Chapter 5
Mean-Field Dynamical Models for Spin-1 Systems

5.1 Introduction

In Chapters 3 and 4, we have described numerical results obtained from Monte Carlo (MC) simulations of kinetic spin models. Typically, there is no intrinsic dynamics for these spin models as the Poisson bracket of the spin variable and the Hamiltonian is zero. Therefore, we introduce stochastic kinetics by placing the system in contact with a heat bath that induces thermal fluctuations in the system. There are two prototypical examples of stochastic kinetics, i.e., (a) Glauber spin-flip (nonconserved) kinetics, which we considered in Chapter 3; and (b) Kawasaki spin-exchange (conserved) kinetics, which we considered in Chapter 4.

These kinetic spin models constitute microscopic-level descriptions of the relevant physical phenomena. We can also formulate macroscopic-level or coarse-grained descriptions of physical phenomena via time-dependent Ginzburg-Landau (TDGL) models – both with and without conservation laws [68]. This coarse-grained (or phenomenological) level of description usually consists of nonlinear stochastic partial differential equations for the macroscopic order parameters, which describe the system.

There are advantages and disadvantages associated with both levels of description. For example, it is relatively easy to formulate a microscopic model for a given physical problem. Furthermore, these microscopic models are ideally suited to numerical simu-
tions. However, these models are usually analytically intractable because of the discrete nature of variables and update processes. Moreover, it is difficult to access asymptotic behaviors using microscopic models because of computational restrictions on system sizes and simulation times. In contrast, macroscopic models are somewhat harder to formulate as there may be some ambiguity about the appropriate free energy functional. However, phenomenological models are analytically more tractable, and it is generally easier to probe asymptotic behaviors using these models.

This chapter focuses on the connection between the microscopic and macroscopic levels of description for spin-1 models with stochastic kinetics. In particular, we will use an approach based on master-equation techniques [78] to obtain a coarse-grained phenomenological model from a given microscopic model. The resultant coarse-grained models are referred to as “mean-field (MF) dynamical models” [78], and constitute a convenient description of various physical phenomena [4]. In earlier work, Puri and Sharma [7, 8] have obtained MF dynamical models in the context of spin-1 Ising models with conserved kinetics (as discussed in Chapter 4). In this chapter, we will generalize their approach to the case of spin-1 models with arbitrary interactions. Furthermore, we will also formulate MF dynamical models for the nonconserved evolution discussed in Chapter 3.

This chapter is organized as follows. Section 5.2 discusses MF dynamical models for nonconserved domain growth with local barriers (see Chapter 3). In Section 5.3 we present analogous models for the arbitrary spin-1 model with conserved kinetics (see Chapter 4). Finally, Section 5.4 concludes this chapter with a summary and discussion of our results.
5.2 Systems with Nonconserved Kinetics

5.2.1 General Definitions

Recall the most general version of the Blume-Emery-Griffiths (BEG) Hamiltonian [28]:

$$H = -J \sum_{(ij)} s_is_j - K \sum_{(ij)} s_i^2s_j^2 - \frac{M}{2} \sum_{(ij)} (s_i^2s_j + s_i^2s_j^2) - h \sum_i s_i - \Delta \sum_i s_i^2, \quad s_i = \pm 1, 0, \quad (5.1)$$

where the angular brackets refer to a sum over nearest-neighbor pairs; $J(>0)$ is the ferromagnetic exchange interaction strength; $K$ is the quadrupole exchange strength; $h$ is an external magnetic field; and $\Delta$ is an external anisotropy field.

In Chapter 3, we focused on Eq. (5.1) with intrinsic parameters $K = M = 0$. Furthermore, we considered the dynamical evolution of this system in a fixed $(T, h, \Delta)$-ensemble, with nonconserved constrained kinetics such that $\pm 1 \to 0$ and $0 \to \pm 1$ were the only transitions permitted. We will consider the symmetric case $h = 0$, and $\gamma(= \beta\Delta)$ positive and large, i.e., there is a strong local barrier to the microscopic processes which enable domain growth. Thus, Eq. (5.1) reduces to the Blume-Capel (BC) model [34, 35]:

$$H = -J \sum_{(ij)} s_is_j - \Delta \sum_i s_i^2, \quad (5.2)$$

The spin configuration at a time $t$ is denoted by $\{s_i\} \equiv \{s_1, s_2, ..., s_{N-1}, s_N\}$, where there are $N$ spins in the system. We will consider the evolution of the probability distribution $P(\{s_i\}; t)$ for a configuration $\{s_i\}$ at time $t$, and thereby obtain coupled equations for the order parameters $\langle s_i \rangle \equiv m$ and $\langle s_i^2 \rangle \equiv \rho$.

5.2.2 The Master Equation

The spin system evolves with time due to stochastic microscopic processes. The corresponding probability for a given configuration $\{s_i\}$ also evolves with time, according to a phenomenological kinetic equation referred to as the master equation [79], viz.,

$$\frac{\partial}{\partial t} P(\{s_i\}; t) = -\sum_{i=1}^{N} \sum_{s=0,\pm 1} W(s \to s) P(\{s_1, ..., s_i, ..., s_N\}; t)$$
5.2. Systems with Nonconserved Kinetics

\[ + \sum_{i=1}^{N} \sum_{s=0, \pm 1} W(s \to s_i) P(\{s_1, \ldots, s, \ldots, s_N\}; t). \quad (5.3) \]

The first term on the right-hand-side (RHS) of Eq. (5.3) is the loss of the probability by spins flipping out of the specified configuration, and the second term on the RHS represents the probability gain by spins flipping into the specified configuration.

The transition probabilities \( W(s_i \to s) \) must satisfy the detailed-balance relation, so that evolution governed by the master equation results in thermal equilibrium with a canonical distribution. The detailed-balance condition assumes the form

\[ W(s \to s_i) P^0(\{s_1, \ldots, s, \ldots, s_N\}) = W(s_i \to s) P^0(\{s_1, \ldots, s_i, \ldots, s_N\}), \quad (5.4) \]

where \( P^0(\{s_i\}) \) denotes the equilibrium distribution, i.e., \( P^0(\{s_i\}) = Z^{-1} \exp[-\beta H(\{s_i\})] \), where \( \beta = T^{-1} \) (we set \( k_B = 1 \); \( H(\{s_i\}) \) is the spin Hamiltonian; and \( Z \) is the partition function. Any transition probability \( W(s_i \to s) \) which is consistent with the detailed-balance condition leads to correct thermal equilibrium. Various functional forms for \( W(s_i \to s) \) have been used in the literature [80]. We will consider the Suzuki-Kubo form [81]:

\[ W(s_i \to s) = \frac{1}{\tau} \left[ 1 - \tanh \left( \frac{\beta \Delta H(s_i \to s)}{2} \right) \right], \quad (5.5) \]

where \( \tau \) is the time-scale of the microscopic process; and \( \Delta H(s_i \to s) \) is the energy change associated with the process \( (s_i \to s) \), i.e.,

\[ \Delta H(s_i \to s) = H(\{s_1, \ldots, s, \ldots, s_N\}) - H(\{s_1, \ldots, s_i, \ldots, s_N\}). \quad (5.6) \]

It is easily confirmed that the transition probability in Eq. (5.5) satisfies the detailed-balance condition. We are interested in the constrained kinetics \( \pm 1 \to 0 \) and \( 0 \to \pm 1 \). Therefore, in our subsequent discussion, we will use the following functional form of the transition probability:

\[ W(s_i \to s) = \frac{1}{\tau} (1 + s_i s) \left[ 1 - \tanh \left( \frac{\beta \Delta H(s_i \to s)}{2} \right) \right]. \quad (5.7) \]
The prefactor \((1 + s_i s) = 0\) when \(s_i = \pm 1\) and \(s = \mp 1\), and thereby eliminates transitions \(\pm 1 \rightarrow \mp 1\). The modified transition probability in Eq. (5.7) also satisfies the detailed-balance condition. In the context of the BC Hamiltonian in Eq. (5.2), we have

\[
\Delta H(s_i \rightarrow s) = J(s_i - s) \sum_{L_i} s_{L_i} + \Delta (s_i^2 - s^2),
\]

where \(L_i\) denotes the nearest-neighbors of \(i\). The evolution equations for the moments of the spin variables (i.e., the order parameters) can now be easily derived from Eq. (5.3) with the transition probabilities from Eq. (5.7). This will be the content of the next subsection.

### 5.2.3 Evolution Equations for Order Parameters

The \(k\)th moment of the spin variable \(s_i\) at time \(t\) is defined by

\[
\langle s_i^k \rangle = \sum_{\{s\}} s_i^k P(\{s_1, \ldots, s_i, \ldots, s_N\}; t),
\]

where the sum is taken over all possible configurations of the spins. Using the definition of the moment in Eq. (5.9) and the master equation, we can write the evolution equation for the first moment as follows:

\[
\frac{\tau}{\partial t} \langle s_i \rangle = - \sum_{\{s_i\}} \sum_{i=1}^{N} \sum_{s=0,\pm 1} s_n (1 + s s_i) \left\{ 1 - (s_i - s) \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_i} s_{L_i} + \Delta (s_i + s) \right) \right] \right\} \times P(\{s_1, \ldots, s_i, \ldots, s_N\}; t)
\]

\[
+ \sum_{\{s_i\}} \sum_{i=1}^{N} \sum_{s=0,\pm 1} s_n (1 + s s_i) \left\{ 1 + (s_i - s) \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_i} s_{L_i} + \Delta (s_i + s) \right) \right] \right\} \times P(\{s_1, \ldots, s, \ldots, s_N\}; t).
\]

(5.10)

Exchanging \(s_i\) and \(s\) in the second sum and doing the sum over \(i\), we see that the only term which survives cancellations is the one with \(i = n\). Thus, we have (absorbing \(\tau\) in the definition of \(t\))

\[
\frac{\partial}{\partial t} \langle s_n \rangle = - \sum_{\{s_i\}} \sum_{s=0,\pm 1} (s_n - s)(1 + s s_n) \left\{ 1 - (s_n - s) \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_n} s_{L_n} + \Delta (s_n + s) \right) \right] \right\} \times P(\{s_1, \ldots, s_i, \ldots, s_N\}; t).
\]

(5.11)
Recall that \( s_i^3 = s_i \), so that

\[
(s_n - s)(1 + ss_n) = s_n - s + s_n^2s - s_n^2,
\]

\[
(s_n - s)^2(1 + ss_n) = s_n^2 + s^2 - 2s_n^2s^2.
\] (5.12)

Using Eq. (5.12), the evolution equation for \( \langle s_n \rangle \) becomes

\[
\frac{\partial}{\partial t} \langle s_n \rangle = - \sum_{\{s_i\}} \sum_{s=0,\pm 1} \left\{ (s_n - s + s_n^2s - s_n^2s^2) - (s_n^2 + s^2 - 2s_n^2s^2) \right\} \times \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_n} s_{L_n} + \Delta(s_n + s) \right) \right] P(\{s_i\}; t). \tag{5.13}
\]

Next, we sum over \( s \) and use the facts that \( s_n = 0 \) when \( s = \pm 1 \); and

\[
\tanh \left[ \frac{\beta}{2} \left( J \sum_{L_n} s_{L_n} + \Delta s_n \right) \right] = \frac{(1 + s_n)}{2} \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_n} s_{L_n} + \Delta \right) \right] + \frac{(1 - s_n)}{2} \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_n} s_{L_n} - \Delta \right) \right]. \tag{5.14}
\]

This yields an exact evolution equation for \( \langle s_n \rangle \) as follows:

\[
\frac{\partial}{\partial t} \langle s_n \rangle = - \langle s_n \rangle + \frac{1}{2} \left( 2 - s_n^2 + \langle s_n \rangle \right) \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_n} s_{L_n} + \Delta \right) \right] + \frac{1}{2} \left( 2 - s_n^2 - \langle s_n \rangle \right) \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_n} s_{L_n} - \Delta \right) \right]. \tag{5.15}
\]

So far, our discussion is exact in the framework of our modeling. However, Eq. (5.15) is part of an infinite hierarchy of coupled equations and is analytically and numerically intractable. We simplify the model in Eq. (5.15) by invoking the MF approximation, where expectation values of functions of the spin variables are replaced by functions of the expectation values of the spins. In the MF approximation, we obtain

\[
\frac{\partial}{\partial t} \langle s_n \rangle = - \langle s_n \rangle + \frac{1}{2} \left( 2 - \langle s_n^2 \rangle + \langle s_n \rangle \right) \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_n} \langle s_{L_n} \rangle + \Delta \right) \right] + \frac{1}{2} \left( 2 - \langle s_n^2 \rangle - \langle s_n \rangle \right) \tanh \left[ \frac{\beta}{2} \left( J \sum_{L_n} \langle s_{L_n} \rangle - \Delta \right) \right]. \tag{5.16}
\]
Chapter 5. Mean-Field Dynamical Models for Spin-1 Systems

The MF dynamical equation for the evolution of the second moment of the spin variable, i.e., the quadrupole moment, is obtained in a similar fashion. To this end, we start with the definition in Eq. (5.9) and the master equation (5.3) to write down the evolution equation for the quadrupole moment:

\[ \tau \frac{\partial}{\partial t} \langle s_n^2 \rangle = - \sum_{\{s_i\}} \sum_{s=0,\pm 1}^N s_n^2 (1 + ss_i) \left\{ 1 - (s_i - s) \tanh \left[ \frac{\beta}{2} \left( J \sum_{b_i} s_{b_i} + \Delta (s_i + s) \right) \right] \cdot P(\{s_i, ..., s_i, ..., s_N\}; t) \right. \\
+ \left. \sum_{\{s_i\}} \sum_{s=0,\pm 1}^N s_n^2 (1 + ss_i) \left\{ 1 + (s_i - s) \tanh \left[ \frac{\beta}{2} \left( J \sum_{b_i} s_{b_i} + \Delta (s_i - s) \right) \right] \cdot P(\{s_i, ..., s_i, ..., s_N\}; t) \right\} \right\}
\]

Again, we exchange \( s_i \) and \( s \) in the second term on the RHS. The only terms that survive the cancellation are terms with \( i = n \). Furthermore, noting that

\[ (s^2 - s_n^2)(1 + ss_n) = s^2 - s_n^2 \] \hspace{1cm} (5.18)
\[ (s^2 - s_n^2)(1 + ss_n)(s_n - s) = s^2 s_n + ss_n^2 - s - s_n \] \hspace{1cm} (5.19)

Eq. (5.17) can be written as (absorbing \( \tau \) into the definition of \( t \))

\[ \frac{\partial}{\partial t} \langle s_n^2 \rangle = \sum_{\{s_i\}} \sum_{s=0,\pm 1} \left\{ (s^2 - s_n^2) - (s^2 s_n + ss_n^2 - s - s_n) \tanh \left[ \frac{\beta}{2} \left( J \sum_{b_i} s_{b_i} + \Delta (s_n + s) \right) \right] \cdot P(\{s_i\}; t) \right\}
\]

Summing over \( s \), and invoking the MF approximation, we obtain the dynamical equation

\[ \frac{\partial}{\partial t} \langle s_n^2 \rangle = 2 - 3 \langle s_n^2 \rangle + \frac{1}{2} \left( 2 - \langle s_n^2 \rangle + \langle s_n \rangle \right) \tanh \left[ \frac{\beta}{2} \left( J \sum_{b_i} \langle s_{b_i} \rangle + \Delta \right) \right] \\
- \frac{1}{2} \left( 2 - \langle s_n^2 \rangle - \langle s_n \rangle \right) \tanh \left[ \frac{\beta}{2} \left( J \sum_{b_i} \langle s_{b_i} \rangle - \Delta \right) \right]. \hspace{1cm} (5.21)
\]

Eqs. (5.16) and (5.21) constitute our MF dynamical model for the coupled evolution of the two order parameters, \( \langle s_k \rangle \) and \( \langle s_n^2 \rangle \). The MF approximation was crucial to obtain closed-form evolution equations which are analytically and numerically tractable.
Finally, let us check that Eqs. (5.16) and (5.21) do indeed contain the static mean-field solutions for \( \langle s_n \rangle \) and \( \langle s_n^2 \rangle \), the magnetization and the quadrupole moment, respectively. Recall that the static mean-field equations, when \( K = M = h = 0 \) (see Chapter 2), are given by

\[
\langle s_n \rangle^0 = \frac{2 \sinh (\beta J \sum_L \langle s_L \rangle^0)}{\exp(-\beta \Delta) + 2 \cosh (\beta J \sum_L \langle s_L \rangle^0)},
\]

\[
\langle s_n^2 \rangle^0 = \frac{2 \cosh (\beta J \sum_L \langle s_L \rangle^0)}{\exp(-\beta \Delta) + 2 \cosh (\beta J \sum_L \langle s_L \rangle^0)}. \tag{5.22}
\]

Let's introduce new variables \( x \) and \( y \) such that

\[
x = \exp \left( \beta \sum_L \langle s_L \rangle^0 \right),
\]

\[
y = \exp(-\beta \Delta). \tag{5.23}
\]

In terms of \( x \) and \( y \), we can write the following relations:

\[
\tanh \left[ \frac{\beta}{2} \left( J \sum_L \langle s_L \rangle^0 + \Delta \right) \right] = \frac{x - y}{x + y},
\]

\[
\tanh \left[ \frac{\beta}{2} \left( J \sum_L \langle s_L \rangle^0 - \Delta \right) \right] = \frac{xy - 1}{xy + 1},
\]

\[
\langle s_n \rangle^0 = \frac{x^2 - 1}{x^2 + xy + 1},
\]

\[
\langle s_n^2 \rangle^0 = \frac{x^2 + 1}{x^2 + xy + 1},
\]

\[
\langle s_n^2 \rangle^0 - \langle s_n \rangle^0 = \frac{2}{x^2 + xy + 1},
\]

\[
\langle s_n^2 \rangle^0 + \langle s_n \rangle^0 = \frac{2x^2}{x^2 + xy + 1}. \tag{5.25}
\]

Substituting Eq. (5.25) into the RHS of the evolution equations (Eqs. (5.16) and (5.21)), we see that the RHS of each equation become identically zero, as expected.

### 5.2.4 Continuum Version of Dynamical Equations

In Eqs. (5.16) and (5.21), let us introduce continuum variables,

\[
\langle s_n \rangle \equiv m(\vec{r}, t),
\]
where \( \vec{r} \) is the (continuum) vector which labels the discrete space-point \( n \). In terms of these continuum variables, Eq. (5.16) can be written as

\[
\frac{\partial m(\vec{r}, t)}{\partial t} = -m + \frac{1}{2} (2 - \rho + m) \tanh \left[ \frac{\beta}{2} (J q m + J \nabla^2 m + \ldots + \Delta) \right]
+ \frac{1}{2} (2 - \rho - m) \tanh \left[ \frac{\beta}{2} (J q m + J \nabla^2 m + \ldots - \Delta) \right],
\]

(5.27)

where \( q \) is the coordination number of a lattice site. We introduce the notation \( z = \frac{\beta}{2} (J q m + J \nabla^2 m + \ldots) \) and \( \alpha = \tanh(\beta \Delta/2) \). Then, making use of the identity,

\[
\tanh(A + B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B},
\]

(5.28)

we obtain

\[
\tanh \left( z + \frac{\beta \Delta}{2} \right) = (\alpha + \tanh z)[1 - \alpha \tanh z + (\alpha \tanh z)^2 - (\alpha \tanh z)^3 + \ldots]
+ (\alpha \tanh z)^4 - (\alpha \tanh z)^5 + \ldots
\]

\[
= \alpha + (1 - \alpha^2) \tanh^2 z - \alpha(1 - \alpha^2) \tanh^4 z + \alpha^2(1 - \alpha^2) \tanh^6 z + \ldots
+ (-1)^n \alpha^n (1 - \alpha^2) \tanh^{n+1} z + \ldots
\]

(5.29)

Similarly, we have

\[
\tanh \left( z - \frac{\beta \Delta}{2} \right) = -\alpha + (1 - \alpha^2) \tanh z + \alpha(1 - \alpha^2) \tanh^2 z + \alpha^2(1 - \alpha^2) \tanh^3 z + \ldots
+ \alpha^n (1 - \alpha^2) \tanh^{n+1} z + \ldots
\]

(5.30)

Therefore,

\[
\frac{1}{2} (2 - \rho + m) \tanh \left( z + \frac{\beta \Delta}{2} \right) + \frac{1}{2} (2 - \rho - m) \tanh \left( z - \frac{\beta \Delta}{2} \right)
= \frac{(2 - \rho)}{2} \left[ 2(1 - \alpha^2) \tanh z + 2\alpha^2(1 - \alpha^2) \tanh^3 z + 2\alpha^4(1 - \alpha^2) \tanh^5 z + \ldots \right]
+ \frac{m}{2} \left[ 2\alpha - 2\alpha(1 - \alpha^2) \tanh^2 z - 2\alpha^2(1 - \alpha^2) \tanh^4 z - 2\alpha^5(1 - \alpha^2) \tanh^6 z + \ldots \right]
= (2 - \rho)(1 - \alpha^2) \left[ \tanh z + \alpha^2 \tanh^3 z + \alpha^4 \tanh^5 z + \ldots \right]
+ m\alpha \left[ 1 - (1 - \alpha^2) \tanh^2 z - \alpha^2(1 - \alpha^2) \tanh^4 z + \ldots \right].
\]

(5.31)
Now, using the Taylor expansion of the “tanh” function, and collecting terms in powers of \( z \) yields

\[
\frac{1}{2} (2 - \rho + m) \tanh \left( z + \frac{\beta \Delta}{2} \right) + \frac{1}{2} (2 - \rho - m) \tanh \left( z - \frac{\beta \Delta}{2} \right)
\]

\[= (2 - \rho)(1 - \alpha^2) \left[ z + \left( \alpha^2 - \frac{1}{3} \right) z^3 + \left( \alpha^4 - \alpha^2 + \frac{2}{15} \right) z^5 + \ldots \right]
+ \alpha m \left[ 1 - (1 - \alpha^2) z^2 + \left( \frac{2}{3} - \alpha^2 \right) (1 - \alpha^2) z^4 + \ldots \right]
\]

\[= \left[ (2 - \rho)(1 - \alpha^2) \frac{q \beta J}{2} + \alpha \right] m + (2 - \rho)(1 - \alpha^2) \frac{\beta J}{2} \nabla^2 m
+ \left[ (2 - \rho)(1 - \alpha^2) \left( \alpha^2 - \frac{1}{3} \right) \left( \frac{q \beta J}{2} \right)^3 - \alpha(1 - \alpha^2) \left( \frac{q \beta J}{2} \right)^2 \right] m^3
+ \left[ (2 - \rho)(1 - \alpha^2) \left( \alpha^4 - \alpha^2 + \frac{2}{15} \right) \left( \frac{q \beta J}{2} \right)^5 + \alpha \left( \frac{2}{3} - \alpha^2 \right) (1 - \alpha) \left( \frac{q \beta J}{2} \right)^4 \right] m^5
+ \text{higher order terms.} \quad (5.32)
\]

Thus, the dynamical equation for \( m(\vec{r}, t) \) assumes the form

\[
\frac{\partial m(\vec{r}, t)}{\partial t} = \left[ (2 - \rho)(1 - \alpha^2) \frac{q \beta J}{2} + \alpha - 1 \right] m
+ (1 - \alpha^2) \left( \frac{q \beta J}{2} \right)^2 \left[ (2 - \rho) \left( \alpha^2 - \frac{1}{3} \right) \frac{q \beta J}{2} - \alpha \right] m^3
+ (1 - \alpha^2) \left( \frac{q \beta J}{2} \right)^4 \left[ (2 - \rho) \left( \alpha^4 - \alpha^2 + \frac{2}{15} \right) \frac{q \beta J}{2} + \alpha \left( \frac{2}{3} - \alpha^2 \right) \right] m^5
+ (2 - \rho)(1 - \alpha^2) \frac{\beta J}{2} \nabla^2 m + \text{higher order terms.} \quad (5.33)
\]

The corresponding dynamical equation for \( \rho(\vec{r}, t) \) can be obtained from

\[
\frac{\partial \rho(\vec{r}, t)}{\partial t} = 2 - 3 \rho + \frac{1}{2} (2 - \rho + m) \tanh \left( z + \frac{\beta \Delta}{2} \right)
- \frac{1}{2} (2 - \rho - m) \tanh \left( z - \frac{\beta \Delta}{2} \right). \quad (5.34)
\]

The subsequent calculation is analogous to that described earlier. For brevity, we do not reproduce the calculation here and merely present the resultant evolution equation.
for $\rho(\vec{r}, t)$, which is as follows:

$$
\frac{\partial \rho(\vec{r}, t)}{\partial t} = 2(1 + \alpha) - (3 + \alpha)\rho 
+ (1 - \alpha^2) \frac{q^2 J}{2} \left[ 1 - (2 - \rho)\alpha \frac{q^2 J}{2} \right] m^2 
+ (1 - \alpha^2) \left( \frac{q^2 J}{2} \right)^3 \left[ \left( \alpha^2 - \frac{1}{3} \right) + (2 - \rho)\alpha \left( \frac{2}{3} - \alpha^2 \right) \frac{q^2 J}{2} \right] m^4 
+ (1 - \alpha^2) \frac{\beta J}{2} m \nabla^2 m + \text{higher order terms.} \tag{5.35}
$$

### 5.2.5 Dynamics in the $\beta \Delta \to \infty$ Limit

Finally, let us focus on our MF dynamical models in the limit $\gamma = \beta \Delta \to \infty$. Recall that this is the physical situation we considered in Chapter 3, where high values of $\gamma$ presented a local barrier to domain growth. It is relevant to examine the implications of our MF models in this simple limit.

In our MF dynamical equations, introduce the notation $z = \beta J \sum_{L_n} \langle s_{L_n} \rangle / 2$, as in subsection 5.2.4. In the limit $\gamma \to \infty$, we have

$$
\tanh \left( z + \frac{\beta \Delta}{2} \right) = \frac{e^{2z} - e^{-\beta \Delta}}{e^{2z} + e^{-\beta \Delta}} 
= \left( 1 - \frac{\theta}{e^{2z}} \right) \left( 1 + \frac{\theta}{e^{2z}} \right)^{-1} 
= 1 - 2 \frac{\theta}{e^{2z}} + O(\theta^2), \tag{5.36}
$$

where $\theta = e^{-\beta \Delta}$ is a small expansion parameter. Similarly, we have

$$
\tanh \left( z - \frac{\beta \Delta}{2} \right) = -1 + 2 \theta e^{2z} + O(\theta^2). \tag{5.37}
$$

This gives the equation of motion for $\langle s_n \rangle$, correct to $O(\theta)$, as

$$
\frac{\partial \langle s_n \rangle}{\partial t} = 2\theta \left[ (2 - \langle s^2_n \rangle) \sinh \left( \beta J \sum_{L_n} \langle s_{L_n} \rangle \right) - \langle s_n \rangle \cosh \left( \beta J \sum_{L_n} \langle s_{L_n} \rangle \right) \right]. \tag{5.38}
$$

In the same way, we can write the equation of motion for $\langle s^2_n \rangle$, correct to $O(\theta)$, as

$$
\frac{\partial \langle s^2_n \rangle}{\partial t} = 4(1 - \langle s^2_n \rangle) - \cdots
$$
Next, let us examine the static solutions in this limit. For \( I = 0 \), the 0's are entirely eliminated from the system, which reverts to the spin-1/2 Ising model. For \( I \to 0 \), an expansion of Eq. (5.22) yields (correct to \( O(\gamma) \))

\[
\langle s_n \rangle^0 = \tanh \left( \beta J \sum_{L_n} \langle s_{L_n} \rangle^0 \right) - \frac{\theta}{2} \sech \left( \beta J \sum_{L_n} \langle s_{L_n} \rangle^0 \right) \tanh \left( \beta J \sum_{L_n} \langle s_{L_n} \rangle^0 \right)
\]

\[
\langle s_n^2 \rangle^0 = 1 - \frac{\theta}{2} \sech \left( \beta J \sum_{L_n} \langle s_{L_n} \rangle^0 \right).
\]

We get the same result from Eqs. (5.38) and (5.39) by setting \( \partial \langle s_n \rangle / \partial t = 0 \), \( \partial \langle s_n^2 \rangle / \partial t = 0 \). It is relevant to examine the dynamics of the homogeneous system in the limit \( I \to 0 \). In the homogeneous case \( (\langle s_n \rangle = m(t), \langle s_n^2 \rangle = \rho(t)) \), Eqs. (5.38) and (5.39) can be written as

\[
\frac{dm(t)}{dt} = 2\theta \left[ (2 - \rho) \sinh(q\beta Jm) - m \cosh(q\beta Jm) \right],
\]

\[
\frac{d\rho(t)}{dt} = 4(1 - \rho) - 2\theta \left[ (2 - \rho) \cosh(q\beta Jm) - m \sinh(q\beta Jm) \right].
\]

Since \( \rho \) rapidly relaxes to a value in the vicinity of 1 (with \( O(\theta) \) corrections), the following equations should be reasonable approximations to Eq. (5.41):

\[
\frac{dm(t)}{dt} \simeq 2\theta \left[ \sinh(q\beta Jm) - m \cosh(q\beta Jm) \right],
\]

\[
\frac{d\rho(t)}{dt} \simeq 2\theta \left[ - \sinh(q\beta Jm) \tanh(q\beta Jm) + m \sinh(q\beta Jm) \right],
\]

suggesting that \( \theta^{-1} = e^\gamma \) is merely a scale for the time variable in the evolution of the system. Of course, this is precisely what we found in the MC simulations discussed in Chapter 3.

Finally, we examine the relaxational dynamics to the homogeneous fixed point \( (m_0, \rho_0) \). The relevant eigenvalues are obtained by linearizing Eq. (5.41) about the fixed point as
Chapter 5. Mean-Field Dynamical Models for Spin-1 Systems

follows:
\[
\frac{d}{dt} \delta m(t) = M_{11} \delta m(t) + M_{12} \delta \rho(t),
\]
\[
\frac{d}{dt} \delta \rho(t) = M_{21} \delta m(t) + M_{22} \delta \rho(t),
\] (5.43)

where \( \delta m = m - m_0 \) and \( \delta \rho = \rho - \rho_0 \). The linearized coefficients are

\[
M_{11} = 2 \theta \left[ (2 - \rho_0) \beta J - 1 \right] \cosh(\beta J m_0) - \beta J m_0 \sinh(\beta J m_0),
\]
\[
M_{12} = -2 \theta \beta J \cosh(\beta J m_0),
\]
\[
M_{21} = -2 \theta \left[ (2 - \rho_0) \beta J - 1 \right] \sinh(\beta J m_0) - \beta J m_0 \cosh(\beta J m_0),
\]
\[
M_{22} = -4 + 4 \theta \cosh(\beta J m_0).
\] (5.44)

Let us focus on the physical situation below the ordering temperature. All our expansions are valid up to \( O(\theta) \), so we can set \( \rho_0 = 1 \) in the above equations for \( M_{ij} \); and choose \( m_0 \) as the solution of \( m_0 = \tanh(\beta m_0) \). Hence, we have (correct to \( O(\theta) \))

\[
M_{11} = 2 \theta \left[ \beta J \text{sech}(\beta J m_0) - \cosh(\beta J m_0) \right],
\]
\[
M_{12} = -2 \theta \beta J \cosh(\beta J m_0),
\]
\[
M_{21} = 2 \theta \sinh(\beta J m_0),
\]
\[
M_{22} = -4 + 4 \theta \cosh(\beta J m_0).
\] (5.45)

Clearly, \( M_{11}, M_{12}, M_{22} < 0 \) and the sign of \( M_{21} \) is determined by the sign of \( m_0 \). The eigenvalues for relaxational dynamics in Eq. (5.43) are obtained as

\[
\lambda_{1,2} = \frac{M_{11} + M_{22}}{2} \pm \sqrt{\left( \frac{M_{11} + M_{22}}{2} \right)^2 - (M_{11}M_{22} - M_{12}M_{21})}.
\] (5.46)

Therefore, \( \lambda_{1,2} < 0 \), as expected from the relaxational nature of the dynamics.

5.3 Systems with Conserved Kinetics

5.3.1 General Definitions

In the previous section, we developed MF kinetic models for the BEG model in the case where the order parameters are not conserved. In this section, we develop MF models for
the case where the microscopic dynamics conserves the order parameters. The models developed in this section are coarse-grained counterparts of the MC models studied in Chapter 4.

We define our lattice model as in Section 5.2, and our earlier discussion is applicable here also – except that the numbers of individual species are now conserved. As in Chapter 4, we impose the kinetic constraint that $\pm 1 \leftrightarrow 0$ interchanges are allowed, and $\pm 1 \leftrightarrow \mp 1$ interchanges are forbidden. The most general form of the BEG Hamiltonian with the composition constraint is

$$H = -J \sum_{(ij)} s_i s_j - K \sum_{(ij)} s_i^2 s_j^2 - \frac{M}{2} \sum_{(ij)} (s_i^2 s_j^2 + s_i^2 s_j^2).$$

(5.47)

In accordance with our discussion in Chapter 4, we will obtain MF dynamical models for the case with $M = 0$. Of course, our models are trivially generalized to the case with $M \neq 0$ also.

### 5.3.2 The Master Equation

In this case, the spin system evolves due to the stochastic interchange of neighboring spins, subject to kinetic constraints. As before, the probability of a given configuration $\{s_i\}$ evolves with time according to the master equation, which has a somewhat different form in the present context:

$$\frac{\partial}{\partial t} P(\{s_i\};t) = -\sum_{i=1}^{N} \sum_{j \in L_i} W(s_i \leftrightarrow s_j) P(\{s_1, \ldots, s_i, s_j, \ldots, s_N\};t)$$

$$+ \sum_{i=1}^{N} \sum_{j \in L_i} W(s_j \leftrightarrow s_i) P(\{s_1, \ldots, s_j, s_i, \ldots, s_N\};t),$$

(5.48)

where $L_i$ denotes the neighbors of $i$. The first term on the RHS is the loss of probability through the exchange $s_i \leftrightarrow s_j$, and the second term on the RHS is the gain in probability by the opposite exchange. As before, the transition probabilities must satisfy the detailed-balance condition and we use the Suzuki-Kubo form (with kinetic constraints included)

$$W(s_i \leftrightarrow s_j) = \frac{1}{2\tau} (1 + s_i s_j) \left[ 1 - \tanh \left( \frac{\beta \Delta H(s_i \leftrightarrow s_j)}{2} \right) \right],$$

(5.49)
where \( \tau \) is a characteristic time governing the transitions; and \( \Delta H(s_i \leftrightarrow s_j) \) is the change in energy associated with the exchange \( s_i \leftrightarrow s_j \).

For the BEG Hamiltonian considered here, the change in energy due to the exchange \( s_i \leftrightarrow s_j \) is given by

\[
\Delta H(s_i \leftrightarrow s_j) = J(s_i - s_j) \left( \sum_{L_i \neq j} s_{L_i} - \sum_{L_j \neq i} s_{L_j} \right) + K(s_i^2 - s_j^2) \left( \sum_{L_i \neq j} s_{L_i}^2 - \sum_{L_j \neq i} s_{L_j}^2 \right).
\]

With this, we can write the transition rates as

\[
W(s_i \leftrightarrow s_j) = \frac{1}{2\tau} \left(1 + s_i s_j\right) \left\{ 1 - (s_i - s_j) \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_i \neq j} s_{L_i} - \sum_{L_j \neq i} s_{L_j} \right) \right] \right. \\
+ \left. \frac{\beta K}{2} \left( s_i + s_j \right) \left( \sum_{L_i \neq j} s_{L_i}^2 - \sum_{L_j \neq i} s_{L_j}^2 \right) \right\},
\]

where we used the fact that \( (s_i - s_j) = \pm 1 \) to factor it out from the argument of the "tanh" function. Thus, the master equation can be written as

\[
\frac{\partial}{\partial t} P(\{s_i\}; t) = -\frac{1}{2\tau} \sum_{i=1}^{N} \sum_{j \in L_i} \left(1 + s_i s_j\right) \left\{ 1 - (s_i - s_j) \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_i \neq j} s_{L_i} - \sum_{L_j \neq i} s_{L_j} \right) \right] \right. \\
+ \left. \frac{\beta K}{2} \left( s_i + s_j \right) \left( \sum_{L_i \neq j} s_{L_i}^2 - \sum_{L_j \neq i} s_{L_j}^2 \right) \right\} P(\{s_1, \ldots, s_i, s_j, \ldots, s_N\}; t) \\
+ \frac{1}{2\tau} \sum_{i=1}^{N} \sum_{j \in L_i} \left(1 + s_i s_j\right) \left\{ 1 - (s_j - s_i) \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_i \neq j} s_{L_i} - \sum_{L_j \neq i} s_{L_j} \right) \right] \right. \\
+ \left. \frac{\beta K}{2} \left( s_i + s_j \right) \left( \sum_{L_i \neq j} s_{L_i}^2 - \sum_{L_j \neq i} s_{L_j}^2 \right) \right\} P(\{s_1, \ldots, s_j, s_i, \ldots, s_N\}; t).
\]
5.3.3 Evolution Equations for Order Parameters

We recall that the $n^{th}$ moment of spin $s_k$ is given by

$$\langle s_k^n \rangle = \sum_{\{s\}} s_k^n P(\{s\}; t). \quad (5.53)$$

We use the master equation to obtain an evolution equation for $\langle s_k \rangle$ as

$$2\tau \frac{\partial \langle s_k \rangle}{\partial t} = - \sum_{\{s\}} \sum_{j=1}^{N} s_k(1 + s_i s_j) \left\{ 1 - (s_i - s_j) \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_i \neq j} s_{L_i} - \sum_{L_j \neq i} s_{L_j} \right) + \frac{\beta K}{2} (s_i + s_j) \left( \sum_{L_i \neq j} s_{L_i}^2 - \sum_{L_j \neq i} s_{L_j}^2 \right) \right] \right\} P(\{s_1, \ldots, s_i, s_j, \ldots, s_N\}; t)$$

$$+ \sum_{\{s\}} \sum_{j=1}^{N} s_k(1 + s_i s_j) \left\{ 1 - (s_i - s_j) \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_i \neq j} s_{L_i} - \sum_{L_j \neq i} s_{L_j} \right) + \frac{\beta K}{2} (s_i + s_j) \left( \sum_{L_i \neq j} s_{L_i}^2 - \sum_{L_j \neq i} s_{L_j}^2 \right) \right] \right\} P(\{s_1, \ldots, s_i, s_j, \ldots, s_N\}; t). \quad (5.54)$$

Next, we exchange $i$ and $j$ in the second sum on the RHS and find that there is a large-scale cancellation of terms on the RHS. The only terms which survive are those with $i = k$. Therefore, after rearranging the remaining terms, we arrive at (absorbing the factor $\tau$ into the definition of time)

$$\frac{\partial \langle s_k \rangle}{\partial t} = \sum_{\{s\}} \sum_{j \in L_k} (s_j - s_k)(1 + s_k s_j) \left\{ 1 - (s_k - s_j) \right\} \times \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_k \neq j} s_{L_k} - \sum_{L_j \neq k} s_{L_j} \right) + \frac{\beta K}{2} (s_k + s_j) \left( \sum_{L_k \neq j} s_{L_k}^2 - \sum_{L_j \neq k} s_{L_j}^2 \right) \right]. \quad (5.55)$$

Furthermore, we notice that

$$\langle s_j - s_k \rangle (1 + s_k s_j) = s_j - s_k + s_k s_j^2 - s_k^2 s_j, \quad (5.56)$$

$$\langle s_j - s_k \rangle (1 + s_k s_j) (s_k - s_j) = 2s_k^2 s_j^2 - s_k^2 - s_j^2. \quad (5.57)$$
and $s_k + s_j = \pm 1$. This enables us to write the “tanh” function as

$$\tanh \left[ \frac{\beta J}{2} \left( \sum_{L_k \neq j} s_{L_k} - \sum_{L_j \neq k} s_{L_j} \right) + \frac{\beta K}{2} (s_k + s_j) \left( \sum_{L_k \neq j} s_{L_k}^2 - \sum_{L_j \neq k} s_{L_j}^2 \right) \right]$$

$$= \frac{1 - (s_k + s_j)}{2} \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_k \neq j} s_{L_k} - \sum_{L_j \neq k} s_{L_j} \right) - \frac{\beta K}{2} \left( \sum_{L_k \neq j} s_{L_k}^2 - \sum_{L_j \neq k} s_{L_j}^2 \right) \right]$$

$$+ \frac{1 + (s_k + s_j)}{2} \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_k \neq j} s_{L_k} - \sum_{L_j \neq k} s_{L_j} \right) + \frac{\beta K}{2} \left( \sum_{L_k \neq j} s_{L_k}^2 - \sum_{L_j \neq k} s_{L_j}^2 \right) \right].$$

(5.58)

Substituting the above expressions in the evolution equation, we obtain

$$\frac{\partial \langle s_k \rangle}{\partial t} = \left\langle \sum_{j \in L_k} \left( s_j - s_k + s_j^2 s_k - s_j s_k^2 \right) \right\rangle$$

$$+ \frac{1}{2} \left\langle \sum_{j \in L_k} \left( s_k + s_j + s_k^2 + s_j^2 - s_k s_j^2 - s_k^2 s_j - 2 s_k s_j^2 \right) \right\rangle$$

$$\times \tanh \left\{ \frac{\beta}{2} \left[ J \left( \sum_{L_k \neq j} s_{L_k} - \sum_{L_j \neq k} s_{L_j} \right) + K \left( \sum_{L_k \neq j} s_{L_k}^2 - \sum_{L_j \neq k} s_{L_j}^2 \right) \right] \right\}$$

$$+ \frac{1}{2} \left\langle \sum_{j \in L_k} \left( -s_k - s_j + s_k^2 + s_j^2 + s_k s_j^2 + s_j^2 s_k - 2 s_k^2 s_j \right) \right\rangle$$

$$\times \tanh \left\{ \frac{\beta}{2} \left[ J \left( \sum_{L_k \neq j} s_{L_k} - \sum_{L_j \neq k} s_{L_j} \right) - K \left( \sum_{L_k \neq j} s_{L_k}^2 - \sum_{L_j \neq k} s_{L_j}^2 \right) \right] \right\}.$$

(5.59)

At this stage of our calculation, the evolution equation is exact within the framework of our model. However, as we discussed earlier, the equation is analytically and numerically intractable. As usual, we invoke the MF approximation to obtain the following closed equation for the evolution of $\langle s_k \rangle$:

$$\frac{\partial \langle s_k \rangle}{\partial t} = \sum_{j \in L_k} \left( \langle s_j \rangle - \langle s_k \rangle + \langle s_j^2 \rangle \langle s_k \rangle - \langle s_j \rangle \langle s_k^2 \rangle \right)$$

$$+ \frac{1}{2} \sum_{j \in L_k} \left( \langle s_k \rangle + \langle s_j \rangle + \langle s_k^2 \rangle + \langle s_j^2 \rangle - \langle s_k \rangle \langle s_j \rangle \langle s_k \rangle - \langle s_j \rangle \langle s_k \rangle - 2 \langle s_k \rangle \langle s_j \rangle \langle s_j \rangle \right).$$
\[ \times \tanh \left\{ \frac{\beta}{2} \left[ J \left( \sum_{L_k} \langle s_{L_k} \rangle - \sum_{L_j} \langle s_{L_j} \rangle \right) + K \left( \sum_{L_k} \langle s_{L_k}^2 \rangle - \sum_{L_j} \langle s_{L_j}^2 \rangle \right) \right] \right\} \]

\[ + \frac{1}{2} \sum_{j \in L_k} \left( -\langle s_k \rangle - \langle s_j \rangle + \langle s_k^2 \rangle + \langle s_j^2 \rangle + \langle s_k \rangle \langle s_j \rangle - 2 \langle s_k \rangle \langle s_j \rangle \right) \]

\[ \times \tanh \left\{ \frac{\beta}{2} \left[ J \left( \sum_{L_k} \langle s_{L_k} \rangle - \sum_{L_j} \langle s_{L_j} \rangle \right) - K \left( \sum_{L_k} \langle s_{L_k}^2 \rangle - \sum_{L_j} \langle s_{L_j}^2 \rangle \right) \right] \right\}. \]

(5.60)

The evolution equation for the second moment \( \langle s_k^2 \rangle \) can be derived in the same way as that for \( \langle s_k \rangle \). From the master equation, we obtain

\[ 2\tau \frac{\partial \langle s_k^2 \rangle}{\partial t} = -\sum_{\{s_i\}} \sum_{j \in L_i} \sum_{i = 1}^{N} s_k^2 (1 + s_i s_j) \left\{ 1 - (s_i - s_j) \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_i \neq j} s_{L_i} - \sum_{L_j \neq i} s_{L_j} \right) \right] \right\}

\[ + \frac{\beta K}{2} (s_i + s_j) \left( \sum_{L_i \neq j} s_{L_i}^2 - \sum_{L_j \neq i} s_{L_j}^2 \right) \right\} P(\{s_1, ..., s_i, s_j, ..., s_N\}; t) \]

\[ + \sum_{\{s_i\}} \sum_{j \in L_i} \sum_{i = 1}^{N} s_k^2 (1 + s_i s_j) \left\{ 1 - (s_j - s_i) \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_i \neq j} s_{L_i} - \sum_{L_j \neq i} s_{L_j} \right) \right] \right\}

\[ + \frac{\beta K}{2} (s_i + s_j) \left( \sum_{L_i \neq j} s_{L_i}^2 - \sum_{L_j \neq i} s_{L_j}^2 \right) \right\} \right\} P(\{s_1, ..., s_j, s_i, ..., s_N\}; t). \]

(5.61)

Exchanging \( i \) and \( j \) in the second sum on the RHS and canceling terms between the first and second sums, we obtain (absorbing \( \tau \) into \( t \))

\[ \frac{\partial \langle s_k^2 \rangle}{\partial t} = \sum_{\{s_i\}} \sum_{j \in L_k} \left\{ s_j^2 - s_k^2 \right\}

\[ \times \tanh \left[ \frac{\beta J}{2} \left( \sum_{L_k \neq j} s_{L_k} - \sum_{L_j \neq k} s_{L_j} \right) + \frac{\beta K}{2} (s_k + s_j) \left( \sum_{L_k \neq j} s_{L_k}^2 - \sum_{L_j \neq k} s_{L_j}^2 \right) \right] \right\}

\times P(\{s_1, ..., s_i, s_j, ..., s_N\}; t). \]

(5.62)

Using Eq. (5.58), and after invoking the MF approximation, we arrive at the following equation for the evolution of \( \langle s_k^2 \rangle \) :

\[ \frac{\partial \langle s_k^2 \rangle}{\partial t} = \sum_{j \in L_k} (\langle s_j^2 \rangle - \langle s_k^2 \rangle) \]
As in the nonconserved case, we have explicitly confirmed that the RHS of Eqs. (5.60) and (5.63) becomes identically zero on substituting the static solution from Eq. (5.22). For the sake of brevity, we do not present these calculations here.

### 5.4 Summary and Conclusion

Let us conclude this chapter with a summary and discussion of the results presented here. We have developed a general formalism to obtain coarse-grained phenomenological models equivalent to kinetic spin-1 models. This formalism employs the master-equation methodology and provides a simple and mechanical way of obtaining a reasonable macroscopic model for a given physical problem. The resultant mean-field (MF) dynamical models have several advantages. Firstly, they are continuum models and therefore analytically more tractable than the corresponding microscopic models. Secondly, these models are better able to access asymptotic behaviors of the systems we are interested in.

More specifically, we have obtained MF dynamical models for spin-1 systems with (a) nonconserved kinetics, as considered in Chapter 3; and (b) conserved kinetics, as considered in Chapter 4. In the nonconserved problem, our analysis of the MF models demonstrates the existence of a natural time-scale associated with local barriers. This time-scale is identical to that identified heuristically from our MC simulations in Chapter 3. For the conserved problem, we are presently studying the MF models in greater detail.
and will present our results elsewhere.

Finally, we should point out that the MF models presented in this chapter provide a convenient alternative for numerical simulations of the problems discussed in Chapters 3 and 4. Our simulations of MF models for the nonconserved and conserved cases yield results, which are similar to those, presented in Chapters 3 and 4, respectively. For brevity, we do not duplicate these results here.