Chapter 5

MATHEMATICAL PROBLEMS IN RC ELECTRIC CIRCUIT: ADOMIAN DECOMPOSITION METHOD

The content of this chapter is published in the following paper.

5.1 Introduction

A resistor capacitor circuit (RC circuit) or RC network, is an electric circuit composed of resistor and capacitor driven by a voltage source. A first order RC circuit is composed of one resistor and one capacitor and is the simplest type of RC circuit. A first order RC circuit is one of the simplest analogues infinite impulse response electronic filters. It consists of a resistor and capacitor connected in series driven by voltage source. These circuits exhibit important types of behavior that are fundamental to analogue electronics. In particular, they are able to act as passive filters. This chapter presents the RC circuit. Depending on whether the reactive element $C$ is in series with the load will dictate whether the filter is low pass or high pass. If a current $I$ flow through RC circuit then the voltage $V$ across the capacitor is $Q/C$ and voltage across resistor is $RI$.

A series RL circuit expressed by charge equation, is a second order linear differential equation. The solution of second order linear differential equation is obtained by Adomian decomposition method.

The real strength of ADM is demonstrated by solving series LC circuit with linear $L$ and nonlinear $C$. The solution is non-sinusoidal
(nonlinear) periodic steady state. The solution is without any assumptions of number of harmonics as in classical harmonic balance techniques.

In this chapter, we study the first order, second order electric circuits of linear and nonlinear type with the help of Adomian decomposition method. It is organised as follows: Section 5.2 consists the study of first order circuits (RC circuits) which are governed by first order differential equations and solved by Adomian decomposition method. In section 5.3, second order circuit and in section 5.4, nonlinear second order circuit are studied. Adomian decomposition method is successfully applied to solve them and concluding remark is given in the last section.

5.2 RC Electric Circuits

It is well known that the study of RC electric circuits \([14, 24, 33]\) is governed by first and second linear and nonlinear differential equations with suitable initial conditions. Applying Adomian decomposition method \([1, 11]\), we successfully discussed the behavior of charge and current in the following results.
Proposition 5.2.1. If an electric circuit having resistance $R$ ohms, capacitor $C$ farad and voltage or electromotive force $V$ in volts. Then the behavior of the charge $Q(t)$ is expressed as

$$Q(t) = CV \left[1 - e^{-\frac{t}{RC}}\right]$$

Figure 5.1: R-C Circuit

Formation of mathematical problem: By Kirchhoff’s Voltage Law, we get differential equations

$$Ri + \frac{Q}{C} = V$$

$$R \frac{dQ}{dt} + \frac{Q}{C} = V$$

$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{V}{R}$$

(5.2.1)

with initial condition

$$Q(0) = 0$$

(5.2.2)
which is IVP for linear nonhomogeneous first order differential equation with constant coefficients. The equation (5.2.1) can be written as
\[
\frac{dQ}{dt} = \frac{V}{R} - \frac{1}{RC}Q
\]
The above equation can be written in operator form
\[
\mathbb{L}Q = \frac{V}{R} - \frac{1}{RC}Q \tag{5.2.3}
\]
where \(\mathbb{L} = \frac{d}{dt}\) is differential operator. We look for series solution
\[
Q(t) = \sum_{n=0}^{\infty} Q_n = Q_0 + Q_1 + Q_2 + Q_3 + \ldots \tag{5.2.4}
\]
By applying the inverse operator \(\mathbb{L}^{-1} = \int_0^t (.) dt\) on both sides of the equation (5.2.3) and using initial condition, we have
\[
\mathbb{L}^{-1}\mathbb{L}Q = \mathbb{L}^{-1}\frac{V}{R} - \frac{1}{RC}\mathbb{L}^{-1}Q
\]
\[
Q(t) - Q(0) = \mathbb{L}^{-1}\frac{V}{R} - \frac{1}{RC}\mathbb{L}^{-1}Q
\]
\[
Q(t) = \frac{V}{R}t - \frac{1}{RC}\mathbb{L}^{-1}Q
\]
\[
\sum_{n=0}^{\infty} Q_n = \frac{V}{R}t - \frac{1}{RC}\mathbb{L}^{-1}\left(\sum_{n=0}^{\infty} Q_n\right)
\]
\[
Q_0 + Q_1 + Q_2 + Q_3 + \ldots = \frac{V}{R}t - \frac{1}{RC}\mathbb{L}^{-1}Q_0 - \frac{1}{RC}\mathbb{L}^{-1}Q_1 - \frac{1}{RC}\mathbb{L}^{-1}Q_2 - \ldots
\]
Now, we compare LHS and RHS terms by term and also proceed to
compute $Q_0, Q_1, Q_2, Q_3, ...$ with the help of the following recursive scheme.

\[
Q_0 = \frac{V}{R} t \\
Q_1 = -\frac{1}{RC} \int Q_0 \\
Q_2 = -\frac{1}{RC} \int Q_1 \\
Q_3 = -\frac{1}{RC} \int Q_2 \\
Q_4 = -\frac{1}{RC} \int Q_3 \\
\vdots
\]

We find $Q_1, Q_2, Q_3, ...$

\[
Q_1 = -\frac{1}{RC} \int Q_0 = -\frac{1}{RC} \int_0^t \left( \frac{V}{R} t \right) dt = -\frac{V}{R^2C} \frac{t^2}{2} \\
Q_2 = -\frac{1}{RC} \int Q_1 = -\frac{1}{RC} \int_0^t \left( -\frac{V}{R^2C} \frac{t^2}{2} \right) dt = -\frac{V}{R^3C^2} \frac{t^3}{6} \\
Q_3 = -\frac{1}{RC} \int Q_2 = -\frac{1}{RC} \int_0^t \left( -\frac{V}{R^3C^2} \frac{t^3}{6} \right) dt = -\frac{V}{R^4C^3} \frac{t^4}{24} \\
Q_4 = -\frac{1}{RC} \int Q_3 = -\frac{1}{RC} \int_0^t \left( -\frac{V}{R^4C^3} \frac{t^4}{24} \right) dt = -\frac{V}{R^5C^4} \frac{t^5}{120} \\
\vdots
\]
Substitute $Q_0, Q_1, Q_2, Q_3, Q_4, \ldots$ in (5.2.4), we have

$$Q(t) = \frac{V}{R} t - \frac{V}{R^2 C} \frac{t^2}{2} + \frac{V}{R^3 C^2} \frac{t^3}{6} - \frac{V}{R^4 C^3} \frac{t^4}{24} + \frac{V}{R^5 C^4} \frac{t^5}{120} - \ldots$$

$$= CV - CV + \frac{VC}{R C} t - \frac{VC}{R^2 C} \frac{t^2}{2} + \frac{VC}{R^3 C^2} \frac{t^3}{6} - \frac{VC}{R^4 C^3} \frac{t^4}{24} + \ldots$$

$$= CV - CV \left[ 1 - \frac{1}{RC} t + \frac{1}{R^2 C} \frac{t^2}{2} - \frac{1}{R^3 C^2} \frac{t^3}{3!} + \frac{1}{R^4 C^3} \frac{t^4}{4!} - \ldots \right]$$

$$= CV - CV e^{-\frac{t}{RC}}$$

$$Q(t) = CV \left[ 1 - e^{-\frac{t}{RC}} \right]$$

which is the solution of IVP (5.2.1) − (5.2.2).

![Figure 5.2: The graphical representation of solution of IVP (5.2.1) − (5.2.2).](image)

**Remark 5.2.1.** Under steady state condition circuit appear only as a capacitor which means it act as an open circuit with electromotive force or voltage $V$ while $Ve^{-\frac{t}{RC}}$ gives transient response of circuit.
5.2.1 Test Problem:

Now, we discuss an interesting engineering problem of practical interest.

Example 5.2.1. A condenser of capacity $C = 5 \times 10^{-5}$ farad is charged through a resistance $R = 200$ ohms by steady voltage $V = 2000$ volts. Calculate the current at the instant of closing the switch.

![Figure 5.3: RC-circuit.](image)

In this example, we have $R = 200$ ohms, $C = 5 \times 10^{-5}$ farad and $V = 2000$ volts.

**Formation of mathematical problem:** By Kirchhoff’s Voltage Law, we get the differential equations

\[
R \frac{dQ}{dt} + \frac{Q}{C} = V
\]
\[
R \frac{dQ}{dt} + \frac{1}{RC}Q = \frac{V}{R}
\]  

(5.2.5)
with initial condition

\[ Q(0) = 0 \quad (5.2.6) \]

which is IVP for linear non homogeneous first order differential equation.

\[ \frac{dQ}{dt} + \frac{1}{200 \times 5 \times 10^{-5}} Q = \frac{2000}{200} \]

\[ \frac{dQ}{dt} + 100Q = 10 \quad (5.2.7) \]

The above equation can be written in operator form

\[ \mathbb{L}Q = 10 - 100Q \quad (5.2.8) \]

where \( \mathbb{L} = \frac{d}{dt} \) is differential operator.

We look for series solution

\[ Q(t) = \sum_{n=0}^{\infty} Q_n = Q_0 + Q_1 + Q_2 + Q_3 + ... \quad (5.2.9) \]

By applying the inverse operator \( \mathbb{L}^{-1} = \int_0^t . dt \) on both sides of the equation \( (5.2.8) \) and using initial condition \( (5.2.6) \), we have

\[ \mathbb{L}^{-1}\mathbb{L}Q = \mathbb{L}^{-1}(10) - \mathbb{L}^{-1}(100Q) \]

\[ Q(t) - Q(0) = 10t - \mathbb{L}^{-1}(100Q) \]

\[ Q(t) = 10t - 100\mathbb{L}^{-1}Q \]
\[ \sum_{n=0}^{\infty} Q_n = 10t - 100L^{-1}\left(\sum_{n=0}^{\infty} Q_n\right) \]

\[ Q_0 + Q_1 + Q_2 + Q_3 + ... = 10t - 100 \int_0^t Q_0 dt - 100 \int_0^t Q_1 dt - ... \]

Now we compare LHS and RHS terms by term and also proceed to compute \( Q_0, Q_1, Q_2, Q_3, ... \) with the help of the following recursive scheme.

\[ Q_0 = 10t \]
\[ Q_1 = -100 \int_0^t Q_0 dt \]
\[ Q_2 = -100 \int_0^t Q_1 dt \]
\[ Q_3 = -100 \int_0^t Q_2 dt \]
\[ \vdots \]
\[ \vdots \]

we find \( Q_1, Q_2, Q_3, ... \)

\[ Q_1 = -100 \int_0^t Q_0 dt \]
\[ = -100 \int_0^t (10t) dt \]
\[ = -1000 \frac{t^2}{2} \]
\[ Q_2 = -100 \int_0^t Q_1 \, dt = -100 \int_0^t \left( -1000 \frac{t^2}{2} \right) \, dt = -100000 \frac{t^3}{6} \]

\[ Q_3 = -100 \int_0^t Q_2 \, dt = -100 \int_0^t \left( -100000 \frac{t^3}{6} \right) \, dt = -10000000 \frac{t^4}{24} \]

Substitute \( Q_0, Q_1, Q_2, Q_3 \) in equation (5.2.9),

\[ Q(t) = Q_0 + Q_1 + Q_2 + Q_3 + \ldots \]

\[ = 10t - 1000 \frac{t^2}{2} + 100000 \frac{t^3}{6} - 10000000 \frac{t^4}{24} + \ldots \]

\[ = \frac{10 \times 10}{10} t - \frac{1000 \times 10 \frac{t^2}{2}}{10} + \frac{100000 \times 10 \frac{t^3}{6}}{10} - \frac{10000000 \times 10 \frac{t^4}{24}}{10} + \ldots \]

\[ = \frac{1}{10} - \frac{1}{10} + \frac{10 \times 10}{10} t - \frac{1000 \times 10 \frac{t^2}{2}}{10} + \frac{100000 \times 10 \frac{t^3}{6}}{10} - \ldots \]

\[ = \frac{1}{10} - \frac{1}{10} \left[ 1 - 100t + 1000 \frac{t^2}{2} - 100000 \frac{t^3}{6} + 10000000 \frac{t^4}{24} + \ldots \right] \]

\[ = \frac{1}{10} - \frac{1}{10} \left[ 1 - 100t + (100)^2 \frac{t^2}{2} - (100)^3 \frac{t^3}{6} + (100)^4 \frac{t^4}{24} + \ldots \right] \]

\[ Q(t) = \frac{1}{10} - \frac{1}{10} e^{-100t} \]

differentiate with respect to \( t \)

\[ I = \frac{dQ}{dt} = 0 - \frac{1}{10} e^{-100t} \times (-100) \]

\[ I = 10 e^{-100t} \]
current at closing the switch i.e at $t = 0$ seconds,

$$I = 10e^{-100\times 0} = 10e^0 = 10 \text{ amp}$$

$I = 10 \text{ amp}$ which is the solution of IVP $(5.2.5) - (5.2.6)$.

![Graphical representation of solution of $(5.2.5) - (5.2.6)$](image)

Figure 5.4: The graphical representation of solution of $(5.2.5) - (5.2.6)$.

### 5.3 Second Order Linear Electric Circuit:

We know that, second order electric circuits are governed by second order differential equation. In this section we solve second order linear differential equation by Adomian decomposition method [1].

**Proposition 5.3.1.** If a electric circuit having resistance $R$ ohms, inductance $L$ henries and voltage or electromotive force $E$ in volts then discuss the behavior of charge.
Figure 5.5: Series RL-circuit.

**Formation of mathematical problem:** We defining the relation between charge $q$ and current $I$ is $\frac{dI}{dt} = \frac{dq}{dt^2}$ and $I = \frac{dq}{dt}$. According to Kirchhoff’s Voltage Law, the charge $Q$ satisfies the differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} = E \quad (5.3.1)$$

with initial conditions

$$Q(0) = 0, \quad Q'(0) = 0 \quad (5.3.2)$$

which is the IVP of second order linear differential equation, and equation (5.3.1) can be written in operator form

$$\mathbb{L}Q = \frac{E}{L} - \frac{R}{L}\mathbb{R}Q \quad (5.3.3)$$

where $\mathbb{L} = \frac{d^2}{dt^2}$ and $\mathbb{R} = \frac{d}{dt}$ be differential operators. We look for series solution
\[ Q(t) = \sum_{n=0}^{\infty} Q_n = Q_0 + Q_1 + Q_2 + Q_3 + \ldots \quad (5.3.4) \]

By applying the inverse operator \( \mathbb{L}^{-1} = \int_0^t \int_0^t \cdot \cdot \cdot dt \cdot dt \) on both sides of the equation (5.3.3) and using initial condition (5.3.2), we have

\[ \mathbb{L}^{-1} \mathbb{L} Q = \mathbb{L}^{-1} \left( \frac{E}{L} \right) - \frac{R}{L} \mathbb{L}^{-1} \mathbb{R} Q \]

\[ Q(t) - Q(0) - tQ'(0) = \frac{Et^2}{L^2} - \frac{R}{L} \mathbb{L}^{-1} \mathbb{R} Q \]

\[ Q(t) = \frac{Et^2}{L^2} - \frac{R}{L} \mathbb{L}^{-1} \mathbb{R} Q \]

\[ \sum_{n=0}^{\infty} Q_n = \frac{Et^2}{L^2} - \frac{R}{L} \int_0^t \left( \sum_{n=0}^{\infty} Q_n \right) dt \]

Now we compare LHS and RHS terms by term and also proceed to compute \( Q_0, Q_1, Q_2, Q_3, \ldots \) with the help of the following recursive scheme.

\[ Q_0 = \frac{Et^2}{L^2} \]

\[ Q_1 = -\frac{R}{L} \int_0^t Q_0 dt \]

\[ Q_2 = -\frac{R}{L} \int_0^t Q_1 dt \]

\[ Q_3 = -\frac{R}{L} \int_0^t Q_2 dt \]

\[ \vdots \]

\[ \vdots \]
we find $Q_1, Q_2, Q_3, Q_4, \ldots$

\[
Q_1 = -\frac{R}{L} \int_0^t Q_0 dt = -\frac{R}{L} \int_0^t \left( \frac{E t^2}{L} \right) dt = -\frac{R E t^3}{L^2} \frac{1}{6}
\]

\[
Q_2 = -\frac{R}{L} \int_0^t Q_1 dt = -\frac{R}{L} \int_0^t \left( -\frac{R E t^3}{L^2} \frac{1}{6} \right) dt = \frac{R^2 E t^4}{L^3} \frac{1}{24}
\]

\[
Q_3 = -\frac{R}{L} \int_0^t Q_2 dt = -\frac{R}{L} \int_0^t \left( \frac{R^2 E t^4}{L^3} \frac{1}{24} \right) dt = -\frac{R^3 E t^5}{L^4} \frac{1}{120}
\]

\[
Q_4 = -\frac{R}{L} \int_0^t Q_3 dt = -\frac{R}{L} \int_0^t \left( -\frac{R^3 E t^5}{L^4} \frac{1}{120} \right) dt = \frac{R^4 E t^6}{L^5} \frac{1}{720}
\]

\[
Q_5 = -\frac{R}{L} \int_0^t Q_4 dt = -\frac{R}{L} \int_0^t \left( \frac{R^4 E t^6}{L^5} \frac{1}{720} \right) dt = -\frac{R^5 E t^7}{L^6} \frac{1}{5040}
\]
Substitute \( Q_0, Q_1, Q_2, Q_3, Q_4, Q_5 \) in equation (5.3.4), we have

\[
Q(t) = \sum_{n=0}^{\infty} Q_n
\]

\[
= Q_0 + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + \ldots
\]

\[
= \frac{E t^2}{L^2} - \frac{R E t^3}{L^2} \frac{t^2}{6} + \frac{R^2 E t^4}{L^3} \frac{t^2}{24} - \frac{R^3 E t^5}{L^4} \frac{t^2}{120} + \frac{R^4 E t^6}{L^5} \frac{t^2}{720} - \frac{R^5 E t^7}{L^6} \frac{t^2}{5040} + \ldots
\]

\[
Q(t) = \frac{E L R^2 t^2}{R^2 L^2} - \frac{R E L R^2 t^3}{R^2 L^2} \frac{t^2}{6} + \frac{R^2 E L R^2 t^4}{R^2 L^3} \frac{t^2}{24} - \frac{R^3 E L R^2 t^5}{R^2 L^4} \frac{t^2}{120} + \frac{R^4 E L R^2 t^6}{R^2 L^5} \frac{t^2}{720} - \ldots
\]

\[
Q(t) = \frac{E L}{R^2} \left[ \frac{1}{R} \frac{t}{L} + \frac{R^2 t^2}{L^2} 2 - \frac{R^3 t^3}{L^3} 6 + \frac{R^4 t^4}{L^4} 24 - \frac{R^5 t^5}{L^5} 120 + \frac{R^6 t^6}{L^6} 720 - \ldots \right] - \frac{E L}{R^2} + \frac{E}{R}
\]

\[
Q(t) = \frac{E L}{R^2} e^{-\frac{t}{R}} - \frac{E L}{R^2} + \frac{E}{R}
\]

which is the solution of IVP (5.3.1) – (5.3.2).
Remark 5.3.1. Charge flowing in series $RL$ circuit during transient state has two components-linear with time due to resistance and exponential with time due to inductance. Inductor has property to oppose the change of magnetic flux and hence the change of charge. At initial condition charge is zero and rate of change of charge with time (current) is zero. With time the current builds up in an inductor to constant value and hence rate of charge is constant with time, thereafter decided by series resistance.

5.3.1 Test Problem:

Now we solve interesting problem of practical interest.

Example 5.3.1. A circuit having resistance is 15 ohms and an inductance is 3 henry and electromotive force 120 volts. Find the charge
In this example, we have $R = 15$ ohms, $L = 3$ henry and $E = 120$ volts.

**Formation of mathematical problem:** By Kirchhoff’s Voltage Law, we get differential equation

$$3 \frac{d^2 Q}{dt^2} + 15 \frac{dQ}{dt} = 120$$

$$\frac{d^2 Q}{dt^2} + 5 \frac{dQ}{dt} = 40 \quad (5.3.5)$$

with initial conditions

$$Q(0) = 0, \quad Q'(0) = 0 \quad (5.3.6)$$

The equation (5.3.5) can be written in operator form
\[ LQ = 40 - 5 \frac{dQ}{dt} \]

\[ LQ = 40 - 5RQ \quad (5.3.7) \]

where \( L = \frac{d^2}{dt^2} \) and \( R = \frac{d}{dt} \) be differential operators. We look for series solution

\[ Q(t) = \sum_{n=0}^{\infty} Q_n = Q_0 + Q_1 + Q_2 + Q_3 + \ldots \quad (5.3.8) \]

By applying the inverse operator \( L^{-1} = \int_0^t \int_0^t (.) (.) dt dt \) on both sides of the equation (5.3.7) and using initial condition (5.3.6), we have

\[ L^{-1}LQ = L^{-1}(40) - 5L^{-1}RQ \]

\[ Q(t) - Q(0) - tQ'(0) = 40 \frac{t^2}{2} - 5L^{-1}RQ \]

\[ \sum_{n=0}^{\infty} Q_n = 40 \frac{t^2}{2} - 5 \int_0^t \left( \sum_{n=0}^{\infty} Q_n \right) dt \]

Now we compare LHS and RHS terms by term and also proceed to compute \( Q_0, Q_1, Q_2, Q_3, \ldots \) with the help of the following recursive scheme.

\[ Q_0 = 40 \frac{t^2}{2} \]

\[ Q_1 = -5 \int_0^t Q_0 dt \]
\[ Q_2 = -5 \int_0^t Q_1 dt \]
\[ Q_3 = -5 \int_0^t Q_2 dt \]
\[ Q_4 = -5 \int_0^t Q_3 dt, \quad \text{and so on.} \]

We find \( Q_1, Q_2, Q_3, Q_4, \ldots \)

\[ Q_1 = -5 \int_0^t Q_0 dt 
= -5 \int_0^t (40 \frac{t^2}{2}) dt 
= -200 \frac{t^3}{6} \]

\[ Q_2 = -5 \int_0^t Q_1 dt 
= -5 \int_0^t (-200 \frac{t^3}{6}) dt 
= 1000 \frac{t^4}{24} \]

\[ Q_3 = -5 \int_0^t Q_2 dt 
= -5 \int_0^t (1000 \frac{t^4}{24}) dt 
= -5000 \frac{t^5}{120} \]

\[ Q_4 = -5 \int_0^t Q_3 dt 
= -5 \int_0^t (-5000 \frac{t^5}{120}) dt 
= 25000 \frac{t^6}{720} \]
Substitute $Q_0, Q_1, Q_2, Q_3, Q_4, ...$ in (5.3.8), we have

$$Q(t) = \sum_{n=0}^{\infty} Q_n$$

$$= Q_0 + Q_1 + Q_2 + Q_3 + Q_4 + ...$$

$$= 40 \frac{t^2}{2} - 200 \frac{t^3}{6} + 1000 \frac{t^4}{24} - 5000 \frac{t^5}{120} + 25000 \frac{t^6}{720} - ...$$

$$Q(t) = 8 \frac{5}{5} - 8t + 40 \frac{t^2}{2} - 200 \frac{t^3}{6} + 1000 \frac{t^4}{24} - 5000 \frac{t^5}{120} + 25000 \frac{t^6}{720} - ... - \frac{8}{5} + 8t$$

$$Q(t) = \frac{8}{5} \left[ 1 - 5t + 25 \frac{t^2}{2} - 125 \frac{t^3}{6} + ... \right] - \frac{8}{5} + 8t$$

$$Q(t) = \frac{8}{5} e^{-5t} - \frac{8}{5} + 8t$$

$$Q(t) = \frac{8}{5} (e^{-5t} - 1) + 8t$$

which is the solution of IVP (5.3.5) – (5.3.6).

Figure 5.8: The graphical representation the solution of IVP (5.3.5) – (5.3.6).
5.4 Second Order Nonlinear Electric Circuit

In this section we study second order nonlinear differential equation governed by \( LC \) circuit. Consider a capacitor of two terminals as a dipole in which a functional relationship between the electric charge, the voltage and the time has the following form [10]:

\[
f(Q, V, t) = 0
\]

A nonlinear capacitor is said to be controlled by charge when it is possible to express the tension as a function of charge:

\[
V = V(Q)
\]

Such phenomena’s are studied in the following results.

**Proposition 5.4.1.** If a \( LC \) series circuit having linear inductance \( L \) and nonlinear capacitance \( C \), the voltage across the loop can be expressed as voltage across the inductor plus voltage across the nonlinear capacitor. Then the charge \( Q(t) \) can be expressed as

\[
Q(t) = k\cos\sqrt{\alpha t} + \frac{\beta k^3}{\alpha} \cos\sqrt{\alpha t} - \frac{\beta k^3}{\alpha} + 3\alpha\beta k^3 t^4 \frac{4!}{4!} - 24\alpha^2 \beta k^3 t^3 \frac{6!}{6!} + ...
\]
Formation of Mathematical Problem: The $LC$ circuit consists of a linear inductor in a series with a nonlinear capacitor. The relationship between the charge of the nonlinear capacitor and the voltage drop across it may be approximated by the cubic equation

\[ V_c = sQ + aQ^3 \]

where $V_c$ is the potential across the plates of the nonlinear capacitor, $Q$ is the charge, and $s$ and $a$ are constants. The equation of the circuit can be written as

\[ L \frac{dI}{dt} + sQ + aQ^3 = 0 \]

Dividing by $L$ and we obtain the cubic Duffing equation in the form

\[ \frac{d^2Q}{dt^2} + \alpha Q + \beta Q^3 = 0 \quad (5.4.1) \]

with initial conditions
\[ Q(0) = k \quad \text{and} \quad Q'(0) = 0 \]  \hspace{1cm} (5.4.2)

where \( \alpha = s/L = \frac{1}{LC} \) and \( \beta = a/L = \frac{1}{LCq_0} \) constants.

The equation (5.4.1) can be written in operator form

\[ \mathbb{L}Q = -\alpha Q - \beta Q^3 \]  \hspace{1cm} (5.4.3)

where \( \mathbb{L} = \frac{d^2}{dt^2} \) is differential operator. We look for series solution

\[ Q(t) = \sum_{n=0}^{\infty} Q_n. \]  \hspace{1cm} (5.4.4)

By applying the inverse operator

\[ \mathbb{L}^{-1} = \int_0^t \int_0^t (...)dtdt \]

on both sides of equation (5.4.3) by using initial condition (5.4.2), we have

\[ \mathbb{L}^{-1}\mathbb{L}Q = -\alpha \mathbb{L}^{-1}Q - \beta \mathbb{L}^{-1}Q^3 \]

\[ Q(t) - Q(0) - tQ'(0) = -\alpha \mathbb{L}^{-1}Q - \beta \mathbb{L}^{-1}Q^3 \]

\[ Q(t) = k - \alpha \mathbb{L}^{-1}Q - \beta \mathbb{L}^{-1}Q^3 \]

\[ \sum_{n=0}^{\infty} Q_n = k - \alpha \mathbb{L}^{-1}\left( \sum_{n=0}^{\infty} Q_n \right) - \beta \mathbb{L}^{-1}\left( \sum_{n=0}^{\infty} A_n \right) \]
Now we compare LHS and RHS term by term and also proceed to compute $Q_0$, $Q_1$, $Q_2$, $Q_3$,... with the help of recursive scheme

\[
Q_0 = k
\]
\[
Q_1 = -\alpha L^{-1}Q_0 - \beta L^{-1}A_0
\]
\[
Q_2 = -\alpha L^{-1}Q_1 - \beta L^{-1}A_1
\]
\[
Q_3 = -\alpha L^{-1}Q_2 - \beta L^{-1}A_2
\]
\[
. 
\]
\[
. 
\]
\[
. 
\]

We find the Adomian polynomials $A_0$, $A_1$, $A_2$, $A_3$,....

\[
A_0 = Q_0^3
\]
\[
A_1 = 3Q_0^2Q_1
\]
\[
A_2 = 3Q_0^2Q_2 + 3Q_0Q_1^2
\]
\[
A_3 = 3Q_0^2Q_3 + 6Q_0Q_1Q_2 + Q_1^3, \text{ and so on.}
\]

We find

\[
Q_0 = k
\]
\[
Q_1 = -\alpha L^{-1}Q_0 - \beta L^{-1}A_0
\]
\[
= -\alpha \int_0^t \int_0^t (k)dtdt - \beta \int_0^t \int_0^t (k^3)dtdt
\]
\[
= -\alpha k \frac{t^2}{2} - \beta k^3 \frac{t^2}{2}
\]
\[ Q_2 = -\alpha L^{-1}Q_1 - \beta L^{-1}A_1 \]
\[ = -\alpha \int_0^t \int_0^t \left[ -\alpha k^2 t^2 - \beta k^3 t^2 \right] \, dt \, dt - \beta \int_0^t \int_0^t \left[ -\alpha k^2 t^2 - \beta k^3 t^2 \right] \, dt \, dt \]
\[ = \alpha^2 k^4 t^4/4! + \alpha \beta k^3 t^4/4! - 3\beta k^2 t_0^t \int_0^t \int_0^t \left[ -\alpha k^2 t^2 - \beta k^3 t^2 \right] \, dt \, dt \]
\[ = \alpha^2 k^4 t^4/4! + \alpha \beta k^3 t^4/4! - 3\beta k^2 \left( -\alpha k^4 t^4/4! - \beta k^3 t^4/4! \right) \]
\[ = \alpha^2 k^4 t^4/4! + \alpha \beta k^3 t^4/4! + 3\alpha k^3 t^4/4! + 3\beta^2 k^5 t^4/4! \]
\[ Q_2 = \alpha^2 k^4 t^4/4! + 4\alpha \beta k^3 t^4/4! + 3\beta^2 k^5 t^4/4! \]

\[ Q_3 = -\alpha L^{-1}Q_2 - \beta L^{-1}A_2 \]
\[ = -\alpha \int_0^t \int_0^t \left[ \alpha^2 k^4 t^4/4! + 4\alpha \beta k^3 t^4/4! + 3\beta^2 k^5 t^4/4! \right] \, dt \, dt - \beta \int_0^t \int_0^t \left[ 3k^2 q_2 + 3k^2 q_1^2 \right] \, dt \, dt \]
\[ = \left[ -\alpha^3 k^6 t^6/6! - 4\alpha^2 \beta k^3 t^6/6! - 3\alpha \beta^2 k^5 t^6/6! \right] - \beta \int_0^t \int_0^t \left[ 3k^2 \left( \alpha^2 k^4 t^4/4! + 4\alpha \beta k^3 t^4/4! + 3\beta^2 k^5 t^4/4! \right) + 3k \left( \alpha^2 k^2 + 2\alpha \beta k^4 + \beta^2 k^6 \right) \right] \, dt \, dt \]
\[ = -\alpha^3 k^6 t^6/6! - 4\alpha^2 \beta k^3 t^6/6! - 3\alpha \beta^2 k^5 t^6/6! - 3\alpha^2 \beta k^3 t^6/6! - 12\alpha \beta^2 k^5 t^6/6! - 9\beta^2 k^7 t^6/6! - 18\alpha^2 \beta k^3 t^6/6! - 36\alpha \beta^2 k^5 t^6/6! - 18\beta^3 k^7 t^6/6! \]
\[ = -\alpha^3 k^6 t^6/6! - 25\alpha^2 \beta k^3 t^6/6! - 51\alpha \beta^2 k^5 t^6/6! - 27\beta^3 k^7 t^6/6! \]

Substitute \( Q_0, Q_1, Q_2, Q_3, \ldots \) in equation (5.4.4), we get
\[ Q(t) = Q_0 + Q_1 + Q_2 + Q_3 + \ldots \]
\[ = k - \frac{\alpha k t^2}{2!} - \frac{\beta k^3 t^6}{6!} + \frac{\alpha^2 k^4 t^4}{4!} + 4\alpha\beta k^3 t^4 + \alpha^3 k^2 t^6 - \frac{\alpha^2 k^3 t^6}{6!} - \frac{25\alpha^2 \beta k^3 t^6}{6!} - \frac{51\alpha^2 k^5 t^6}{6!} - \frac{27\beta^3 k^7 t^6}{6!} + \ldots \]
\[ = k - \frac{\alpha k t^2}{2!} - \frac{\beta k^3 t^6}{6!} - \frac{\alpha^2 k^4 t^4}{4!} + 4\alpha\beta k^3 t^4 + 3\beta^2 k^5 t^4 + \frac{\alpha^3 k^2 t^6}{6!} - \frac{25\alpha^2 \beta k^3 t^6}{6!} - \frac{51\alpha^2 k^5 t^6}{6!} - \frac{27\beta^3 k^7 t^6}{6!} + \ldots \]
\[ = k \left[ 1 - \frac{\alpha t^2}{2!} + \frac{\alpha^2 t^4}{4!} - \frac{\alpha^3 t^6}{4!} + \ldots \right] + \frac{\beta k^3}{\alpha} \left[ 1 - \frac{\alpha t^2}{2!} + \frac{\alpha^2 t^4}{4!} - \frac{\alpha^3 t^6}{4!} + \ldots \right] - \frac{\beta k^3}{\alpha} + 3\alpha\beta k^3 t^4 + \frac{24\alpha^2 \beta k^3 t^6}{6!} - 24\alpha^2 \beta k^3 t^6 + \ldots \]

\[ Q(t) = k\cos\sqrt{\alpha t} + \frac{\beta k^3}{\alpha} \cos\sqrt{\alpha t} - \frac{\beta k^3}{\alpha} + 3\alpha\beta k^3 t^4 + \frac{24\alpha^2 \beta k^3 t^6}{6!} + \ldots \]

which is the solution of IVP \((5.4.1) - (5.4.2)\).

**Figure 5.10:** The graphical representation of solution of IVP \((5.4.1) - (5.4.2)\).

**Remark 5.4.1.** At the initial time, the current in the circuit is zero \((Q'(0) = 0)\) and charge is constant. All the energy is stored in the capacitor. Capacitor discharges and current builds up to maximum
when all the energy is stored in the inductor. Current starts decaying and capacitor starts charging in reverse direction till it is fully charged. All the energy is stored in the capacitor at this point, same cycle repeats but this time with opposite direction of current. Thus there are oscillations across capacitor and inductor for linear capacitor and linear inductor. The charge across time is pure sinusoidal for nonlinear capacitor, the charge across time is non-sinusoidal but periodic.

5.5 Concluding remark:

Adomian decomposition method provides robust solution for $RC$, $RL$ and $LC$ circuits with nonlinear and linear components without any assumptions on quality factor $Q$ or limitations of number of harmonics in case of periodic steady state problems. Some interesting engineering problems of practical interest are discussed and their complete solutions are obtained. The graphical representation of solution is also given using MATLAB.