)
is the quotient of $k_3$ when it is divided by 3, by Lemma 3.5.

\[
\text{Therefore } w(f) = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{d} n_i - 3 \sum_{i=1}^{d} r_i \\
\geq (3k_1 + k_2 + k_3) - 3(k_1 + l_1 + l_2) \\
= (3l_1 + 3l_2 + 2) - 3(l_1 + l_2) \\
= 2.
\]

Therefore $\gamma_{bst}(G) = \min w(f) = 2$.

Assume that this result is true for $n = m - 1$.

Let $n = m$ and $n_m = 3s_m$. Then $r_m \leq s_m$, by equation (3.52).

\[
\text{Therefore } w(f) = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{m} n_i - 3 \sum_{i=1}^{m} r_i \\
= \sum_{i=1}^{m-1} n_i + n_m - 3(\sum_{i=1}^{m-1} r_i + r_m) \\
= (\sum_{i=1}^{m-1} n_i - 3 \sum_{i=1}^{m-1} r_i) + n_m - 3r_m \\
\geq 2 + 3s_m - 3s_m (\text{since by our assumption}) \\
= 2.
\]

Therefore $\gamma_{bst}(G) = \min w(f) = 2$.

Hence $\gamma_{bst}(G) = 2$ if $3m - 1$ or $3m$ times $n_i = 3s_i + 1$ for all $m > 0$.

**Subcase (ii):** 0 or $3m + 1$ times $n_i = 3s_i + 1$ for all $m \geq 0$.

We claim that $\gamma_{bst}(G) = 5$.

Let $k_2 + k_3 = 3l_1 + 3l_2 + 2$, where $l_1$ is the quotient of $k_2$ when it is divided by 3 and $l_2$ is the quotient of $k_3$ when it is divided by 3.

We prove this by induction on $n$. 77
For $n = 3$.

$|V_i| = n_i$ for $i = 1, 2, 3$.

Then $r_1 + r_2 + r_3 \leq k_1 + l_1 + l_2 - 1$, where $l_1$ is the quotient of $k_2$ when it is divided by 3 and $l_2$ is the quotient of $k_3$ when it is divided by 3, by Lemma 3.5.

\[
\begin{align*}
\text{Therefore } w(f) &= \sum_{v \in V(G)} f(v) = n_1 + n_2 + n_3 - 3(r_1 + r_2 + r_3) \\
&\geq (3k_1 + k_2 + k_3) - 3(k_1 + l_1 + l_2 - 1) \\
&= (3l_1 + 3l_2 + 2) - 3(l_1 + l_2 - 1) \\
&= 5.
\end{align*}
\]

Therefore $\gamma_{bat}(G) = \min w(f) = 5$.

For $n = 4$.

$|V_i| = n_i$ for $i = 1, 2, 3, 4$.

Then $\sum_{i=1}^{4} r_i \leq k_1 + l_1 + l_2 - 1$, where $l_1$ is the quotient of $k_2$ when it is divided by 3 and $l_2$ is the quotient of $k_3$ when it is divided by 3, by Lemma 3.5.

\[
\begin{align*}
\text{Therefore } w(f) &= \sum_{v \in V(G)} f(v) = \sum_{i=1}^{4} n_i - 3 \sum_{i=1}^{4} r_i \\
&\geq (3k_1 + k_2 + k_3) - 3(k_1 + l_1 + l_2 - 1) \\
&= (3l_1 + 3l_2 + 2) - 3(l_1 + l_2 - 1) \\
&= 5.
\end{align*}
\]

Therefore $\gamma_{bat}(G) = \min w(f) = 5$.

Assume that this result is true for $n = m - 1$. 

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Let \( n = m \) and \( n_m = 3s_m \). Then \( r_m \leq s_m \), by equation (3.52).

\[
\text{Therefore } w(f) = \sum_{v \in V(G)} f(v) = \sum_{i=1}^{m} n_i - 3 \sum_{i=1}^{m} r_i
\]
\[
= \sum_{i=1}^{m-1} n_i + n_m - 3(\sum_{i=1}^{m-1} r_i + r_m)
\]
\[
= (\sum_{i=1}^{m-1} n_i - 3 \sum_{i=1}^{m-1} r_i) + n_m - 3r_m
\]
\[
\geq 5 + 3s_m - 3s_m \ (\text{since by our assumption})
\]
\[
= 5.
\]

Therefore \( \gamma_{bst}(G) = \min w(f) = 5. \)

Thus \( \gamma_{bst}(G) = 5 \) if 0 or \( 3m + 1 \) times \( n_i = 3s_i + 1 \) for all \( m \geq 0. \)

2 if \( \sum_{i=1}^{n} n_i = 3k + 2 \) and \( 3m - 1 \) or \( 3m \) times \( n_i = 3s_i + 1, \forall m > 0. \)

3 if \( \sum_{i=1}^{n} n_i = 3k \) and at least one \( n_i \) is not a multiple of 3.

Hence \( \gamma_{bst}(G) = \begin{cases} 
4 \text{ if } \sum_{i=1}^{n} n_i = 3k + 1. \\
5 \text{ if } \sum_{i=1}^{n} n_i = 3k + 2 \text{ and } 0 \text{ or } 3m + 1 \text{ times } n_i = 3s_i + 1, \forall m \geq 0. \\
6 \text{ if } \sum_{i=1}^{n} n_i = 3k \text{ and all } n_i \text{ are multiple of } 3,
\end{cases} \)

where \( k, s_i, m \) are any positive integers.

\[
\star\star\star\star\star
\]