Chapter-3

SOME MORE RESULTS ON THE LATTICE $K_1(S)$

Introduction

In this chapter, we discuss some special types of congruences on a topological semigroup $S$ and related results about the lattice $K_1(S)$ of semigroup compactifications of a topological semigroup $S$, also some results about atoms and dual atoms of $K_1(S)$ are obtained. These arose as a result of our attempt (though not successful) to obtain at least some partial converse of the theorem— for topological semigroups $S_1$ and $S_2$ if $(\beta_1,B_1)$ and $(\beta_2,B_2)$ are topologically isomorphic then the lattices $K_1(S_1)$ and $K_1(S_2)$ are isomorphic.

In Section 3.1, we prove that

(i) A topological semigroup $S$ with Bohr compactification $(\beta,B)$ has a semigroup compactification $(\alpha,A)$ determined by 'n' disjoint closed proper weak ideals (ideals) of $B$, at least one of which is non-singleton only if $S$ has a semigroup compactification strictly bigger than $(\alpha,A)$. 
(ii) A topological semigroup $S$ with Bohr compactification $(\beta, B)$ has an $n$-point compactification determined by \('n'\) non-empty subsets of $B$ does not imply that it has an $(n-1)$-point compactification, nor does it imply that there is a semigroup compactification strictly bigger than $(\alpha, A)$ different from $(\beta, B)$.

(iii) If a topological semigroup $S$ with $(\beta, B)$ has an $n$-point compactification $(\alpha, A)$ determined by \('n'\) non-empty weak ideals (ideals) of $B$, then there exists semigroup compactification strictly bigger than $(\alpha, A)$, but it does not imply that $S$ has an $(n-1)$-point compactification.

In Section 3.2 we describe the dual atoms and atoms of $K_1(S)$, when $B$ is finite.

3.1 Special types of congruences

In this section we introduce weak ideals, joint weak ideals and complementary joint ideals of a semigroup $S$ and describe special types of congruences on $S$.

Let $S$ be a semigroup and $\omega$ an ideal of $S$, then $(\omega \times \omega) \cup \Delta$ is a congruence on $S$ [HOW]. But converse need not be true. i.e., if $R$ is a congruence of the form $(\omega \times \omega) \cup \Delta$, then $\omega$ need not be an ideal of $S$. 
Example 3.1.1.

1) $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ with multiplication modulo 4 is a semigroup.

Here $\{1, 3\} \times \{1, 3\} \cup \triangle$ is a congruence on $\mathbb{Z}_4$, but $\{1, 3\}$ is not an ideal of $\mathbb{Z}_4$.

2) Let $T = \{o, e, f, g, x, y\}$ with usual matrix multiplication where

$$
o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$
f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$
x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here $\{f, g\} \times \{f, g\} \cup \triangle$ is a congruence, but $\{f, g\}$ is not an ideal.
In this situation we introduce the following definitions.

Definition 3.1.2.

A non-empty subset $\omega$ of a semigroup $S$ is said to be a

(i) weak right ideal of $S$

if either $ax, bx \in \omega$ or $ax = bx$ for all $a, b \in \omega$
and for all $x \in S$.

(ii) weak left ideal of $S$

if either $xa, xb \in \omega$ or $xa = xb$ for all $a, b \in \omega$
and for all $x \in S$.

(iii) weak ideal of $S$, if it is both weak right and weak left ideal of $S$

i.e., if either $ax, bx \in \omega$ or $ax = bx$ and either $xa, xb \in \omega$ or $xa = xb$
for all $a, b \in \omega$ and for all $x \in S$

Result 3.1.3

A topological semigroup $S$ has a non-trivial closed congruence of the form $\omega \times \omega \cup \Delta$ if and only if $\omega$ is a closed non-singleton proper weak ideal of $S$. 
Proof

Assume that $S$ has a non-trivial closed congruence of the form $\omega \times \omega \cup \Delta = R$ (say)

i.e., $\triangle \subseteq R \subseteq S \times S$

and for all $(a,b) \in R$ both $a,b \in \omega$ or $a = b$.

Since $R$ is non-trivial, there exist at least one $(a,b)$ such that $a \neq b \in \omega$.

i.e., $\omega$ is a non-singleton proper subset of $S$.

$\omega$ is a weak ideal. For,

since $R$ is a congruence,

both $(ax,bx), (xa,xb) \in R$ for all $a,b \in \omega$ and

for all $x \in S$.

i.e., either $ax, bx \in \omega$ or $ax=bx$

and either $xa, xb \in \omega$ or $xa = xb$

for all $a,b \in \omega$ and for all $x \in S$.

i.e., $\omega$ is a weak ideal.

Again $\omega$ is closed; for,

let $(x_\alpha)$ be a net in $\omega$, $(x_\alpha) \longrightarrow x \in S$.

Since $\omega$ is non-singleton, let $y (\neq x) \in \omega$. 
\( (x_\alpha, y) \) is a net in \( R \), which is closed.

\[ i.e., (x, y) \in \omega \times \omega \quad (\therefore x \neq y) \]

\[ \therefore x \in \omega. \]

On the other hand, consider \( \omega \) as a closed non-singleton proper weak ideal of \( S \), then clearly \( R = \omega \times \omega \cup \Delta \) is closed, since \( \Delta \) is closed in \( S \times S \) and \( \omega \) is closed in \( S \).

Clearly \( R \) is an equivalence.

Again \( R \) is compatible. For,

since \( \omega \) is a weak ideal both \( (ax, bx), (xa, xb) \in R \)

for all \( a, b \in \omega \) and for all \( x \in S \).

Clearly \( R \) is non-trivial, since \( \omega \) is a non-singleton proper subset of \( S \).

Hence the result.

Remark 3.1.4.

If \( B \) is the Bohr compactification of a topological semigroup \( S \), then \( B/(\omega \times \omega) \cup \Delta \) is called the semigroup compactification of \( S \) determined by \( \omega \). Thus \( S \) has a semigroup compactification defined by \( \omega \) if and only if \( \omega \) is a closed non-singleton proper weak ideal of \( B \).
Definition 3.1.5.

Two non-empty disjoint subsets \( \omega_1 \) and \( \omega_2 \) of a semigroup \( S \) are said to be

(i) joint weak right ideals if
   either \( ax, bx \in \omega_1 \) or \( ax, bx \in \omega_2 \) or \( ax = bx \)
   for all \( a, b \in \omega_1 \) or \( a, b \in \omega_2 \) and for all \( x \in S \).

(ii) joint weak left ideals if
     either \( xa, xb \in \omega_1 \) or \( xa, xb \in \omega_2 \) or \( xa = xb \)
     for all \( a, b \in \omega_1 \) or \( a, b \in \omega_2 \) and for all \( x \in S \).

(iii) joint weak ideals if they are both joint weak right and joint weak left ideals of \( S \).

i.e., either \( ax, bx \in \omega_1 \) or \( ax, bx \in \omega_2 \) or \( ax = bx \)
and either \( xa, xb \in \omega_1 \) or \( xa, xb \in \omega_2 \) or \( xa = xb \)
for all \( a, b \in \omega_1 \) or \( a, b \in \omega_2 \), and for all \( x \in S \).

Result 3.1.6

A topological semigroup \( S \) has a non-trivial closed congruence of the form \( \omega_1 \times \omega_1 \cup \omega_2 \times \omega_2 \cup \bigtriangleup \) (\( \cup \) indicates the sets whose union is taken, are disjoint) if and only if \( \omega_1 \) and \( \omega_2 \) are disjoint closed proper joint weak ideals, at least one of which is non-singleton.
Proof

Suppose that $S$ has a non-trivial closed congruence of the form

$$\omega_1 \times \omega_1 \cup \omega_2 \times \omega_2 \cup \triangle = R \text{ (say)}$$

$$\triangle \subseteq R \subseteq S \times S,$$ since $R$ is non-trivial

and for all $(a, b) \in R$, both $a, b \in \omega_1$ or both $a, b \in \omega_2$ or $a = b$.

Clearly $\omega_1$, $\omega_2$ are disjoint proper subsets of $S$ and at least one of them is non-singleton.

$\omega_1$, $\omega_2$ are joint weak ideals. For,

suppose first that $a, b \in \omega_1$.

Then both $(ax, bx), (xa, xb) \in R$ for all $x \in S$

(\because R is compatible)

i.e., either $ax, bx \in \omega_1$ or $ax, bx \in \omega_2$ or $ax = bx$

and either $xa, xb \in \omega_1$ or $xa, xb \in \omega_2$ or $xa = xb$ for all $x \in S$.

i.e., $\omega_1$, $\omega_2$ are joint weak ideals.

Similarly if for all $a, b \in \omega_2$ and for $a = b$.

Also they are closed. If $\omega_1$ is a singleton, then clearly it is closed. If not, we proceed as follows.
Let \((x_\alpha)\) be a net in \(\omega_1\), \((x_\alpha) \longrightarrow x \in S\).

Since \(\omega_1\) is non-singleton, let \(y(\neq x) \in \omega_1, (x_\alpha, y)\) be a net in \(\omega_1 \times \omega_1\).

\(\therefore \ (x_\alpha, y)\) be a net in \(\omega_1 \times \omega_1 \cup \omega_2 \times \omega_2 \cup \Delta = R\),

which is closed.

\(\therefore \) the limit \((x, y)\) of \((x_\alpha, y)\) belongs to \(R\)

\(\therefore \ (x, y) \in \omega_1 \times \omega_1\).

i.e., both \(x, y \in \omega_1\). \(\therefore x \in \omega_1\). Thus \(\omega_1\) is closed. Similarly \(\omega_2\) is closed.

Hence the result.

On the other hand, if \(\omega_1, \omega_2\) are disjoint closed proper joint weak ideals of \(S\), at least one of which is non-singleton, then \(\omega_1 \times \omega_1 \cup \omega_2 \times \omega_2 \cup \Delta = R\) is closed, since \(\omega_1, \omega_2\) are closed in \(S\) and \(\Delta\) is closed in \(S \times S\).

\(R\) is clearly reflexive and symmetric.

\(R\) is transitive. For,

\((a, b) \in R \text{ and } (b, c) \in R\)

imply either both \(a, b \in \omega_1\) or both \(a, b \in \omega_2\) or \(a=b\) and either both \(b, c \in \omega_1\) or both \(b, c \in \omega_2\) or \(b=c\).
Since \( \omega_1, \omega_2 \) are disjoint, the following cases are not possible.

1. \( a, b \in \omega_1 \) and \( b, c \in \omega_2 \)
2. \( a, b \in \omega_2 \) and \( b, c \in \omega_1 \)

All other cases imply \((a,c) \in R\)

R is compatible. For,

Since \( \omega_1, \omega_2 \) are joint weak ideals

either both \( ax, bx \in \omega_1 \) or both \( ax, bx \in \omega_2 \) or \( ax = bx \)

and either both \( xa, xb \in \omega_1 \) or both \( xa, xb \in \omega_2 \) or \( xa = xb \)

for all \((a, b) \in R \) and for all \( x \in S \)

I.e., both \((ax, bx)\) and \((xa, xb) \in R\)

Also R is non-trivial since at least one of \( \omega_1, \omega_2 \) is non-singleton.

Hence the result.

Remark 3.1

A topological semigroup \( S \) with Bohr compactification \((\beta, B)\) has a semigroup compactification "determined" by \( \{\omega_1, \omega_2\} \) if and only if \( \omega_1, \omega_2 \) are disjoint closed proper joint weak ideals of \( B \), at least one of which is non-singleton.
Definition 3.1.8.

A finite disjoint family \( \{ \omega_1, \omega_2, \ldots, \omega_n \} \) of \( S \) is said to be joint weak ideals if either both \( ax, bx \in \omega_1 \) or both \( ax, bx \in \omega_2 \) or \( \ldots \) or both \( ax, bx \in \omega_n \) or \( ax = bx \) and either both \( xa, xb \in \omega_1 \) or both \( xa, xb \in \omega_2 \) or \( \ldots \) both \( xa, xb \in \omega_n \) or \( xa = xb \), for all \( a, b \in \omega_1 \) or in \( \omega_2 \) or in \( \omega_3 \) or \( \ldots \) in \( \omega_n \) for all \( x \in S \).

By a similar argument as to that in result (3.1.6) we obtain the following.

Result 3.1.9.

A topological semigroup \( S \) has a non-trivial closed congruence of the form \( \bigcup \limits_{i=1}^{n} \omega_i x \omega_i \cup \Delta \leftrightarrow \omega_i \)'s are disjoint closed proper joint weak ideals of \( S \), at least one of which is non-singleton.

Definition 3.1.10.

Two non-empty subsets \( \omega_1 \) and \( \omega_2 \) of a semigroup \( S \) are said to be

(i) joint right ideals, if either \( ax, bx \in \omega_1 \) or \( ax, bx \in \omega_2 \) for all \( a, b \in \omega_1 \) or \( a, b \in \omega_2 \) and for all \( x \in S \).
(ii) joint left ideals, if either $xa, xb \in \omega_1$ or $xa, xb \in \omega_2$ for all $a, b \in \omega_1$ or $a, b \in \omega_2$ and for all $x \in S$.

(iii) joint ideals, if both joint right and joint left ideals.

i.e., either $ax, bx \in \omega_1$ or $ax, bx \in \omega_2$
and either $xa, xb \in \omega_1$ or $xa, xb \in \omega_2$ for all $a, b \in \omega_1$
or $a, b \in \omega_2$ and for all $x \in S$.

Definition 3.1.11 Complementary joint ideals.

Two joint ideals $\omega_1$ and $\omega_2$ of a semigroup $S$ are said to be complementary if they are disjoint and $\omega_1 \cup \omega_2 = S$.

Result 3.1.12.

A topological semigroup $S$ has a non-trivial closed congruence of the form $\omega_1 \times \omega_1 \cup \omega_2 \times \omega_2$ if and only if $\omega_1$ and $\omega_2$ are disjoint closed proper complementary joint ideals of $S$, at least one of which is non-singleton.

Suppose $S$ has a non trivial closed congruence of the form $\omega_1 \times \omega_1 \cup \omega_2 \times \omega_2 = R$ (say).

Then,

1) $\omega_1, \omega_2$ are proper subsets of $S$, at least one of which is non-singleton.
2) \( w_1, w_2 \) are complementary joint ideals for,

for all \( a \in S \) \( (a,a) \in w_1 \times w_1 \cup w_2 \times w_2 = R \)

(\( \therefore \) R is a congruence)

i.e., either \( a \in w_1 \) or \( w_2 \), for all \( a \in S \)

i.e., \( S = w_1 \cup w_2 \).

Clearly \( w_1, w_2 \) are disjoint (\( R \) being an equivalence)

Again \( w_1, w_2 \) are joint ideals for,

for all \( a, b \in w_1 \) or \( a, b \in w_2 \) and for all \( x \in S \)

both \( (ax, bx), (xa, xb) \in w_1 \times w_1 \cup w_2 \times w_2 \)

(\( \therefore \) R is compatible)

i.e., either both \( ax, bx \in w_1 \) or \( ax, bx \in w_2 \)

and either both \( xa, xb \in w_1 \) or \( xa, xb \in w_2 \)

\( \implies \) \( w_1, w_2 \) are joint ideals.

\( w_1, w_2 \) are closed. For, consider \( w_1 \).

If \( w_1 \) is a singleton, then clear. If \( w_1 \) is not a singleton, we proceed as follows.

Let \( (x_\alpha) \) be a net in \( w_1 \), \( (x_\alpha) \implies x \in S \).

Since \( w_1 \) is non-singleton, there exists \( y \in w_1, y \notin w_2 \).
\[ (x_\alpha, y) \text{ be a net in } \omega_1 \times \omega_1 \psi \omega_2 \times \omega_2, \]
which is closed.

\[ \text{the limit } (x, y) \text{ of } (x_\alpha, y) \text{ belongs to } \omega_1 \times \omega_1 \psi \omega_2 \times \omega_2. \]
i.e., both \((x, y) \in \omega_1 \times \omega_1 \psi \omega_1 \Rightarrow y \in \omega_1).\]

\[ x \in \omega_1. \text{ Thus } \omega_1 \text{ is closed.} \]

Similarly \(\omega_2\) is closed.

Hence the result.

On the other hand, if \(\omega_1, \omega_2\) are disjoint closed proper complementary joint ideals of \(S\), at least one of which is non-singleton, then \(R = \omega_1 \times \omega_1 \psi \omega_2 \times \omega_2\) is a closed non-trivial subset of \(S \times S\).

\(R\) is an equivalence for,

for all \(a \in S\), either \(a \in \omega_1\) or in \(\omega_2\)

\[ (\therefore \omega_1 \cup \omega_2 = B, \omega_1 \cap \omega_2 = \emptyset) \]

\[ \therefore (a, a) \in \omega_1 \times \omega_1 \psi \omega_2 \times \omega_2 \]
i.e., \(\Delta \subseteq \omega_1 \times \omega_1 \psi \omega_2 \times \omega_2\)
i.e., \(R\) is reflexive.

Clearly \(R\) is symmetric. Also \(R\) is transitive for,

let \((a, b), (b, c) \in R\).
i.e., either both \( a, b \in \omega_1 \) or both \( a, b \in \omega_2 \)

and either both \( b, c \in \omega_1 \) or both \( b, c \in \omega_2 \).

Since \( \omega_1, \omega_2 \) are disjoint, the possible cases are

\[
\begin{align*}
\text{a, b } & \in \omega_1, \quad \text{b, c } \in \omega_1 \\
\text{and a, b } & \in \omega_2, \quad \text{b, c } \in \omega_2
\end{align*}
\]

\[
\therefore \quad (a, c) \in \omega_1 \times \omega_1 \uplus \omega_2 \times \omega_2
\]

Again,

\( R \) is compatible.

For,

since \( \omega_1, \omega_2 \) are joint ideals, by the definition, we have either

\[
\begin{align*}
\text{ax, bx } & \in \omega_1 \quad \text{or ax, bx } \in \omega_2 \\
\text{and either}
\end{align*}
\]

\[
\begin{align*}
\text{xa, xb } & \in \omega_1 \quad \text{or xa, xb } \in \omega_2 \\
\text{for all a, b } & \in \omega_1 \quad \text{or a, b } \in \omega_2 \\
\text{and for all } x & \in S.
\end{align*}
\]

i.e., \( (ax, bx), (xa, xb) \in \omega_1 \times \omega_1 \uplus \omega_2 \times \omega_2 \).

Hence the result.
Remark 3.1.13.

A topological semigroup \( S \) with Bohr compactification \((\beta, B)\) has a semigroup compactification determined by a non-trivial closed congruence of the form \( \omega_1 \times \omega_1 \cup \omega_2 \times \omega_2 \) if and only if \( \omega_1 \) and \( \omega_2 \) are disjoint closed proper complementary joint ideals of \( B \), at least one of which is non-singleton.

Theorem 3.1.14.

A topological semigroup \( S \) has a non-trivial closed congruence of the form \( \bigcup_{i=1}^{n} \omega_i \times \omega_i \) if and only if \( \omega_i \)'s are disjoint closed proper complementary joint ideals of \( S \) (i.e., \( \omega_i \cap \omega_j = \emptyset \) for \( i \neq j \) and \( \bigcup_{i=1}^{n} \omega_i = S \)), at least one of which is non-singleton.

As a result we get

Theorem 3.1.15.

A topological semigroup \( S \) has an \( n \)-point compactification if and only if its Bohr compactification has \( n \) disjoint closed proper complementary joint ideals, at least one of which is non-singleton.
Remark 3.1.16.

A semigroup $S$ has a set $\{\omega_i\}_{i=1}^n$ of finite number of joint weak ideals does not imply that any of the $\omega_i$'s is a weak ideal, nor does it imply that a proper subset of $\{\omega_i\}_{i=1}^n$ forms joint weak ideals.

Example.

$\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with multiplication modulo 8 is a semigroup

(1) $\{\omega_1, \omega_2\} = \{\{1, 5\}, \{3, 7\}\}$ is a set of joint weak ideals, but neither $\{1, 5\}$ nor $\{3, 7\}$ is a weak ideal of $\mathbb{Z}_8$.

(2) $\{\omega_1, \omega_2, \omega_3\} = \{\{1, 7\}, \{2, 6\}, \{3, 5\}\}$ is a set of joint weak ideals, but $\{\{1, 7\}, \{3, 5\}\}$ is not a set of joint weak ideals. Also $\{1, 7\}, \{3, 5\}$ are not weak ideals.

Result 3.1.17

A semigroup $S$ has a congruence of the form

$\bigcup_{i=1}^n (\omega_i \times \omega_i) \cup \triangle$ does not imply

(1) $(\omega_i \times \omega_i) \cup \triangle$ is a congruence on $S$ for any $i=1, 2, \ldots, n$.

(2) $\bigcup_{j \in A} (\omega_j \times \omega_j) \cup \triangle$ is a congruence on $S$ for some proper subset $A$ of $\{1, 2, \ldots, n\}$.
Theorem 3.1.18.

A semigroup $S$ has a congruence of the form

$$\bigcup_{i=1}^{n} (\bigcup \omega_i x \omega_i) \cup \Delta,$$

with $\omega_i$'s weak ideals (ideals), then $S$ has a congruence of the form $$\bigcup_{j \in A} (\bigcup \omega_j x \omega_j) \cup \Delta,$$ where $A$ is any proper subset of $\{1, 2, \ldots, n\}$, contained in

$$\bigcup_{i=1}^{n} (\bigcup \omega_i x \omega_i) \cup \Delta.$$

Proof.

Given $$\bigcup_{i=1}^{n} (\bigcup \omega_i x \omega_i) \cup \Delta$$ is a congruence, with $\omega_i$'s weak ideals (ideals).

i.e., $\omega_i$'s are disjoint closed proper weak ideals (ideals)

$\therefore$ $$\bigcup (\omega_i x \omega_i) \cup \Delta$$ is a congruence for any $i=1, 2, \ldots, n$

Consider $$\{\omega_j\}_{j \in A},$$ where $A$ is any proper subset of $\{1, 2, \ldots, n\}$.

Then $$\bigcup_{j \in A} (\omega_j x \omega_j) \cup \Delta$$ is a congruence for each $j \in A$.

i.e., $$\bigcup_{j \in A} (\omega_j x \omega_j) \cup \Delta$$ is a congruence contained in $$(\bigcup_{i=1}^{n} \omega_i x \omega_i) \cup \Delta,$$

since $$\{\omega_j\}_{j \in A} \subset \{\omega_i\}_{i=1}^{n}$$

Hence the result.
Theorem 3.1.19

Let $S$ be a topological semigroup with closed
congruence $\bigcup_{i=1}^{n} \omega_{i} \times \omega_{i} \cup \triangle$, where $\omega_{i}$'s are either
ideals or weak ideals. Then $S$ has a closed congruence
of the form $\bigcup_{j \in A} \omega_{j} \times \omega_{j} \cup \triangle$ contained in $\bigcup_{i=1}^{n} \omega_{i} \times \omega_{i} \cup \triangle$,
where $A$ is any proper subset of $\{1,2,\ldots,n\}$.

Proof.

Given $\bigcup_{i=1}^{n} \omega_{i} \times \omega_{i} \cup \triangle$ is a closed congruence
with $\omega_{i}$'s are weak ideals (ideals).

i.e., $\omega_{i}$'s are closed disjoint proper weak ideals, at
least one of which is non-singleton [3.1.9]
i.e., each $\omega_{i} \times \omega_{i} \cup \triangle$ is a closed congruence
i.e., $R = \bigcup_{j \in A} \omega_{j} \times \omega_{j} \cup \triangle$, where $A$ is any proper
subset of $\{1,2,\ldots,n\}$ is a congruence contained in
$\bigcup_{i=1}^{n} \omega_{i} \times \omega_{i} \cup \triangle$ [3.1.18].

Also $R = \bigcup_{j \in A} \omega_{j} \times \omega_{j} \cup \triangle$ is closed, since $\omega_{i}$'s are
closed in $S$ and $\triangle$ is closed in $S \times S$. 
We obtained the following theorem about the lattice $K_1(S)$ of a given topological semigroup $S$ with Bohr compactification $(\beta,B)$.

**Theorem 3.1.20.**

Let $S$ be a topological semigroup with Bohr compactification $(\beta,B)$. If $S$ has a semigroup compactification $(\alpha,A)$ determined by 'n' disjoint closed proper weak ideals $\{\omega_i\}_{i=1}^n$ of $B$, at least one of which is non-singleton, then there is a semigroup compactification in $K_1(S)$ strictly bigger than $(\alpha,A)$.

**Proof.**

Since $\{\omega_i\}_{i=1}^n$ are disjoint closed proper weak ideals (ideals) of $B$, at least one of which is non-singleton, $B$ has a non-trivial closed congruence of the form

$$R = \left(\bigcup_{i=1}^n \omega_i \times \omega_j\right) \cup \Delta.$$ 

Let $(\alpha,A)$ denote the semigroup compactification determined by $R$. i.e., $(\beta,B) > (\alpha,A) \in K_1(S)$

Again since $\omega_i$'s are weak ideals (ideals) for each $i \in \{1,2,\ldots,n\}$, by theorem (3.1.19), $B$ has a
closed congruence of the form

\[ R' = \bigcup_{j \in I} \omega_j \times \omega_j \cup \Delta, \]

where \( I \) is any proper subset of \( \{1, 2, \ldots, n\} \) and \( R' \) is contained in \( R \).

Let \((\alpha_1, A_1)\) denotes semigroup compactification determined by \( R' \) and \((\beta, B) > (\alpha_1, A_1) > (\alpha, A)\).

Hence the result.

Remark 3.1.21.

Theorem (3.1.20) need not be true, if \((\alpha, A)\) is determined by \( \{\omega_i\}_{i=1}^{n} \) closed disjoint proper weak ideals, at least one of which is non-singleton.

For example,

Let \( S \) be a topological semigroup with \((\beta, B)\), where \( B = \mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\} \) with discrete topology and multiplication modulo 8.

(1) Here \( \{\omega_1, \omega_2, \omega_3\} = \{\{1, 7\}, \{2, 6\}, \{3, 5\}\} \) set of joint weak ideals and \( \mathbb{Z}_8 \) has a congruence

\[ \{1, 7\} \times \{1, 7\} \cup \{2, 6\} \times \{2, 6\} \cup \{3, 5\} \times \{3, 5\} \cup \Delta \]

but \( \{1, 7\} \times \{1, 7\} \cup \{3, 5\} \times \{3, 5\} \cup \Delta \) are not congruences.
(2) \(\{\{1,5\}, \{3,7\}\}\) set of joint weak ideals and
\(\{1,5\} \times \{1,5\} \cup \{3,7\} \times \{3,7\} \cup \Delta\) is a congruence
but \(\{1,5\} \times \{1,5\} \cup \Delta\) and \(\{3,7\} \times \{3,7\} \cup \Delta\) are not congruences.

Remark 3.1.22.

A semigroup \(S\) has a set \(\{\omega_i\}_{i=1}^{n}\) of finite
number of complementary joint ideals does not imply that
any of the \(\omega_i\)'s is a weak ideal, nor does it imply that
a proper subset of \(\{\omega_i\}_{i=1}^{n}\) forms joint weak ideals, or
complementary joint ideals.

Example.

(1) Let \(S = \{e, a, f, b\}\) with multiplication defined
below is a semigroup

\[
\begin{array}{c|cccc}
  & e & a & f & b \\
\hline
  e & e & a & f & b \\
  a & a & e & b & f \\
  f & f & b & f & b \\
  b & b & f & b & f \\
\end{array}
\]

Here \(\{\omega_1, \omega_2\} = \{\{e, f\}, \{a, b\}\}\) is a set of complementary
joint ideals, but neither \(\{e, f\}\) nor \(\{a, b\}\) is a weak ideal.

(2) \(\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}\) with multiplication modulo 6
is a semigroup.
Here \( \{w_1, w_2, w_3\} = \{\{2,5\}, \{1,4\}, \{0,3\}\} \) is a set of complementary joint ideals but \( \{\{2,5\}, \{1,4\}\} \) are not sets of joint weak ideals, nor complementary joint ideals.

From the above remark we have the following result.

Result 3.1.23.

A semigroup \( S \) has a congruence of the form

\[
\bigcup_{i=1}^{n} w_i \times w_i \text{ does not imply } (1) \bigcup_{j \in A} \omega_j \times \omega_j \cup A \text{ is a congruence on } S, \text{ where } A \text{ is any proper subset of } \{1,2,\ldots,n\}.
\]

(2) \( \bigcup_{j \in A} \omega_j \times \omega_j \) is a congruence on \( S \), where \( A \) is any proper subset of \( \{1,2,\ldots,n\} \).

Theorem 3.1.24.

A semigroup \( S \) has a non-trivial congruence of the form

\[
\bigcup_{i=1}^{n} w_i \times w_i \text{ with } w_i \text{'s weak ideals (ideals) then } S \text{ has a congruence of the form } (\bigcup_{j \in A} \omega_j \times \omega_j) \cup A, \text{ where } A \text{ is any proper subset of } \{1,2,\ldots,n\}. \text{ But } \bigcup_{j \in A} \omega_j \times \omega_j \text{ is not a congruence.}
Proof

Given \( \bigcup_{i=1}^{n} \omega_i \times \omega_i \) is a non-trivial congruence with \( \omega_i \)'s weak ideals (ideals).

i.e., \( \omega_i \)'s are disjoint proper weak ideals for each \( i = 1, \ldots, n \), at least one of which is non-singleton.

\[ \therefore \ (\omega_i \times \omega_i) \cup \Delta \text{ is a congruence for each } i = 1, \ldots, n. \]

i.e., \( (\omega_j \times \omega_j) \cup \Delta \text{ is a congruence for each } j \in A \), where

\( A \) is any proper subset of \( 1, \ldots, n \).

\[ \therefore \ (\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta \text{ is a congruence contained in } \bigcup_{i=1}^{n} \omega_i \times \omega_i. \]

But \( \bigcup_{j \in A} \omega_j \times \omega_j \) is not a congruence, since it is not reflexive.

Theorem 3.1.25

Let \( S \) be a topological semigroup with non-trivial closed congruence \( \bigcup_{i=1}^{n} \omega_i \times \omega_i \), where \( \omega_i \)'s are weak ideals (ideals), then \( S \) has a closed congruence of the form \( (\bigcup_{j \in A} \omega_j \times \omega_j) \cup \Delta \) contained in \( \bigcup_{i=1}^{n} \omega_i \times \omega_i \), where \( A \) is any proper subset of \( \{1, \ldots, n\} \).
Proof

This is immediate from (3.1.12) and (3.1.24).

Result 3.1.26.

A topological semigroup $S$ with $(\beta, B)$ has an $n$-point compactification does not imply that it has an $(n-1)$-point compactification, nor does it imply that there is a semigroup compactification strictly bigger than $(\alpha, A)$ and different from $(\beta, B)$.

For example,

Let $S$ be a topological semigroup with $(\beta, B)$ where $B = \{e, a, f, b\}$ with discrete topology and multiplication defined below

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Closed congruences on $B$ are

\[
R_1 = \Delta
\]

\[
R_2 = \{f, b\} \times \{f, b\} \cup \Delta
\]

\[
R_3 = \{e, f\} \times \{e, f\} \cup \{a, b\} \times \{a, b\}
\]

\[
R_4 = \{e, a\} \times \{e, a\} \cup \{f, b\} \times \{f, b\}
\]

\[
R_5 = B \times B
\]
$R_3$ determines a 2-point compactification say $(α, A)$, where $\{\{e,f\}, \{a,b\}\}$ is a set of closed proper disjoint non-singleton complementary joint ideals but $\{e,f\} \times \{e,f\}$ and $\{a,b\} \times \{a,b\}$ are not closed congruences on $B$.

i.e., two point compactification does not imply the existence of one-point compactification.

Also,

$\{e,f\} \times \{e,f\} \cup \Delta$ and $\{a,b\} \times \{a,b\} \cup \Delta$ are not closed congruences contained in $R_3$. So 2-point compactification does not imply there exist a semigroup compactification strictly bigger than $(α, A)$ and different from $(β, B)$.

Next theorem shows that if a topological semigroup $S$ with $(β, B)$ has an $n$-point compactification $(α, A)$ determined by 'n' weak ideals (ideals) of $B$, then there exists semigroup compactification strictly bigger than $(α, A)$. And in this case also it does not imply that $S$ has an $(n-1)$-point compactification.
Theorem 3.1.27.

A topological semigroup $S$ with $(\beta, B)$ has an $n$-point compactification $(\alpha, A)$ determined by 'n' weak ideals (ideals) of $B$, then there exists semigroup compactification strictly bigger than $(\alpha, A)$. And in this case also it does not imply that $S$ has an $(n-1)$-point compactification.

Proof.

Since $(\alpha, A)$ is an $n$-point compactification of $S$, $(\alpha, A)$ is determined by a non-trivial closed congruence of the form $\bigcup_{i=1}^{n} \omega_i \times \omega_i$, where $\omega_i$'s closed proper complementary joint ideals of $B$, at least one of which is non-singleton. Also given that $\omega_i$'s are weak ideals (ideals).

i.e., $\omega_i$'s are closed disjoint proper weak ideals (ideals) of $B$, at least one of which is non-singleton.

By theorem (3.1.19) $B$ has a closed congruence of the form $\bigcup_{j \in A} (\omega_j \times \omega_j) \cup \Delta$, where $A$ is any proper subset of $\{1, 2, \ldots, n\}$.

Also it determines a semigroup compactification $(\alpha_1, A_1)$ such that $(\beta, B) \succ (\alpha_1, A_1) \succ (\alpha, A)$.

i.e., there is a semigroup compactification strictly bigger than $(\alpha, A)$.
But it does not imply that $S$ has an $(n-1)$-point compactification.

For example,

Let $S$ be a topological semigroup with $(\beta,B)$, where $B = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 1\}$ with discrete topology and multiplication defined by $xy = \max \{\frac{1}{2}, xy\}$

Here $S$ has a 2-point compactification determined by

$$R = \left\{ \frac{1}{2}, 1 \right\} \times \left\{ \frac{1}{2}, 1 \right\} \cup \left\{ \frac{1}{3}, \frac{1}{4} \right\} \times \left\{ \frac{1}{3}, \frac{1}{4} \right\}$$

where,

$$\left\{ \left\{ \frac{1}{2}, 1 \right\}, \left\{ \frac{1}{3}, \frac{1}{4} \right\} \right\}$$

is a set of disjoint proper closed non-singleton complementary joint ideals.

Also $\left\{ \frac{1}{2}, 1 \right\}$ and $\frac{1}{3}, \frac{1}{4}$ are weak ideals.

But $\left\{ \frac{1}{2}, 1 \right\} \times \left\{ \frac{1}{2}, 1 \right\}$ and $\left\{ \frac{1}{3}, \frac{1}{4} \right\} \times \left\{ \frac{1}{3}, \frac{1}{4} \right\}$ are not congruences.

.'. 2-point compactifications determined by weak ideals does not imply existence of one-point compactification.
3.2 Some results about atoms and dual atoms of $K_1(S)$

In this section, we describe the dual atoms and atoms of $K_1(S)$, family of all semigroup compactifications of a topological semigroup $S$ with $(\beta, B)$, where $B$ is finite.

An element $(\alpha, A) \in K_1(S)$ is a dual atom of $K_1(S)$ provided $(\alpha, A) < (\beta, B)$ and there does not exist $(\alpha_1, A_1) \in K_1(S)$ for which $(\alpha, A) < (\alpha_1, A_1) < (\beta, B)$.

An element $(\alpha_0, A_0) \in K_1(S)$ is an atom of $K_1(S)$ provided $(\alpha_0, A_0) > (\alpha, \{0\})$, where $(\alpha, \{0\})$ is the smallest semigroup compactification of $S$ and there does not exist $(\alpha_1, A_1) \in K_1(S)$ for which $(\alpha_0, A_0) > (\alpha_1, A_1) > (\alpha, \{0\})$.

Theorem 3.2.1.

Let $S$ be a topological semigroup with Bohr compactification $(\beta, B)$, where $B$ is finite, and $\omega^*$ be the collection of all weak ideals, joint weak ideals, complementary joint ideals of $B$. If there exists a closed non-singleton proper weak ideal $\omega$ minimal (maximal) in $\omega^*$, then $(\alpha, A)$ the semigroup compactification determined by $\omega$ is a dual atom (atom) of $K_1(S)$.

Proof.

Let $|B| = n$, where $n$ is finite and $\omega^*$ be the collection of all weak ideals, joint weak ideals, complementary joint ideals of $B$. 
(a) Let 'w' be a closed non-singleton proper weak ideal of B minimal in w*.

i.e., there exists no weak ideal, no joint weak ideals, no complementary joint ideals properly contained in ω and (ωxω) ⨈ Δ is a non-trivial closed congruence on B.

i.e., Δ ⊆ (ωxω) ⨈ Δ, and there exists no non-trivial closed congruence properly contained in (ωxω) ⨈ Δ.

If not, let R' be a non-trivial closed congruence properly contained in (ωxω) ⨈ Δ.

i.e., R' ⊆ (ωxω) ⨈ Δ ⊆ BxB

Since R' is a non-trivial closed congruence, R' is determined by at least one non-singleton subset A (say) of B; if not, let |A| = 1, R' determined by A is Δ, this is not possible since R' ≠ Δ. Then the possible cases of R' are the following:

Case-1

R' is determined by a subset A of B with

1 < |A| < n

If |A| = 2, i.e., A = {a, b} (say)

Then R' = {a, b} × {a, b} ⨈ Δ ⊆ (ωxω) ⨈ Δ
Since \( a \neq b \), \( \{a, b\} \subset \omega \)
and since \( R' \) is a congruence, for all \( a, b \in A \)
\( ax, bx \in A \) or \( ax = bx \)
and \( xa, xb \in A \) or \( xa = xb \) for all \( x \in B \).

i.e., \( A = \{a, b\} \) is a weak ideal, also we have
\( \{a, b\} \subset \omega \), which is a contradiction.

Similarly we have a contradiction if \( R' \) is
determined by any non-empty subset \( A \) of \( B \), with
\( 1 < |A| < n \).

Case-2

If \( R' \) is determined by two non-singleton subsets
say \( A_1 = \{a, b\} \), \( A_2 = \{c, d\} \)

i.e., \( \{a, b\} \times \{a, b\} \cup \{c, d\} \times \{c, d\} \cup \Delta \) is a closed
congruence contained in \( \omega \times \omega \cup \Delta \)

Since \( a \neq b \), \( c \neq d \), \( \{a, b, c, d\} \subset \omega \)
and since \( R' \) is a congruence, for all \( x \in B \)
and for all \( a, b \in \{a, b\} \) or in \( \{c, d\} \)
\( ax, bx \in \{a, b\} \) or \( ax, bx \in \{c, d\} \) or \( ax = bx \)
and \( xa, xb \in \{a, b\} \) or \( xa, xb \in \{c, d\} \) or \( xa = xb \)
i.e. \{\{a,b\}, \{c,d\}\} is a set of joint weak ideals contained in \(\omega\), which is a contradiction.

Similarly, we have a contradiction if \(R'\) is determined by any collection of subsets of \(B\), at least one of which is non-singleton.

**Case-3**

If \(R'\) is determined by any two non-singleton subsets \(A_1, A_2\) of \(B\) such that \(A_1 \cup A_2 = B\).

Let \(A_1 = \{a,b\}, A_2 = \{c,d\}\)

\[ R' = \{a,b\} \times \{a,b\} \cup \{c,d\} \times \{c,d\} \] is a closed congruence and

\[ \{a,b\} \times \{a,b\} \cup \{c,d\} \times \{c,d\} \subseteq \omega \times \omega \Delta \]

Since \(a \neq b, c \neq d\), \(\{a,b,c,d\}\) \(\subseteq \omega\)

Since \(R'\) is a congruence for all \(x \in B\)

and for all \(a,b \in A_1\) or in \(A_2\)

\[ ax, bx \in A_1 \text{ or } ax, bx \in A_2 \]

and \(xa, xb \in A_1\) or \(xa, xb \in A_2\)

i.e., \(\{A_1, A_2\}\) is a set of complementary joint ideals contained in \(\omega\), which is a contradiction.
Similarly we have a contradiction, if \( R' \) is determined by any disjoint collection of subsets \( B \), whose union is \( B \), at least one of which is non-singleton.

Thus in all these possible cases, there exists no non-trivial closed congruence properly contained in \((\omega \times \omega) \cup \Delta\).

\[
\therefore (\alpha, A) \text{ the semigroup compactification determined by } (\omega \times \omega) \cup \Delta \text{ is a dual atom of } K_1(S).
\]

(b) Let \( \omega \) be a closed non-singleton proper weak ideal of \( B \) maximal in \( \omega^* \).

\[
\therefore \Delta \subseteq (\omega \times \omega) \cup \Delta \subseteq B \times B \text{ is a closed congruence on } B \text{ and there exists no proper closed congruence properly contains } (\omega \times \omega) \cup \Delta.
\]

If not, let \( R' \) be a closed congruence properly contains \((\omega \times \omega) \cup \Delta\) i.e., \( \Delta \subseteq (\omega \times \omega) \cup \Delta \subseteq R' \subseteq B \times B \)

Since \( R' \) is non-trivial, the possible cases of \( R' \) are same as that in (a) and we have a contradiction

1. if \( R' \) is determined by any non-singleton subset \( A \) of \( B \) with \( 1 < |A| < n \).

2. if \( R' \) is determined by any disjoint collection of subsets of \( B \) at least one of which is non-singleton.
there exists no proper closed congruence properly contains\((\omega \times \omega) \cup \triangle\)

(\alpha, A) the semigroup compactification corresponding to \((\omega \times \omega) \cup \triangle\) is an atom of \(K_1(S)\).

By similar argument we have the following.

Remark-1

If \(\omega\) is a set of closed joint weak ideals of \(B\) at least one of which is non-singleton minimal (maximal) in \(\omega^*\), then \((\alpha, A)\) the semigroup compactification corresponding to \((\omega \times \omega) \cup \triangle\) is a dual atom (atom) of \(K_1(S)\).

Remark-2

If \(\omega\) is a set of closed complementary joint ideals of \(B\) at least one of which is non-singleton minimal (maximal) in \(\omega^*\), then \((\alpha, A)\) the semigroup compactification corresponding to \((\omega \times \omega) \cup \triangle\) is a dual atom (atom) of \(K_1(S)\).