CHAPTER III

THEORETICAL ANALYSIS

3.1.1 Introduction

Analysis of airspring is developed in this chapter right from the simplest airspring consisting of a piston in a cylinder to the model of two-degrees-of-freedom system of vehicle suspensions. As will be seen an airspring is a nonlinear system. Conventionally it is linearised by making assumptions of small motions and small changes in pressures. These assumptions are true in most practical cases. However linearised systems can be studied only for steady state response. For response to initial disturbances or shock inputs it is necessary to solve nonlinear equations numerically.

Theory of linearised system as available in the literature is given in this chapter and the same is extended to the analysis of vehicle suspensions which includes bounce and pitch motions. This analysis has been done for the first time and since the use of airsprings on vehicles is increasing it is felt that these results will be useful in suspension design.

Nonlinear equations have been developed in the dimensionless parameters for various cases. These need numerical solution. Computer Programs for these cases are developed and are given the appendices. Numerical results obtained are discussed in the next chapter.

3.2.1 A Cylinder Piston Arrangement

Consider a piston carrying a mass m floating in a cylinder of cross-section area A.

In static condition

$$P_0 A = mg \qquad \ldots(1)$$

where P_0 is the initial pressure of gas in the cylinder. Let h_0

be the height of the gas column in the cylinder. Hence initial volume of gas in the cylinder is

$$v_o = Ah_o \qquad \ldots(2)$$

when the mass vibrates, gas undergoes a process of adiabatic expansion or compression.

$$pv^{\gamma} = \text{constant} \qquad \ldots(3)$$

$$p\gamma v^{\gamma-1}\,dv + dp\,v^{\gamma} = 0$$

$$dp = -\frac{p\gamma\,A\,dh}{v}$$

$$= -\frac{p\gamma\,A\,dh}{Ah}$$

$$\frac{dp}{dh} = -\frac{p\gamma A}{Ah} = \frac{-p\gamma}{h} \qquad \ldots(4)$$

$$\text{Stiffness} = \frac{dF}{dh} = \frac{A\,dp}{dh} = \frac{+p\gamma A}{h} \qquad \ldots(5)$$

Hence equation of motion for the mass becomes

$$m\ddot{x} = \{p\gamma A/(h_o+x)\}x \qquad \ldots(6)$$

where x is the displacement from the static equilibrium position.

3.2.2 It can be seen that equation is nonlinear in x. If x is assumed to be sufficiently small w.r.t. h_o

$$m\ddot{x} = -\frac{p_o\gamma A x}{h_o} \qquad \ldots(7)$$

Natural frequency of the system is

$$\omega_n^2 = \frac{p_o\gamma A}{m h_o} = \frac{\gamma g}{h_o}$$

$$= \frac{p_o\gamma A^2}{m v_o} = \frac{\gamma g A}{v_o} \qquad \ldots(8)$$

This frequency is used as the reference frequency in the further analysis.

It can be seen that natural frequency can be reduced by increasing the initial height h_o which certainly has practical limitations. Situation can be improved by increasing the volume of the system by connecting it to a tank thus increasing volume v_o while still keeping piston area A of limited value.

3.2.3 However introducing this storage tank makes the system more complicated. This is due to the fact that cylinder and tank are to be connected by some passage for the air flow to occur. If the area of the passage is kept sufficiently large then there is hardly any restriction to the flow and the cylinder and the tank can be considered as one volume.

A large passage will make the system devoid of any damping. This will restrict the use of the system only to a very limited range of frequency and most probably system will not be at all useful under conditions of shock.

Actually as seen from the literature survey the system is attractive because of the inherent damping it can produce due to flow restrictions. If the frequency of vibration is very low there is sufficient time for air to flow from the cylinder to tank and from tank to cylinder. There is energy loss due to the fluid friction. Natural frequency of the system is determined by the combined volume of the cylinder and tank. Thus at low frequencies system behaves as a system having low natural frequency and damping. Damping is determined by passage dimensions. System has good transmissibility characteristics and can take care of resonance conditions as the damping is present.

It is expected that at higher frequencies of vibrations there is no time for the flow to take place between the cylinder and the tank. Thus natural frequency is determined by cylinder volume alone and no damping is present. Hence the system has a dual frequency characteristics.

3.3.1

From Appendix I,

$$\frac{dm}{dt} = \frac{1}{RT}\left(\frac{dp}{dt}\ \frac{u}{\gamma} + p\frac{du}{dt}\right)$$

For small displacements of the piston

$v = v_o$

$p_1 = p_0$

$$\frac{p_1 + p_{t_1}}{2} = p_0$$

For capillary flow from cylinder to tank

$$\frac{dm}{dt} = \frac{cr}{2RT}(p_1^2 - p_t^2)$$

$$= \frac{cr}{2RT}(p_1 + pt)(p_1 - pt)$$

$$= \frac{p_0 cr}{RT}(p_1 - pt)$$

With the approximation $p_1 \approx p_0$ and $v_1 = v_0$

$$\dot{m} = \frac{1}{RT}\left(\frac{v_0}{\gamma}\ \frac{dp_1}{dt} + p_0\frac{dv_1}{dt}\right) \quad \text{for cylinder.}$$

$$\dot{m} = \frac{p_0 cr}{RT}(pt - p_1) \quad \text{(2) for capillary flow.}$$

$$\dot{m} = \frac{-v_t}{\gamma RT}\ \frac{dpt}{dt} \quad \text{(3) for auxilliary tank}$$

From (2) and (3)

$$p_0 cr\,(pt - p_1) = \frac{-v_t}{\gamma}\ \frac{dpt}{dt}$$

putting $d/d_t = D$ and using it as an operator,

$$Pt\left(p_0\,cr + \frac{v_t}{\gamma}D\right) = p_0 cr\, p_1$$

$$P_t = \frac{P_o C_r P_1}{P_o C_r + \frac{\vartheta_t D}{\gamma}} = \frac{\gamma P_o C_r P_1}{\gamma P_o C_r + \vartheta_t D}$$

Equating (1) and (2) and substituting for P_t

$$\frac{\vartheta_o}{\gamma} D P_1 + P_o D \vartheta_1 = P_o C_r P_1 \left(\frac{\gamma P_o C_r}{\gamma P_o C_r + \vartheta_t D} - 1 \right)$$

$$P_1 \times \left(\frac{\vartheta_o}{\gamma} D + P_o C_r \left(\frac{\vartheta_t D}{\gamma P_o C_r + \vartheta_t D} \right) \right) = - P_o D \vartheta_1$$

$$P_1 = \frac{-P_o D \vartheta_1}{\frac{\vartheta_o D}{\gamma} + \frac{P_o C_r \vartheta_t D}{\gamma P_o C_r + \vartheta_t D}}$$

$$= \frac{-\gamma P_o D \vartheta_1}{\vartheta_o D + \frac{\gamma P_o C_r \vartheta_t D}{\gamma P_o C_r + \vartheta_t D}}$$

$$D P_1 = \frac{-\gamma P_o D \vartheta_1}{\left(\vartheta_o + \frac{\gamma P_o C_r \vartheta_t}{\gamma P_o C_r + \vartheta_t D} \right)}$$

Putting $D\vartheta_1 = DAh$ and assuming constant area

$$D P_1 = \frac{-A \gamma P_o D h}{\vartheta_o + \frac{\gamma P_o C_r \vartheta_t}{\gamma P_o C_r + \vartheta_t D}}$$

$\therefore$ Stiffness $K = A \frac{DP_1}{Dh}$

$$= \frac{\gamma P_o A^2}{\vartheta_o + \frac{\gamma P_o C_r \vartheta_t}{\gamma P_o C_r + \vartheta_t D}}$$

3.3.2 Following Bachrach and Rivin(17)

Let $N = \frac{\vartheta_t}{\vartheta_c}$, $\omega_r = \frac{\gamma P_0 C_r}{\vartheta_c}$, $K_c = \frac{\gamma P_0 A^2}{\vartheta_c}$

$D = j\omega$

$$K = \frac{K_c}{1 + \frac{\gamma P_0 C_r N}{\gamma P_0 C_r + j\omega\vartheta_t}} = \frac{K_c}{1 + \frac{\omega_r N}{\omega_r + j\omega N}}$$

$$= \frac{K_c(\omega_r + j\omega N)}{\omega_r + j\omega N + \omega_r N}$$

$$= \frac{K_c(1 + jN\omega/\omega_r)}{1 + N + N(j\omega/\omega_r)}$$

$$\mathrm{Re}\, K = \frac{K_c(1 + (N\omega/\omega_r)^2 + N)}{(1+N)^2 + (N\omega/\omega_r)^2}$$

$$\mathrm{Im}\ K = \frac{K_c\left[(1+N)(N\omega/\omega_r) - (N\omega/\omega_r)\right]}{(1+N)^2 + (N\omega/\omega_r)^2}$$

$$= \frac{K_c N^2 \omega/\omega_r}{(1+N)^2 + (N\omega/\omega_r)^2}$$

Considering K = Re[k] (1 + iη)

$$\eta = \frac{N^2 (\omega/\omega_r)}{1+N+\left(\frac{N\omega}{\omega_r}\right)^2}$$ is the loss factor.

Putting $(\omega/\omega_0) = r$

$$\eta = \frac{N^2 r}{1+N+N^2 r^2}$$

$$\frac{d\eta}{dr} = \frac{N^2(1+N+N^2 r^2) - N^2 r(2N^2 r)}{(1+N+N^2 r^2)^2}$$

for η max,

$$r = \frac{(1+N)^{1/2}}{N}$$

$$\therefore \eta_{max} = \frac{N\sqrt{1+N}}{2(1+N)} = \frac{N}{2\sqrt{1+N}} = \frac{1}{2r}$$

$$\frac{d\eta}{dN} = \frac{2Nr(1+N+N^2 r^2) - N^2 r(1+2Nr^2)}{(1+N+N^2 r^2)^2}$$

For η_{max} this gives

N = -2 which is not possible.

Hence no particular value of N exists for η max

Magnitude of K is

$$|K|^2 = \left(\frac{(1+N+(N\,\omega/\omega_r)^2)^2 + N^4\,\omega/\omega_r}{((1+N)^2+(\frac{N\omega}{\omega_r})^2)^2}\right) K_c^2$$

3.3.3 For a spring in parallel of stiffness 'K''

K' $\quad K' = S_r K_c$

Then $$K_{Real} = K_c\left(\frac{1+N+(N\,\omega/\omega_r)^2}{(1+N)^2+(N\,\omega/\omega_r)^2} + S_r\right)$$

$$\mathrm{Im}\,K = \frac{K_C\ N^2(\omega/\omega_r)}{(1+N)^2+(N\,\omega/\omega_r)^2}$$

$$\mathrm{Re}\,K = k_c\left(\frac{(N+1)(1+S_r(1+N))+(S_r+1)(N\,\omega/\omega_r)^2}{(1+N)^2+(N\,\omega/\omega_r)^2}\right)$$

$$\eta = \frac{N^2(\omega/\omega_r)}{(N+1)(1+S_r(1+N))+(1+S_r)(N\,\omega/\omega_r)^2}$$

In this case also no particular value of N exists for η to be maximum.

Now $$\frac{d\eta}{dr} = N^2\left((N+1)(1+S_r(1+N))+(1+S_r)N^2r^2\right) - N^2 r(2N^2 r(1+S_r)) = 0$$

For maximum η

$$r^2 = \frac{(N+1)(1+S_r(1+N))}{N^2(1+S_r)}$$

For $S_r >> 1$

$$r^2 = \frac{(N+1)^2}{N^2}$$ As in reference [17]

For large $N >> 1$, $r^2 = 1$

$$\eta\, max = \frac{N^2\left(\frac{(N+1)\,(1+Sr(1+N))}{N^2(1+Sr)}\right)^{1/2}}{(N+1)(1+Sr(1+N)) + \frac{(1+Sr)(N+1)(1+Sr(1+N))}{(1+Sr)}}$$

$$= \frac{N}{2\left((N+1)(1+Sr(1+N))\right)^{1/2}} \approx \frac{N}{2\,Sr(N+1)}$$

Hence N does not have much effect on η_{max} while Sr has signified effect. Reference [17] has considered only the case when Sr >> 1. For Sr << 1 formula reduces to

$$r^2 = \frac{(1+N)\;\{1 + Sr(1+N)\}}{N^2}$$

$$\& \quad \eta_{max} = \frac{N}{2(((N+1)\;\{1+Sr(N+1)\}))^{\frac{1}{2}}}$$

Further for large values of N >> 1

$$r^2 \approx \frac{1+SrN}{N} = \frac{1}{N} + Sr$$

Thus the minimum frequency where η max can be made to occur is controlled by Sr.

$$\eta\; max = \frac{N}{2(N(1+SrN))^{1/2}} = \frac{\sqrt{N}}{2\sqrt{1+SrN}}$$

$$\approx \frac{1}{2\sqrt{1/N + Sr}}$$

Thus a smaller value of Sr increases the maximum loss factor. However this will also reduce the frequency at which η_{max} will occur.

3.3.4 Response to Base Motion

Equation of motion for the linearised system is

$$m\ddot{x} + k(1+i\eta)(x-y) = 0$$

$$m\ddot{x} + k(1+i\eta)x = K(1+i\eta)y.$$

Assuming $y = \text{Re}Y \cdot e^{i\omega t}$

and $x = \text{Re}\, X \cdot e^{i\omega t}$

$$X[(K-m\omega^2)+i\eta K] = K(1+i\eta)Y.$$

$$X = \frac{YK(1+i\eta)}{(K-m\omega^2)+i\eta K}$$

$$= \frac{Y(1+i\eta)}{(1-r^2)+i\eta} \quad \text{where } r^2 = \frac{\omega^2}{(K/m)}$$

$$\left|\frac{X}{Y}\right| = \left(\frac{1+\eta^2}{(1-r^2)^2+\eta^2}\right)^{1/2}$$

& phase angle $\tan\phi = \dfrac{-\eta r^2}{(1-r^2)+\eta^2}$

Hence for $r < 1$, ϕ is negative, i.e. mass lags behind the base as it should be.

At $r = 1$, $\tan\phi = -\dfrac{1}{\eta}$

As $r \to \infty$ $\tan\phi = \eta$

However, $r = 0$, $\eta = 0$.

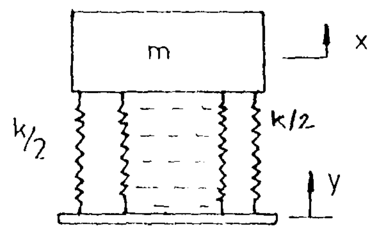
$$r = 1, \quad \eta = \frac{N^2}{(N+1)[1+Sr(N+1)]+N^2(1+Sr)}$$

$t \to \infty \quad , \eta \to 0$

3.4 Non-Linear Equations

3.4.1 Analysis so far given was based on the approximation of small motions and corresponding small changes of pressures. Apart from this assumption since the concept of 'complex spring stiffness' is used because of the operator 'D' used on analysis; only periodic excitations can be analysed for their responses. Transient excitations cannot be studied unless they are converted to frequency spectrum.

However in vehicle vibrations transient response has to be studied for many problems like step function input, vibration amplitudes due to road roughness are large. It is therefore felt that exact analysis as non-linear system may be found to be more useful. Hence exact equations are developed. To make the analysis more useful equations are developed in terms of dimensionless parameters. As any air-spring will work in parallel with ordinary spring (due to stiffness of bellow material or otherwise) that also has been included.



3.4.2 $$p_o A = mg + K\delta \qquad .. (1)$$

Here δ is the initial deflection of the parallel spring. Equation (1) represents the static equilibrium of the system.

Under dynamic conditions,

$$m\ddot{x} = Ap - mg - K(\delta + x - y) \qquad ...(2)$$

where x is displacement of the system above static equilibrium position.

$$m\ddot{x} = Ap - Ap_o - K(x-y)$$

$$= Ap_o(p/p_o - 1) - Kh(X-Y)$$

where X and Y are dimensionless displacements.

$$X'' = Pr(P-1) - (SrPr/\alpha)(X-Y) \qquad \ldots(3)$$

For mass flow from the air spring,

$$\frac{dm_1}{dt} = \frac{1}{RT}\left[\frac{v}{n}\frac{dp}{dt} + p\frac{dv}{dt}\right.$$

Introducing dimensionless time τ , Mass, M putting RT = pv/m.

$$-m_o w_o \frac{dM1}{d\tau} = \left(\frac{1}{\gamma p}\frac{dp}{dt} + \frac{1}{v}\frac{dv}{dt}\right) m$$

$$- = \left[\frac{1}{\gamma p_o p} \quad \frac{w_o p_o dp}{d\tau} + \frac{1}{v_o V} w_o v_o \frac{dV}{d\tau}\right] m_o M1$$

$$\therefore \frac{dM1}{d\tau} = \frac{M_1}{\gamma p}, \frac{dp}{d\tau} + \frac{M_1}{V} \frac{dV}{d\tau} \qquad \text{--- (a)}$$

Now $v = v_o + A(x-y)$

$$V = 1 + \frac{A}{v_o} h(X-Y)$$

$$= 1 + (1 \quad /Vr)(X-Y) \qquad \ldots(b)$$

$$dV/d\tau = (1/Vr)(X'-Y') \qquad \ldots(c)$$

From (a), (b) & (c)

$$\frac{dM1}{d\tau} = \frac{M1}{\gamma p}\frac{dp}{d\tau} + \frac{M1}{Vr+(X-Y)}(X'-Y') \qquad \text{--- (4)}$$

For Mass flow from tank

$$\frac{dmt}{dt} = \frac{vt}{\gamma RT}\ \frac{dpt}{dt}$$

$$m_0 w_0 \frac{dMt}{d\tau} = \frac{v_0 Vt}{\gamma}\ \frac{mt}{pt\,vt}\ \frac{dpt}{dt}$$

$$= \frac{v_0 Vt}{\gamma}\ \frac{m_0 Mt}{p_0 P_t v_0 Vt}\ p_0 w\ \frac{dpt}{d\tau}$$

$$\therefore \frac{dMt}{d\tau} = -\frac{dM1}{d\tau} = \frac{Mt}{\gamma pt}\ \frac{dpt}{d\tau} \qquad \ldots($$

For flow through capillary,

$$-\frac{dm}{dt} = \frac{cr}{2RT}\ (p^2 - Pt^2)$$

$$= \frac{cr}{RT_1 + RTt}\ (p^2 - pt^2)$$

$$= \frac{Cr}{\left(\frac{Pv}{m1} + \frac{Pt\,vt}{mt}\right)}\ (p^2 - pt^2)$$

$$= \frac{C\gamma mo}{p_0 v_0 \left(\frac{PV}{M1} + \frac{PtVt}{Mt}\right)}\ .\ p_0^2 (p - pt)$$

$$= \frac{Crmo}{vo}\ po\ (p^2 - pt^2)\left[\frac{pv}{M1} + \frac{ptvt}{Mt}\right]^{-1}$$

$$-mow_0 \frac{dM1}{d\tau} = \frac{crmopo}{vo}\ (p^2 - Pt^2)\left[\frac{PV}{M1} + \frac{PtVt}{Mt}\right]^{-1}$$

$$-\frac{dM1}{d\tau} = \epsilon \left[\frac{PV}{M1} + \frac{PtVt}{Mt}\right]\ (P^2 - Pt^2)$$

where $\epsilon = \dfrac{Crpo}{w_0 v_0}$ (6)

Equations (3), (4) (5) and (6) constitute the basic four nonlinear equations.

We introduce the following notation to convert the equations into first order equations.

$$X = X_1$$

$$X' = X_2$$

$$X_2 = X_1'$$

$$\frac{dM1}{dt} = M2$$

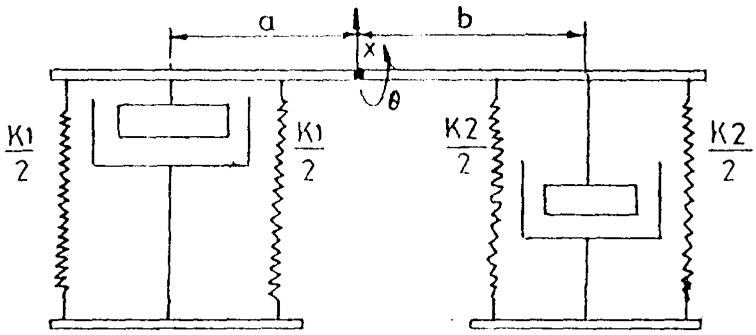
$$\frac{dMt}{dt} = Mt2$$

$$\frac{dP}{dt} = P_2$$

$$\frac{dpt}{dt} = Pt2$$

3.5 Vehicle Model 2 DOF

A vehicle is considered as a 2 DOF system taking into account pitching and bouncing only.



Three different cases are considered viz.

1. Each air spring is conneted to its own tank.

2. Airsprings are connected to each other but no tank.
3. Airsprings are connected together with each airspring also connected to its own tank.

All connections are assumed through a capillary tube.
Equations for static equilibrium are :

$$K_1 \delta_1 + K_2 \delta_2 + mg = A_1 p_{o1} + A_2 p_{o2} \qquad \ldots(1)$$

where δ_1 & δ_2 are static deflections of springs. Subscripts 1 & 2 refer to front and rear springs respectively.

$$(A_1 p_{o1} - K_1 \delta_1).a = (A_2 p_{o2} - K_2 \delta_2)b \qquad \ldots(2)$$

Equation (1) is equilibrium of forces while eqn. (2) is for equilibrium of moments.

In dynamic conditions,

$$m\ddot{x} = -K_1(x + \delta_1 - a\theta - y_1) - K_2(x + \delta_2 + b\theta - y_2) + P_1 A_1 + P_2 A_2 - mg \qquad \ldots(3)$$

$$I\ddot{\theta} = K_1(x + \delta_1 - a\theta - y_1)a - K_2(x + \delta_2 + b\theta - y_2).b - A_1 p_1 a + A_2 p_2 b \qquad (4)$$

Substituting from equations (1) and (2) in equations (3) and (4) respectively,

$$m\ddot{x} = -K_1(x - a\theta - y_1) - K_2(x + b\theta - y_2) + A_1 p_1 + A_2 p_2 - A_1 p_{o1} - A_2 p_{o2} \qquad (5)$$

$$I\ddot{\theta} = K_1 a(x - a\theta - y_1) - K_2 b(x + b\theta - y_2) - A_1 p_1 a + A_2 p_2 b + A_1 p_{o1} a - A_2 p_{o2} b \qquad \ldots(6)$$

In dimensionless form

$$\ddot{X} = -Sr1(X + \theta - Y_1) - Sr2(X + B\theta - Y_2) + P_r(P_1 - 1) + P_r A(P_2 - \lambda)$$

$$\theta'' = Sr_1 R_1(X - \theta - Y_1) - Sr_2 R_1 B(X + B\theta - Y_2)$$
$$- P_r R_1(P_1 - 1) + P_r R_1 B A(P_2 - \lambda)$$

Equations (7) and (8) represents the equations of motions for the 2 DOF system.

3.5.2 Following equations will be useful in the massflow analysis.

For adiabatic compression/expansion

$$P_{01} \vartheta_{01}^{\chi} = P_1 \vartheta_1^{\chi}$$
$$P_{01} \vartheta_{01}^{\chi} = P_{01} \vartheta_{01}^{\chi} P_1 V_1^{\chi}$$
$$\therefore P_1 V_1^{\chi} = 1$$
$$P_{02} \vartheta_{02}^{\chi} = P_2 \vartheta_2^{\chi}$$
$$= P_{01} \vartheta_{01}^{\chi} P_2 V_2^{\chi}$$
$$\therefore P_2 V_2^{\chi} = \lambda(\beta)^{\chi}$$

Mass Flow Equations

Case 1 :- Each spring is connected to its tank through a capillary.

For airspring no. 1 :

$$\frac{dm}{dt} = \frac{1}{RT}\left[\frac{\vartheta_1}{\gamma}\frac{dP_1}{dt} + P_1\frac{d\vartheta_1}{dt}\right]$$

writing $RT = \dfrac{P_1\vartheta_1}{m_1}$

$$\omega_r m_{01}\frac{dM_1}{d\tau} = \frac{m_1}{P_1\vartheta_1}\left[\frac{\vartheta_1}{\gamma}\frac{dP_1}{dt} + P_1\frac{d\vartheta_1}{dt}\right]$$

$$= \frac{m_{01}M_1}{P_1\gamma}\,\omega_r\frac{dP_1}{d\tau} + \frac{m_{01}M_1}{V_{01}V_1}\cdot V_{01}\,\omega_r\frac{dV_1}{d\tau}$$

$$\therefore\ \frac{dM_1}{d\tau} = \frac{M_1}{\gamma P_1}\frac{dP_1}{d\tau} + \frac{M_1}{V_1}\cdot\frac{dV_1}{d\tau}$$

...(15)

For mass flow through the tank 1,

$$\frac{dm_{t1}}{dt} = \frac{-dm_1}{dt} = \frac{m_{t1}}{P_{t1}\vartheta_{t1}}\left[\frac{\vartheta_{t1}}{\gamma}\frac{dP_{t1}}{dt}\right]$$

$$-\frac{dM_1}{d\tau} = \frac{M_{t1}}{\gamma P_{t1}}\frac{dP_{t1}}{d\tau}$$

...(16)

For capillary flow from airsprings to its tank.

$$\frac{-dm}{dt} = \frac{C_{r1}}{2RT}\left(P_1^2 - P_{t1}^2\right)$$

We approximate $2RT = RT_1 + RT_t$

$$-\frac{dm_i}{dt} = Cr_1 \left\{ \frac{P_1 \vartheta_1}{m_1} + \frac{P_{t_1} \vartheta_{t_1}}{mt_1} \right\}^{-1} (P_1^2 - P_t^2)$$

$$-\omega_r \frac{dM_1}{d\tau} = Cr_1 \left\{ \frac{P_1 V_1}{M_1} + \frac{P_{t_1} V_{t_1}}{Mt_1} \right\}^{-1} (P_{01} \cdot \vartheta_{01})^{-1} \cdot P_{01}^2 \left(P_1^2 - P_{t_1}^2 \right)$$

$$-\frac{dM_1}{d\tau} = \frac{Cr_1 P_{01}}{\omega_r \vartheta_{01}} \left\{ \frac{P_1 V_1}{M_1} + \frac{P_{t_1} V_{t_1}}{Mt_1} \right\}^{-1} \cdot \left(P_1^2 - P_{t_1}^2 \right)$$

$$= \epsilon_1 \left\{ \frac{P_1 V_1}{M_1} + \frac{P_t V_{t_1}}{Mt_1} \right\}^{-1} \left(P_1^2 - P_{t_1}^2 \right)$$

From equation (16),

$$\frac{dP_{t_1}}{d\tau} = -\frac{\varkappa P_{t_1}}{Mt_1} \cdot \frac{dM_1}{d\tau} \qquad \ldots (20)$$

From equation (15)

$$\frac{dP_1}{d\tau} = \frac{\varkappa P_1}{M_1} \left\{ \frac{dM_1}{d\tau} - \frac{M_1}{V_1} \cdot \frac{dV_1}{d\tau} \right\} \qquad \ldots (21)$$

For air spring 2,

$$\frac{dm_2}{dt} = \frac{m_2}{P_2 \vartheta_2} \left[\frac{\vartheta_2}{\gamma} \frac{dP_2}{dt} + P_2 \frac{d\vartheta_2}{dt} \right]$$

$$= \frac{m_2}{\gamma P_2} \frac{dP_2}{dt} + \frac{m_2}{\vartheta_2} \frac{d\vartheta_2}{dt}$$

$$\frac{dM_2}{d\tau} = \frac{M_2}{\gamma P_2} \frac{dP_2}{d\tau} + \frac{M_2}{V_2} \frac{dV_2}{d\tau} \qquad \ldots (22)$$

For tank,

$$\frac{dm_{t2}}{dt} = -\frac{dm_2}{dt} = \frac{m_{t2}}{P_{t2}\vartheta_{t2}} \left\{ \frac{\vartheta_{t2}}{\gamma} \frac{dP_{t2}}{dt} \right\}$$

$$\therefore \frac{dM_2}{d\tau} = \frac{M_{t2}}{\gamma P_{t2}} \cdot \frac{dP_{t2}}{d\tau} \qquad \ldots(23)$$

For capillary flow,

$$-\frac{dm_2}{dt} = \frac{C_{r2}}{2RT}\left(P_2^2 - P_{t2}^2\right)$$

putting $2RT = RT_2 + RT_{t2}$

$$-\frac{dm_2}{dt} = \frac{C_{r2}\left(P_2^2 - P_{t2}^2\right)}{\dfrac{P_2\vartheta_2}{m_2} + \dfrac{P_{t2}\vartheta_{t2}}{m_{t2}}} = \frac{C_{r2}P_{01}^2\left(P_2^2 - P_{t2}^2\right)}{\dfrac{P_{01}\vartheta_{01}}{m_{01}}\left[\dfrac{P_2V_2}{M_2} + \dfrac{P_{t2}V_{t2}}{M_{t2}}\right]}$$

$$\frac{dM_2}{d\tau} = \left(\frac{C_{r2}P_{01}}{\omega\vartheta_{01}}\right) \frac{P_2^2 - P_{t2}^2}{\left[\dfrac{P_2V_2}{M_2} + \dfrac{P_{t2}V_{t2}}{M_{t2}}\right]}$$

$$= \epsilon_2 \frac{P_2^2 - P_{t2}^2}{\left[\dfrac{P_2V_2}{M_2} + \dfrac{P_{t2}V_{t2}}{M_{t2}}\right]} \qquad \ldots(24)$$

$$\frac{dP_2}{d\tau} = \frac{\gamma P_2}{M_2}\left\{\frac{dM_2}{d\tau} - \frac{M_2}{V_2}\frac{dV_2}{d\tau}\right\} \qquad \ldots(25)$$

$$\frac{dP_{t2}}{d\tau} = -\frac{\gamma P_{t2}}{M_{t2}} \frac{dM_2}{d\tau} \qquad \ldots(26)$$

Case II

Airsprings connected to each other but not connected to tanks.

Rate of flow of air from the front airspring,

$$\frac{dM_1}{d\tau} = \frac{M_1}{\gamma P_1} \frac{dP_1}{d\tau} + \frac{M_1}{V_1} \frac{dV_1}{d\tau} \qquad \ldots(15)$$

Rate of flow of air from the feart airspring,

$$\frac{dM_2}{d\tau} = \frac{M_2}{\gamma P_2} \frac{dP_2}{d\tau} + \frac{M_2}{V_2} \frac{dV_2}{d\tau} \qquad \ldots(22)$$

For air flow through connecting capillary

$$-\frac{dm_1}{dt} = \frac{C_r}{2RT} \left(P_1^2 - P_2^2\right)$$

writing $2RT = RT_1 + RT_2$

$$-\frac{dm_1}{dt} = \frac{C_r\left(P_1^2 - P_2^2\right)}{\frac{P_1 v_1}{m_1} + \frac{P_2 v_2}{m_2}}$$

$$-\omega_r m_{01} \frac{dM_1}{d\tau} = \frac{C_r P_{01}^2 \left(P_1^2 - P_2^2\right)}{\frac{P_{01} v_{01}}{m_{01}} \left\{\frac{P_1 V_1}{M_1} + \frac{P_2 V_2}{M_2}\right\}}$$

$$-\frac{dM_1}{d\tau} = \frac{C_r P_{01}}{\omega_r v_{01}} \left(P_1^2 - P_2^2\right) \left\{\frac{P_1 V_1}{M_1} + \frac{P_2 V_2}{M_2}\right\}^{-1}$$

$$= \epsilon \left(P_1^2 - P_2^2\right) \left\{\frac{P_1 V_1}{M_1} + \frac{P_2 V_2}{M_2}\right\}^{-1}$$

...(27)

Also we have ...(28)

$$\frac{dM_1}{d\tau} = -\frac{dM_2}{d\tau}$$

From (15)

$$\frac{dP_1}{d\tau} = \frac{\gamma P_1}{M_1}\left\{\frac{dM_1}{d\tau} - \frac{M_1}{V_1}\cdot\frac{dV_1}{d\tau}\right\}$$

From (22)

$$\frac{dP_2}{d\tau} = \frac{\gamma P_2}{M_2}\left\{\frac{dM_2}{d\tau} - \frac{M_2}{V_2}\frac{dV_2}{d\tau}\right\}$$

Case III

Springs connected with each other and with tank :

For front spring,

$$\frac{dM_1}{d\tau} = \frac{M_1}{\gamma P_1}\frac{dP_1}{d\tau} + \frac{M_1}{V_1}\frac{dV_1}{d\tau}$$

For rear spring, ...(22)

$$\frac{dM_2}{d\tau} = \frac{M_2}{\gamma P_2}\frac{dP_2}{d\tau} + \frac{M_2}{V_2}\frac{dV_2}{d\tau}$$

Flow of air from front spring to its tank,

$$\frac{dF_1}{d\tau} = E_1\left[\frac{P_1V_1}{M_1} + \frac{P_{t1}V_{t1}}{M_{t1}}\right]^{-1}\left[P_1^2 - P_{t1}^2\right] \qquad ...(29)$$

Flow of air from front spring to rear spring,

$$\frac{dF_c}{d\tau} = E_2\left[\frac{P_1V_1}{M_1} + \frac{P_2V_2}{M_2}\right]^{-1}\left[P_1^2 - P_2^2\right] \qquad ...(30)$$

Flow of air from rear spring to its tank,

$$\frac{dF_2}{d\tau} = E_3\left[\frac{P_2V_2}{M_2} + \frac{P_{t2}V_{t2}}{M_{t2}}\right]^{-1}\left[P_2^2 - P_{t2}^2\right] \qquad ...(31)$$

Now $$\frac{dM_1}{d\tau} = -\left[\frac{dF_1}{d\tau} + \frac{dF_c}{d\tau}\right] \qquad ...(32)$$

$$\frac{dM_2}{d\tau} = \left[\frac{dF_1}{d\tau} - \frac{dF_2}{d\tau}\right] \qquad \text{...(33)}$$

$$\frac{dM_{t1}}{d\tau} = \frac{dF_1}{d\tau} \qquad \text{...(34)}$$

$$\frac{dM_{t2}}{d\tau} = \frac{dF_2}{d\tau}. \qquad \text{...(35)}$$

Hence we have,

$$\frac{dP_1}{d\tau} = \frac{\gamma P_1}{M_1}\left[\frac{dM_1}{d\tau} - \frac{M_1}{V_1} \cdot \frac{dV_1}{d\tau}\right]$$

$$\frac{dP_2}{d\tau} = \frac{\gamma P_2}{M_2}\left[\frac{dM_2}{d\tau} - \frac{M_2}{V_2}\frac{dV_2}{d\tau}\right]$$

$$\frac{dP_{t1}}{d\tau} = \frac{\gamma P_{t1}}{M_{t1}} \cdot \frac{dM_{t1}}{d\tau}$$

$$\frac{dP_{t2}}{d\tau} = \frac{\gamma P_{t2}}{M_{t2}} \frac{dM_{t2}}{d\tau}$$

3.3.6 Analysis of a vehicle suspension system using linearization :-

System treated is that of a suspension having air springs and conventional springs. Airsprings are connected to each other by a capillary tube. Equations of mass flow for the airsprings are :

$$M_1 = \frac{1}{RT}\left[\frac{v_{01}}{\gamma} D(P_1) \; P_0 D(v_1)\right]$$

$$-m_1 = \frac{P_0 Cr\,(p_1 - p_2)}{RT}$$

$$m_2 = -m_1 = \frac{1}{RT}\left[\frac{v_{02}}{\gamma} D(p_2) + p_0 D(v_2)\right]$$

From equations (2) and (3)

$$P_0 Cr\,(p_1 - p_2) = \frac{v_{02}}{\gamma} D(P_2) + P_0 D(v_2)$$

$$\frac{v_{02}}{\gamma} D(P_2) + p_0 Cr\, p_2 = p_0 Cr\, p_1 - p_0 D(v$$

$$\left[\frac{v_{02}}{\gamma} D + p_0 Cr\right] p_2 = p_0 Cr\, p_1 - p_0 D(v_2)$$

$$p_2 = \left[\frac{v_{02}}{\gamma} D + p_0 Cr\right]^{-1}\left[p_0 Cr\, p_1 - P_0 D(v_2)\right]$$

From equations (1) and (2),

$$\frac{v_{01}}{\gamma} D(p_1) + p_0 D(v_1) = p_0 Cr\left[p_2 - p_1\right]$$

$$\left[\frac{v_{01}}{\gamma} D + p_0 Cr\right] p_1 = p_0 Cr\, p_2 - p_0 D(v_1)$$

$$= p_0 Cr\left[\frac{v_{02}}{\gamma} D + p_0 Cr\right]^{-1}\left[p_0 Cr\, p_1 - p_0 D(v_2)\right] - p_0 D(v_1)$$

$$\left\{\left[\frac{v_{01}}{\gamma} D + p_0 Cr - (p_0 Cr)^2\right]\left[\frac{v_0^2}{\gamma} D + p_0 Cr\right]^{-1}\right\} p_1$$

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$$= -P_o^2 C_r \left[\frac{\vartheta_{o2}}{\gamma} D + P_o C_r \right]^{-1} D(\vartheta_2) - P_o D(V_1)$$

Substituting $\omega_r = \frac{\gamma P_o C_r}{\vartheta_{o1}}$, $V_{o2} = \frac{\vartheta_{o2}}{\vartheta_{o1}}$ we get

$$\left\{ \left(\frac{\vartheta_{o1}}{\gamma} D + P_o C_r \right) - P_o C_r \left[\frac{V_{o2} D}{\omega r} + 1 \right]^{-1} \right\} P_1$$

$$= -P_o \left[\frac{V_{o2}}{\omega r} D + 1 \right]^{-1} D(\vartheta_2) - P_o D(\vartheta_1)$$

$$\left\{ \left(\frac{\vartheta_{o1} D}{\gamma} + P_o C_r \right) \left[\frac{V_{o2} D}{\gamma} + 1 \right] - P_o C_r \right\} P_1$$

$$= -P_o D(\vartheta_2) - P_o \left[\frac{V_{o2} D}{\omega r} + 1 \right] D(\vartheta_1)$$

$$\therefore \left\{ \frac{\vartheta_{o1}}{\gamma} D + \frac{P_o C_r V_{o2} D}{\omega r} + \frac{\vartheta_{o1} D \cdot V_{o2} D}{\gamma \omega r} \right\} P_1 = -P_o D(\vartheta_2) - P_o \left[\frac{V_{o2} D}{\omega r} + 1 \right] D(\vartheta_1)$$

Dividing by $P_o C_r$

$$\left\{ \frac{D}{\omega r} + \frac{V_{o2}}{\omega r} D + \frac{D \cdot V_{o2} D}{\omega r^2} \right\} P_1 = \frac{1}{P_o C_r} \left\{ -P_o D(\vartheta_2) - P_o \left[\frac{V_{o2} D}{\omega_c} + 1 \right] D(\vartheta_1) \right.$$

Multiplying throughout by $\frac{\omega r}{V_{o2}}$

$$\left\{ \frac{1}{V_{o2}} + 1 + \frac{D}{\omega r} \right\} D P_1 = \frac{1}{P_o C_r} \left\{ -\frac{P_o \omega r}{V_{o2}} D(\vartheta_2) - P_o \left[D + \frac{\omega r}{V_{o2}} \right] D(\vartheta_1) \right\}$$

$$= -\frac{P_o \gamma}{\vartheta_{o2}} D(\vartheta_2) - P_o \left[\frac{\gamma / \vartheta_{o1}}{\omega r} D + \frac{\gamma}{\vartheta_{o2}} \right] D(\vartheta_1)$$

$$D(P_1) = \frac{ -\frac{\gamma P_o}{\vartheta_{o1}} \left\{ \frac{\omega r}{\vartheta_{o2}} D(V_2) + \left[D + \frac{\omega r}{V_{o2}} \right] D(\vartheta_1) \right\} }{ \omega r \left(1 + \frac{1}{\ldots} \right) + D } \quad \text{---} (5)$$

$$= \frac{-\gamma P_0}{\vartheta_{c2}} D(\vartheta_2) - \frac{\gamma P_0}{\vartheta_{c1}} \left[\frac{D}{\omega r} + \frac{1}{V_{c2}} \right] D(\vartheta_1)$$

$$\therefore \quad D(P_1) = \frac{\frac{\gamma P_0}{\vartheta_{c1}} \left\{ \frac{1}{V_{02}} D(V_2) + \left[\frac{D}{\omega_c} + \frac{1}{V_{c2}} \right] D(\vartheta_1) \right\}}{\left(1/V_{c2} + 1 + D/\omega r \right)}$$

$$= \frac{\frac{-\gamma P_0}{\vartheta_{c1}} \left\{ \frac{A_2}{V_{02}} D(h_2) + \left[\frac{D}{\omega r} + \frac{1}{V_{c2}} \right] A_1 D(h_1) \right\}}{\left(1/V_{02} + 1 + D/\omega_c \right)}$$

Similarly from (1) and (2)

$$P_0 C r (P_2 - P_1) = \frac{\vartheta_{01}}{\gamma} D(P_1) + P_0 D(\vartheta_1)$$

$$\left[P_0 C r + \frac{\vartheta_{01}}{\gamma} D \right] P_1 = P_0 C r P_2 - P_0 D(\vartheta_1)$$

$$\left[\frac{P_0 C r \gamma}{\vartheta_{01}} + D \right] P_1 = \frac{P_0 C r \gamma}{\vartheta_{01}} P_2 - \frac{P_0 \gamma}{\vartheta_{c1}} D(\vartheta_1)$$

$$P_1 = \left[\omega r + D \right]^{-1} \left\{ \omega r P_2 - \frac{P_0 \gamma}{\vartheta_{c1}} D(\vartheta_1) \right\}$$

...(6)

From (2) and (3),

$$P_0 C r (P_1 - P_2) = \frac{\vartheta_{02}}{\gamma} D(P_2) + P_0 D(\vartheta_2)$$

Multiplying by (γ / V_{01})

$$\omega r (P_1 - P_2) = V_{02} D(P_2) + \frac{P_0 \gamma}{\vartheta_{c1}} D(\vartheta_2)$$

$$(\omega_r + V_{02} D) P_2 = \omega r P_1 - \frac{P_0 \gamma}{\vartheta_{c1}} D(\vartheta_2)$$

Substituting for p_1 from equation (6)

$$\left(\omega r + V_{02} D\right) p_2 = \omega r \left[\omega r + D\right]^{-1} \left\{ \omega_r p_2 - \frac{p_0 r}{v_{01}} D(v_1) \right\} - \frac{p_0 r}{v_{01}} D(v_2)$$

$$\left(\omega r + D\right)\left(\omega r + V_{02} D\right) p_2 = \omega_r^2 p_2 - \frac{\omega r\, p_0 r}{v_{01}} D\left(v_1\right)$$

$$- \frac{p_0 r}{v_{01}} \left(\omega r + D\right) D\left(v_2\right)$$

$$\therefore \left(\omega_r V_{02} D + D\omega_r + D \cdot V_{02} D\right) p_2 = \frac{-p_0 r}{v_{01}} \left\{ \omega_r D(v_1) + (\omega r + D) D(v_2) \right\}$$

$$\therefore \quad D\left(p_2\right) = \frac{\dfrac{-p_0 r}{v_{01}} \left\{ D(v_1) + \left(1 + D/\omega_0\right) D(v_2) \right\}}{\left(V_{02} + 1 + \dfrac{V_{02} D}{\omega r}\right)}$$

$$= \frac{-\dfrac{p_0 r}{v_{01}} \left\{ \dfrac{1}{V_{02}} D\left(v_1\right) + \left(1 + D/\omega_0\right) \dfrac{1}{V_{02}} D\left(v_2\right) \right\}}{\left(1 + 1/V_{02} + D/\omega r\right)}$$

$$= \frac{\dfrac{-p_0 r}{V_{01}} \left\{ \dfrac{\omega r}{V_{02}} D\left(v_1\right) + \left(\omega r + D\right) \dfrac{1}{V_{02}} D\left(v_2\right) \right\}}{\left[\omega\left(1 + \dfrac{1}{V_{02}}\right) + D\right]}$$

Equations of motion for the system hence become :

$$m\ddot{x} = A_1\Delta p_1 + A_2\Delta p_2 - k_1(x - a\theta - y_1) - k_2(x + b\theta - y_2)$$

$$I\ddot{\theta} = k_1 a(x - a\theta - y_1) - k_2 b(x + b\theta - y_2) + A_2\Delta p_2 b - A_1\Delta p_1 a$$

Here p & p represent the pressures in the airsprings above static equilibrium position.

Change of heights of airsprings are respectively

$$h_1 = x = a\Theta - y_1$$

$$h_2 = x + b\Theta - y_2$$

Therefore from (5) and (7)

$$\Delta p_1 = \frac{-\frac{\gamma p_0}{v_{01}}\left\{\frac{A_2}{V_{02}}(x + b\theta - y_2) + \left[\frac{D}{\omega_r} + \frac{1}{V_{02}}\right]A_1(x - a\theta - y_1)\right\}}{1/V_{02} + 1 + D/\omega_r}$$

$$\Delta p_2 = -\frac{\gamma p_0}{v_{01}}\left\{\frac{A_1}{V_{02}}(x - a\theta - y_1) + \left(1 + \frac{D}{\omega_r}\right)\frac{A_2}{V_{02}}(x + b\theta - y_2)\right\} \Big/ \frac{1}{V_{02}} + 1 + \frac{D}{\omega_r}$$

Areas of airsprings are assumed constant at all heights. Writing $\ddot{x} = -\omega^2 X_1$, $\ddot{\theta} = -\omega^2 X_2$ & $D = j\omega$ for harmonic motion

From (8)

$$-m\omega^2 X_1 = \frac{-\frac{A_1\gamma p_0}{v_{01}}\left\{\frac{A_2}{V_{02}}(X_1 + bX_2 - Y_2) + \left[\frac{1}{V_{02}} + \frac{j\omega}{\omega_r}\right]A_1(X_1 - aX_2 - Y_1)\right\}}{1/V_{02} + 1 + j\omega/\omega_r}$$

$$- \frac{\frac{A_2\gamma p_0}{v_{01}}\left\{\frac{A_1}{V_{02}}(X_1 - aX_2 - Y_1) + \left(1 + \frac{j\omega}{\omega_r}\right)\frac{A_2}{V_{02}}(X_1 + bX_2 - Y_2)\right\}}{1/V_{02} + 1 + j\omega/\omega_r}$$

$$- k_1(X_1 - aX_2 - Y_1) - K_2(X_1 + bX_2 - Y_2)$$

$$-m\omega^2 x_1\left[\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega r}\right] = \frac{-A_1A_2\gamma p_o}{\vartheta_{o1}V_{o2}}\left(X_1+bX_2-Y_2\right)-\frac{A_1^2\gamma p_o}{\vartheta_{o1}V_{o2}}\left[1+\frac{j\omega V_{o2}}{\omega r}\right]\left[X_1-aX_2-Y_1\right]$$

$$-\frac{A_1A_2\chi p_o}{\vartheta_{o1}V_{o2}}\left(X_1-aX_2-Y_1\right)-\frac{A_2^2\chi p_o}{\vartheta_{o1}V_{o2}}\left(1+\frac{j\omega}{\omega r}\right)\left[X_1\, bX_2-Y_2\right]$$

$$-\left[k_1\left(X_1-aX_2-Y_1\right)+k_2\left(X_1+bX_2-Y_2\right)\right]\left[\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega r}\right]$$

writing $S = \dfrac{A_1A_2\gamma p_o}{\vartheta_{o1}V_{o2}}$ & $A = \dfrac{A_2}{A_1}$

$$-m\omega^2 x_1\left[\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega r}\right] = -S\left(X_1+bX_2-Y_2\right)-\frac{S}{A}\left[1+\frac{j\omega V_{o2}}{\omega r}\right]\left[X_1-aX_2-Y_1\right]$$

$$-S\left(X_1-aX_2-Y_1\right)-AS\left(1+\frac{j\omega}{\omega r}\right)\left[X_1+bX_2-Y_2\right]$$

$$-\left[k_1\left(X_1-aX_2-Y_1\right)+k_2\left(X_1+bX_2-Y_2\right)\right]\left[\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega r}\right]$$

$$= -X_1\left[S+\frac{S}{A}\left(1+\frac{j\omega V_{o2}}{\omega r}\right)+S+AS\left(1+j\omega/\omega r\right)\right]$$

$$-X_2\left[Sb-\frac{Sa}{A}\left(1+\frac{j\omega V_{o2}}{\omega r}\right)-Sa+AbS\left(1+j\omega/\omega r\right)\right]$$

$$-\left(k_1+k_2\right)x_1\left[\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega r}\right]-\left[k_2b-k_1a\right]x_2\left[\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega r}\right]$$

$$+Y_1\left\{\frac{S}{A}\left(1+\frac{j\omega V_{o2}}{\omega r}\right)+S+k_1\left(\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega r}\right)\right\}$$

$$+Y_2\left\{S+AS\left(1+\frac{j\omega}{\omega r}\right)+k_2\left(\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega r}\right)\right\}$$

$$\therefore \left\{\left[\left(k_1+k_2-m\omega^2\right)\left(1+\frac{1}{V_o}\right)+\left[2S+\frac{S}{A}+SA\right)\right]+j\omega/\omega_r\left[\left(k_1+k_2-m\omega^2\right)+\frac{SV_{o2}}{A}+AS\right]\right\}X_1$$

$$+X_2\left\{\left[\left(k_2b-k_1a\right)\left(\frac{1}{V_{o2}}+1\right)+\left(Sb-\frac{Sa}{A}-Sa+AbS\right)\right]+j\omega\left[k_2b-k_1a-\frac{SaV_{o2}}{A}+AbS\right]\right.$$

$$=Y_1\left\{\left[k_1\left(1+\frac{1}{V_{o2}}\right)+\frac{S}{A}+S\right]+\frac{j\omega}{\omega r}\left[\frac{SV_{o2}}{A}+k_1\right]\right\}$$

$$+Y_2\left\{\left[k_2\left(1+\frac{1}{V_{o2}}\right)+S+AS\right]+\frac{j\omega}{\omega r}\left[AS+k_2\right]\right\}$$

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From equation (9)

$$I\omega^2 X_2 = -k_2 b\,(X_1+bX_2-Y_2)+k_1 a\,(X_1-aX_2-Y_1)$$

$$-\frac{\dfrac{A_2 b\gamma p_0}{v_{o1}}\left\{\dfrac{A_1}{V_{o2}}(X_1-aX_2-Y_1)+\left(1+\dfrac{j\omega}{\omega r}\right)\dfrac{A_2}{V_{o2}}(X_1+bX_2-Y_2)\right\}}{\dfrac{1}{V_{o2}}+1+\dfrac{j\omega}{\omega r}}$$

$$+\frac{\dfrac{A_1 a\gamma p_0}{v_{o1}}\left\{\dfrac{A_2}{V_{o2}}(X_1+bX_2-Y_2)+\left(\dfrac{1}{V_{o2}}+\dfrac{j\omega}{\omega r}\right)A_1(X_1-aX_2-Y_1)\right\}}{\dfrac{1}{V_{o2}}+1+\dfrac{j\omega}{\omega r}}$$

$$-I\omega^2\left[\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega_o}\right]X_2=\left\{-k_2b(X_1+bX_2)+k_1a(X_1-aX_2)\right\}\left\{\frac{1}{V_{o2}}+1+\frac{j\omega}{\omega r}\right\}$$

$$-\frac{A_1A_2\gamma p_0 b}{v_{o1}V_{o2}}(X_1-aX_2-Y_1)-\frac{A_2^2\gamma p_0}{v_{o1}V_{o2}}b\left(1+\frac{j\omega}{\omega r}\right)(X_1+bX_2-Y_2)$$

$$+\frac{A_1A_2\gamma p_0}{v_{o1}V_{o2}}a(X_1+bX_2-Y_2)+\frac{A_1^2 a\gamma p_0}{v_{o1}V_{o2}}\left(1+\frac{j\omega V_{o2}}{\omega r}\right)(X_1-aX_2-Y_1)$$

$$\therefore\Big\{\Big[(k_2b^2+k_1a^2-I\omega^2)\left(\frac{1}{V_{o2}}+1\right)-abS+Ab^2S-aSb+\frac{a^2S}{A}\Big]$$
$$+\frac{j\omega}{\omega r}\Big[(k_2b^2+k_1a^2-I\omega^2)+Ab^2S+\frac{a^2SV_{o2}}{A}\Big]\Big\}X_2$$

$$+\Big\{\Big[(k_2b-k_1a)\left(\frac{1}{V_{o2}}+1\right)+S\Big(b+Ab-a-\frac{a}{A}\Big)\Big]$$
$$+\frac{j\omega}{\omega r}\Big[(k_2b-k_1a)+AbS-\frac{V_{o2}aS}{A}\Big]\Big\}X_1$$

$$=Y_1\Big\{\Big[-k_1a\left(\frac{1}{V_{o2}}+1\right)+bS-\frac{aS}{A}\Big]+\frac{j\omega}{\omega r}\Big[-k_1a-\frac{aSV_{o2}}{A}\Big]\Big\}$$

$$+Y_2\Big\{\Big[k_2b\left(\frac{1}{V_{o2}}+1\right)+AbS-aS\Big]+\frac{j\omega}{\omega r}\Big[k_2b+AbS\Big]\Big\}$$

For a vehicle,

if $y = Y e^{j\omega t}$

then $y_2 = Y_1 e^{j(\omega t-\phi)}$

where $\phi = \omega\left(\frac{a+b}{v}\right)$ where v is the forward velocity of the vehicle.

Also $(a+jb)\,e^{-j\phi} = (a\cos\phi + b\sin\phi) + j(b\cos\phi - a\sin\phi)$

Hence equations (10) and (11) becomes,

$$\left\{\left[(k_1+k_2-m\omega^2)\left(1+\frac{1}{V_{02}}\right)+S\left(2+\frac{1}{A}+A\right)\right]+\frac{j\omega}{\omega_r}\left[(k_1+k_2-m\omega^2)+S\left(\frac{V_{02}}{A}+A\right)\right]\right\}X_1$$

$$+\left\{\left[(k_2b-k_1a)\left(1+\frac{1}{V_{02}}\right)+S\left(b-\frac{a}{A}-a+Ab\right)\right]+\frac{j\omega}{\omega_0}\left[k_2b-k_1a+S\left(Ab-\frac{aV_{02}}{A}\right)\right]\right\}X_2$$

$$=Y_1\left\{\left[k_1\left(1+\frac{1}{V_{02}}\right)+S\left(1+\frac{1}{A}\right)\right]+\frac{j\omega}{\omega_r}\left[\frac{SV_{02}}{A}+k_1\right]\right\}$$

$$+Y_1\left\{\left[k_2\left(1+\frac{1}{V_{02}}\right)+S(1+A)\right]\cos\phi+\frac{\omega}{\omega_r}\left[k_2+AS\right]\sin\phi\right\}$$

$$+jY_1\left\{\frac{\omega}{\omega_0}(k_2+AS)\cos\phi-\left[k_2\left(1+\frac{1}{V_{02}}\right)+S(1+A)\right]\sin\phi\right\}$$

$$=Y_1\left\{\left(1+\frac{1}{V_{02}}\right)(k_1+k_2\cos\phi)+S\left(1+\frac{1}{A}\right)(1+A\cos\phi)+\frac{\omega}{\omega_0}\left[k_2+AS\right]\sin\phi\right\}$$
$$+jY_1\left\{\frac{\omega}{\omega_0}\left[\frac{SV_{02}}{A}+k_1+(k_2+AS)\cos\phi\right]-\left[k_2\left(1+\frac{1}{V_{02}}\right)+S(1+A)\right]\sin\phi\right\}$$

---(12)

$$\left\{\left[(k_2b-k_1a)\left(\frac{1}{V_{02}}+1\right)+S\left(b+Ab-a-\frac{a}{A}\right)\right]+\frac{j\omega}{\omega_r}\left[(k_2b-k_1a)+S\left(Ab-\frac{aV_{02}}{A}\right)\right]\right\}X_1$$

$$+\left\{\left[(k_2b^2+k_1a^2+I\omega^2)\left(\frac{1}{V_{02}}+1\right)+S\left(-ab+Ab^2-ab+\frac{a^2}{A}\right)\right]+\frac{j\omega}{\omega_r}\left[(k_1a^2+k_2b^2-I\omega^2)+S\left(Ab^2+\frac{a^2V_{01}}{A}\right)\right\}X_2$$

$$=Y_1\left\{-k_1a\left(\frac{1}{V_{02}}+1\right)+S\left(b-\frac{a}{A}\right)+\left[k_2b\left(\frac{1}{V_{02}}+1\right)+S(Ab-a)\right]\cos\phi\right.$$
$$\left.+\left[\frac{\omega}{\omega_r}(k_2b+AbS)\right]\sin\phi\right\}$$

$$jY_1\left\{\frac{\omega}{\omega_r}\left[\left(-k_1a-\frac{aSV_{02}}{A}\right)+(k_2b+AbS)\cos\phi\right]-\left[k_2b\left(\frac{1}{V_{02}}+1\right)+S(Ab-a)\right]\sin\phi\right\}$$

---(13)

...85/-

Equations (12) and (13) can be written as

$$\begin{bmatrix} (C+jD) & (E+jF) \\ (E+jF) & (G+jH) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = Y_1 \begin{Bmatrix} L+jM \\ P+jQ \end{Bmatrix}$$

$$\begin{aligned}
\therefore\ \Delta &= (C+jD)(G+jH)-(E+jF)^2 \\
&= (CG-DH)+j(CH+DG)-[E^2-F^2+2jEF] \\
&= (CG-DH-E^2+F^2)+j(CH+DG-2EF) \\
\frac{\Delta 1}{Y_1} &= (L+jM)(G+jH)-(P+jQ)(E+jF) \\
&= (LG-MH)+j(LH+MG)-[PE-QF+j(PF+QE)] \\
&= [LG-MH-PE+QF]+j[LH+MG-QE-PF] \\
\frac{\Delta 2}{Y_1} &= (C+jD)(P+jQ)-(E+jF)(L+jM) \\
&= (CP-DQ)+j(CQ+DP)-[EL-MF+j(EM+LF)] \\
&= [CP-DQ+MF-EL]+j[CQ+DP-EM-FL]
\end{aligned}$$

knowing that

$$\left|\frac{a+jb}{c+jd}\right| = \sqrt{a^2+b^2}\Big/\sqrt{c^2+d^2}$$

We get

$$\left|\frac{X_1}{Y_1}\right| = \left|\frac{\Delta 1}{\Delta Y_1}\right| = \left\{\frac{[LG-MH-PE+QF]^2+[LH+MG-QE-PF]^2}{[CG-DH-E^2+F^2]^2+[CH+DG-2EF]^2}\right\}^{1/2}$$

and $$\left|\frac{X_2}{Y_1}\right| = \left\{\frac{[CP-DQ+MF-EL]^2+[CQ+DP-EM-FL]^2}{[CG-DH-E^2+F^2]^2+[CH+DG-2EF]^2}\right\}^{1/2} \quad \text{— 14}$$

$$\tan\phi_1 = \frac{(LH+MG-QE-PF)(CG-DH-E^2+F^2)-(LG-MH-PE+QF)(CH+DG-2EF)}{(LG-MH-PE+QF)(CG-DH-E^2+F^2)+(LH-MG-QE-PF)(CH+DH-2EF)}$$

$$\tan\phi_2 = \frac{(CQ+DP-EM-FL)(CG-DH-E^2+F^2)-(CP-DQ+MF-EL)(CH+DG-2EF)}{(CP-DQ+MF-EL)(CG-DH-E^2+F^2)+(CQ+DP-EM-FL)(CH+DG-2EF)}$$

Here

$$C = [K_1+K_2-m\omega^2](1+1/V_{o2}) + S(2+1/A+A)$$

$$D = -\frac{\omega}{\omega_o}[(K_1+K_2-m\omega^2)+S(V_{o2}/A+A)]$$

$$E = (K_2 b-K_1 a)(1+1/V_{o2})+S(b-\frac{a}{A}-a+Ab)$$

$$F = \frac{\omega}{\omega_o}[(K_2 b-K_1 a)+S(Ab-\frac{aV_{o2}}{A})]$$

$$G = (K_2 b^2+K_1 a^2-I\omega^2)(\frac{1}{V_{o2}}+1)+S[-b^2+Ab^2-ab+a^2/A]$$

$$H = \frac{\omega}{\omega_o}[(K_1 a^2+K_2 b^2-I\omega^2)+S(Ab^2+a^2 V_{o2}/A)]$$

$$L = (1+1/V_{o2})(K_1+K_2\cos\phi)+S(1+1/A)(1+A\cos\phi)+\frac{\omega}{\omega_o}(K_2+AS)\sin\phi$$

$$M = \frac{\omega}{\omega_o}[\frac{SV_{o2}}{A}+K_1+(K_2+AS)\cos\phi]-[K_2(1+1/V_{o2})+S(1+A)]\sin\phi$$

$$P = -K_1 a(1/V_{o2}+1)+S(b-\frac{a}{A})\ [K_2 b(1/V_{o2}+1)+S(Ab-a)]\cos\phi+\frac{\omega}{\omega_o}(K_2 b+Abs)\sin\phi$$

$$Q = -\frac{\omega}{\omega_o}[K_1 a+\frac{aSV_{o2}}{A})+(K_2 b+Abs)\cos\phi]-[K_2 b(1/V_{o2}+1)+S(Ab-a)]\sin\phi$$

Appendix 1

Flow Equation

$pv = mRT$

$pdv + vdp = dmRT + mRdT$ ....(1)

For adiabatic process

$pv^{\gamma} = \text{constant}$

$\gamma pv^{\gamma-1} dv + v^{\gamma} dp = 0$

$p\gamma dv + vdp = 0$ ...(2)

Also for an adiabatic process

$dq = 0$ No heat flow.

$\therefore dE + dW = 0$

Assuming no change in P.E. & K.E. i.e. $dE = du$

$du + dw = 0$. u is the internal energy.

$mC_v dT + pdv = 0$ ...(3)

We have $Cp - Cv = R$ $Cp/Cv = \gamma$

$$\frac{Cp - Cv}{Cv} = \gamma - 1$$

$$Cv = \frac{R}{\gamma - 1} \qquad ...(4)$$

From (3), $mRdT = -p(\gamma-1)dv$ ...(5)

From (2) $pdv = \dfrac{-vdp}{\gamma}$

$$mRdT = vdp\left(\frac{\gamma-1}{\gamma}\right) \qquad ...(6)$$

From (1), $dmRT = pdv + vdp - mRdT$

$= p\,dv + vdp[1 - \dfrac{\gamma-1}{\gamma}]$

$= pdv + \dfrac{vdp}{\gamma}$

$$\frac{dm}{dt} = \frac{1}{RT}\left\{\frac{v}{\gamma}\frac{dp}{dt} + p\frac{dv}{dt}\right\}$$