Chapter 1

SEMIGROUP COMPACTIFICATIONS

Introduction


In topological spaces, the notion of a compactification was considered for the first time by A. Tychonoff (1930) [TY]. In 1937, E. Čech [CE] and M.H. Stone [ST] independently defined the maximal compactification $\beta X$ and stated its fundamental properties. But in topological semigroups, the theory of semigroup compactification is still in the stage of infancy. However, there are results in special types of compactifications. Also the theory of semitopological semigroups develops in this direction. For example, in [BE-J-M] J.F. Berglund, H.D. Junghenn and P. Milnes develops the theory of compact right topological semigroups and in particular of semigroup compactifications of semitopological semigroups. In 1961,

In this chapter in Section 1.2, we introduce another type of compactification for a given topological semigroup named as "Semigroup Compactification" and discuss some results relating them to the Bohr compactification. Section 1.1 contains some background material needed in later chapters also.

1.1 Preliminaries

Semigroups 1.1.1 A semigroup is a non-empty set $S$ together with an associative multiplication $(x,y) \rightarrow xy$ from $S \times S$ into $S$. If $S$ has a Hausdorff topology such that
(x,y) \mapsto xy \text{ is continuous with the product topology on } S \times S, \text{ then } S \text{ is called a topological semigroup.}

If \( S \) is a compact topological semigroup then \( S \) is called a compact semigroup \([C-H-K_1]\).

Examples 1.1.2

(a) Let \( S \) be a topological space. Define multiplication in \( S \) by \( xy = x (xy = y) \) for every \( x,y \) in \( S \). Then \( S \) is a topological semigroup, called the left zero (right zero) semigroup.

(b) Let \( S \) be a topological space. Let \( z \in S \) be fixed. Define multiplication in \( S \) by \( xy = z \) for every \( x,y \) in \( S \). Then \( S \) is called a zero semigroup which is a topological semigroup with zero '\( z \)'.

(c) Let \( I_u = [0,1] \) with usual topology and usual multiplication. Then \( I_u \) is a compact abelian semigroup.

(d) Let \( I_m = [\frac{1}{2},1] \) with the usual topology and multiplication \((x,y) \mapsto \min \{x,y\}\). Then \( I_m \) is a compact semigroup.

Definition 1.1.3.

A non-empty subset \( T \) of a topological semigroup \( S \) is called a subsemigroup of \( S \) if \( TT \subseteq T \), a left ideal of \( S \) if \( ST \subseteq T \), a right ideal if \( TS \subseteq T \) and an ideal if \( TS \subseteq T \).
If $T$ is a subsemigroup of $S$, $T$ itself is a topological semigroup under the restriction of multiplication on $S$ to $T \times T$ and the closure $\overline{T}$ of $T$ is also a subsemigroup of $S$ [C-H-K1].


Definition 1.1.4

Let $S$ be a semigroup. A relation $R$ on $S$ is said to be left (right) compatible (with the operation on $S$) if $(x,y) \in R \Rightarrow (ax,ay) \in R$ [(xa,ya) \in R] and compatible if $R$ is both left and right compatible.

Definition 1.1.5

A compatible equivalence on a semigroup $S$ is called a congruence [HOW].

Proposition 1.1.6

(a) An equivalence $R$ on a semigroup $S$ is a congruence if and only if $(a,b) \in R$ and $(c,d) \in R \Rightarrow (ac,bd) \in R$.

(b) the intersection of any collection of congruences on a semigroup $S$ is a congruence on $S$. 
(c) $S \times S$ is a congruence on $S$. [HOW]

**Definition 1.1.7**

If $S$ is a semigroup and $I$ is an ideal of $S$ then the semigroup $S/(I \times I) \cup \Delta$ is called the Rees quotient semigroup of $S$ mod $I$ and is denoted as $S/I$ [C-H-K$_1$].

**Definition 1.1.8**

If $R$ is an equivalence (congruence) on a topological space (semigroup) $S$, then $R$ is called a closed equivalence (congruence) if $R$ is a closed subset of $S \times S$ [C-H-K$_1$].

**Definition 1.1.9**

Let $X,Y$ be spaces and $f:X \longrightarrow Y$ a function which is surjective, then $f$ is said to be a quotient map if $W$ being open (closed) in $Y$ is equivalent to $f^{-1}(W)$ being open (closed) in $X$.

**Definition 1.1.10**

A semigroup $S$ is said to be left (right) cancellative provided $x,y,z \in S$ and $xy = xz \implies y = z$ [ $yx=zx \implies y=z$]. If $S$ is both left and right cancellative, then $S$ is said to be cancellative.
Next theorem is an algebraic hypothesis on a compact semigroup which implies that it must be a group [C-H-K1].

Theorem 1.1.11

Let S be a compact cancellative semigroup. Then S is a group [C-H-K1].

Definition 1.1.12

If S and T are semigroups, a function \( \phi: S \rightarrow T \) is called a homomorphism if \( \phi(xy) = \phi(x) \cdot \phi(y) \) for each \( x, y \in S \). If \( \phi \) is surjective, then \( \phi \) is called a surmorphism. If \( \phi \) is also injective then \( \phi \) is called an algebraic isomorphism and S and T are said to be algebraically isomorphic [C-H-K1].

If S and T are topological semigroups and \( \phi:S \rightarrow T \) is both an algebraic isomorphism and a homeomorphism, then \( \phi \) is called a topological isomorphism and S and T are said to be topologically isomorphic [C-H-K1].

If \( \phi: S \rightarrow T \) is a homomorphism, then \( \phi \) preserves subsemigroups and subgroups. In the case that \( \phi \) is a surmorphism then \( \phi \) preserves ideals and minimal ideals of all three types (left, right, two-sided) and \( \phi^{-1} \) preserves subsemigroups, (left, right) ideals.
If $S$ and $T$ are semigroups and $\varnothing : S \rightarrow T$ is a homomorphism, then $K(\varnothing)$ is a relation defined as

\[ \{(x, y) \in S \times S : \varnothing(x) = \varnothing(y)\} \]

Theorem 1.1.13 Induced Homomorphism theorem

Let $A, B$ and $C$ be (topological) semigroups, $\alpha : A \twoheadrightarrow B$ a (quotient) surmorphism, and $\beta : A \rightarrow C$ a (continuous) homomorphism such that $K(\alpha) \subseteq K(\beta)$. Then there exists a unique (continuous) homomorphism $\gamma : B \rightarrow C$ such that the diagram commutes $[C-H-K_1]$

\[ \begin{array}{ccc}
A & \xrightarrow{\omega} & B \\
& \searrow & \swarrow \gamma \\
& \swarrow \beta & \\
C & \xrightarrow{\nu} & C
\end{array} \]

Theorem 1.1.14 First Isomorphism theorem

Let $S$ and $T$ be semigroups and let $\varnothing : S \twoheadrightarrow T$ be a surmorphism. Then $K(\varnothing)$ is a congruence on $S$ and there exists a unique algebraic isomorphism $\psi : S/K(\varnothing) \rightarrow T$ such that the diagram commutes.
Moreover, if $S$ and $T$ are topological semigroups and $\emptyset:S \to T$ is a continuous surmorphism, then $K(\emptyset)$ is a closed congruence on $S$ and the following are equivalent.

(a) $\psi^{-1}$ is continuous  
(b) $\psi$ is a topological isomorphism, and  
(c) $\emptyset$ is quotient

Finally, if these equivalent statements hold, then $S/K(\emptyset)$ is a topological semigroup [C-H-K$_1$].

If $S$ is a topological semigroup and $R$ is a closed congruence on $S$, then $S/R$ with the induced operation and the quotient topology need not be a topological semigroup. This situation has been studied by J.H. Carruth, J.A. Hildebrant and R.J. Koch (1983) [C-H-K$_1$] and some conditions under which $S/R$ is a topological semigroup have been established. This result was established for compact semigroups by Wallace (1955) [WA$_3$] and extended to locally compact $\sigma$-compact semigroups by Lawson and Madison (1971) [LA-M].
Lemma 1.1.15

Let $S$ be a topological semigroup and let $R$ be a closed congruence on $S$ such that $P \times P : S \times S \to S/R \times S/R$ is a quotient map. Then $S/R$ is a topological semigroup [C-H-K$_1$].

Theorem 1.1.16.

Let $S$ be a compact semigroup and let $R$ be a closed congruence on $S$. Then $S/R$ is a compact semigroup.

Let $\{S_i\} \ i \in I$ be a collection of (topological) semigroups. Then co-ordinatewise multiplication on $P \{S_i\} \ i \in I$ is given by $(fg)(j) = f(j) g(j)$, the latter product being taken in $S_j$ for each $j \in I$ [C-H-K$_1$].

Theorem 1.1.17

Let $\{S_i\} \ i \in I$ be a collection of (topological) semigroups and $S = P \{S_i\} \ i \in I$. Then $S$ with coordinate-wise multiplication is a (topological) semigroup and each projection $P_j : S \to S_j$ is a (continuous open) surmorphism.

The concepts of projective (inverse) limits of topological semigroups are developed in [C-H-K$_1$] and some results on compact semigroups are studied by Hofmann and Mostert (1966) [HO-M$_2$], Numakura (1957) [NU$_2$],
A projective system of (topological) semigroups is a triple \((D, \triangleleft), \{S_\alpha\}_{\alpha \in D}, \{\phi^\beta_\alpha\}_{\alpha \triangleleft \beta}\) where

(a) \((D, \triangleleft)\) is a directed set

(b) \(\{S_\alpha\}_{\alpha \in D}\) is a family of (topological) semigroups indexed by \(D\), and

(c) \(\{\phi^\beta_\alpha\}_{\alpha \triangleleft \beta}\) is a family of functions indexed by \(\triangleleft\) such that

(i) \(\phi^\beta_\alpha : S_\beta \longrightarrow S_\alpha\) is a (continuous) homomorphism for each \((\alpha, \beta) \in \triangleleft\)

(ii) \(\phi^\alpha_\alpha = 1_{S_\alpha}\) identity map on \(S_\alpha\), for each \(\alpha \in D\), and

(iii) \(\phi^\beta_\alpha \circ \phi^\gamma_\beta = \phi^\gamma_\alpha\) for all \(\alpha \triangleleft \beta \triangleleft \gamma\) in \(D\). This projective system is denoted by \(\{S_\alpha, \phi^\beta_\alpha\}_{\alpha \in D}\). Each \(\phi^\beta_\alpha\) is called a bonding map and \(\{S_\alpha, \phi^\beta_\alpha\}_{\alpha \in D}\) is said to be strict if each bonding map is surjective [C-H-K_1].

Definition 1.1.19

If \(S = \{x \in P \{S_\alpha\}_{\alpha \in D} : \phi^\beta_\alpha(x(\beta)) = x(\alpha)\) for all \(\alpha \triangleleft \beta\) in \(D\)}
is non-empty, then $S$ is called the projective limit of $\{S_\alpha, \mathcal{O}_\alpha\}_{\alpha \in D}$ and is denoted by

$$S = \lim_{\alpha \in D} \{S_\alpha, \mathcal{O}_\alpha\} \quad \text{or} \quad S = \lim_{\alpha} S_\alpha$$

If $\{S_\alpha, \mathcal{O}_\alpha\}_{\alpha \in D}$ is a strict projective system, then $S$ is called the strict projective limit of $\{S_\alpha, \mathcal{O}_\alpha\}_{\alpha \in D}$ \cite{C-H-K1}.

**Theorem 1.1.20**

Let $\{S_\alpha, \mathcal{O}_\alpha\}_{\alpha \in D}$ be a projective system of (topological) semigroups such that $S = \lim_{\alpha \in D} S_\alpha$ exists. Then $S$ is a (closed) subsemigroup of $P\{S_\alpha\}_{\alpha \in D}$ \cite{HO-M1}.

Some results on compact semigroups that we would require are

**Theorem 1.1.21**

Let $\{S_\alpha, \mathcal{O}_\alpha\}_{\alpha \in D}$ be a projective system of compact semigroups. Then $\lim_{\alpha \in D} S_\alpha$ is a compact semigroup \cite{C-H-K1}.
Theorem 1.1.22.

Let \( \{S_\alpha, \varphi_\alpha^p\}_{\alpha \in D} \) be a projective system of compact semigroups and let \( S = \lim S_\alpha \). Then \( P_\alpha : S \to S_\alpha \) is surjective for each \( \alpha \in D \), where \( P_\alpha \) is the projection map \([C-H-K_1]\).

Note.

Associated with each topological semigroup \( S \), there is a compact semigroup called the Bohr compactification of \( S \) which is universal over the compact semigroups containing dense continuous homomorphic images of \( S \). The existence and uniqueness of Bohr compactification can be proved \([C-H-K_1]\).

Definition 1.1.23 Bohr Compactification

If \( S \) is a topological semigroup, then Bohr compactification of \( S \) is a pair \((\beta, B)\) such that \( B \) is a compact semigroup, \( \beta : S \to B \) is a continuous homomorphism with \( B = \overline{\beta(S)} \) and if \( g : S \to T \) is a continuous homomorphism of \( S \) into a compact semigroup \( T \), then there exists a unique continuous homomorphism \( f : B \to T \) such that the diagram commutes.

\[\begin{array}{c}
\beta \\
\downarrow \\
S \\
\downarrow g \\
T
\end{array}\]

\[f\]

\[\begin{array}{c}
B \\
\downarrow \\
\beta \\
\downarrow \\
S \\
\downarrow g \\
T
\end{array}\]
For each topological semigroup $S$, there exists a Bohr compactification which is unique up to topological isomorphism \([C-H-K_1]\).


The Product Theorem 1.1.24

Let \( \{ S_\alpha : \alpha \in A \} \) be a collection of abelian topological monoids, \((\beta_\alpha, B_\alpha)\) the Bohr compactifications of $S_\alpha$ for each $\alpha \in A$ and $\beta: P \{ S_\alpha \} \longrightarrow P \{ B_\alpha \}$ the function defined by $\beta(x)(\delta) = \beta_\delta P_\delta(x)$, where $P_\delta: P \{ S_\alpha \} \longrightarrow S_\delta$ is projection for each $\delta \in A$.

Then \((\beta, P \{ B_\alpha \})\) is the Bohr compactification of $P \{ S_\alpha \}$.

Remark.

This result is true even in non-abelian case.

1.2. **Semigroup Compactification**

Here we introduce our definition of semigroup compactification.
Definition 1.2.1.

A semigroup compactification of a topological semigroup $S$ is an ordered pair $(g, T)$ where $T$ is a compact semigroup and $g : S \rightarrow T$ is a dense continuous homomorphism of $S$ into $T$. (Here $g$ is dense means $g(S)$ is dense in $T$).

Examples 1.2.2.

(1). Let $N$ be the multiplicative semigroup of +ve integers with the discrete topology.

$$T = \left\{ \frac{1}{n} : n \in N \right\} \cup \{0\}$$

is a closed subsemigroup of $I_u = [0, 1]$ with usual topology and usual multiplication. If $\varnothing : N \rightarrow T$ is defined by $\varnothing(n) = \frac{1}{n}$ for all $n \in N$. Then $(\varnothing, T)$ is a semigroup compactification of $N$.

(2). Let $(R, +)$ be the additive (semi) group of real numbers with the usual topology. Let $T$ be the circle group with usual multiplication and usual topology. If $\varnothing : R \rightarrow T$ is defined by $\varnothing(x) = \exp(2\pi ix)$. Then $(\varnothing, T)$ is a semigroup compactification of $R$.

(3). Bohr compactification $(\beta, B)$ of a topological semigroup $S$ is a semigroup compactification.
Wallace, A.D. has shown that if $B$ is a compact semigroup and $R$ is a closed congruence on $B$, then the quotient space $B/R$ is a compact semigroup [1.1.16]. We prove below that semigroup compactifications of a topological semigroup $S$ are precisely the quotients of the Bohr compactification of $S$ under closed congruences.

Result 1.2.3.

Let $S$ be a topological semigroup with Bohr compactification $(\beta, B)$. If $(\alpha, A)$ is any semigroup compactification of $S$, then

(a) there exists a continuous surmorphism (i.e. surjective homomorphism) $\Theta : B \rightarrow A$ such that $\Theta \beta = \alpha$

(b) and the equivalence defined by $\Theta$ on $B$ is a closed congruence.

(c) $(\alpha, A)$ is the quotient of $(B, \beta)$ with respect to the congruence in (b).

Proof

(a) from the definition of $(\beta, B)$ there exists a continuous homomorphism $\Theta : B \rightarrow A$ such that $\Theta \beta = \alpha$. 
Again $\Theta$ is surjective, for,

$$A = \overline{\alpha(S)} \quad (\therefore \alpha \text{ is dense in } A)$$

$$= \overline{\Theta\beta(S)} = \Theta \overline{\beta(S)} \quad (\therefore \Theta \text{ is a closed map being from a compact space to a } T_2 \text{ space})$$

$$= \Theta(B)$$

We have $\Theta : B \longrightarrow A$ is a continuous surmorphism such that $\Theta\beta = \alpha$.

(b) Let $R$ be relation defined on $B$ by $\Theta$

$$R = \{(x,y) \in B \times B : \Theta(x) = \Theta(y)\} \text{ is clearly an equivalence. }$$

$$R = (\Theta \times \Theta)^{-1}(\Delta_A) \text{ is closed since } \Delta_A \text{ diagonal in } A \times A \text{ of Hausdorff space } A \text{ is closed. }$$

$R$ is a congruence; for,

$$(a,b) \in R \implies \Theta(a) = \Theta(b)$$

$$(c,d) \in R \implies \Theta(c) = \Theta(d)$$

$$.\therefore \Theta(ac) = \Theta(a) \cdot \Theta(c) = \Theta(b) \cdot \Theta(d)$$

$$.\therefore (ac, bd) \in R.$$

Hence $R$ is a closed congruence.
(c) We have $\Theta : B \rightarrow A$ is a quotient map, since it is a closed continuous surmorphism.

Result 1.2.4.

Let $S$ be a topological semigroup with Bohr compactification $(\beta, B)$. If $R$ is a closed congruence on $B$, then there exists a semigroup compactification $(\alpha, A)$ of $S$ so that the congruence defined by this compactification is $R$.

Proof.

Let $R$ be a closed congruence on $B$. Define $\Theta : B \rightarrow B/R$ the natural map. Then $A = B/R$, with the quotient topology and multiplication induced by $\Theta$ is a compact semigroup [1.1.16].

Define $\alpha : S \rightarrow A$ such that $\alpha = \Theta \beta$. Clearly $\alpha$ is a well-defined continuous homomorphism.

Also $\alpha$ is dense; for,

$$\overline{\alpha(S)} = \overline{\Theta \beta(S)} = \Theta \overline{\beta(S)} \quad (\therefore \Theta \text{ is closed})$$

$$= \Theta(B) = A \quad (\therefore \Theta \text{ is surjective})$$

Thus we have $\alpha : S \rightarrow A$ is a dense continuous homomorphism.
(α,A) is a semigroup compactification of S and the congruence defined by (α,A) is that defined by θ which is R.

Remark 1.2.5.

Thus we have proved that if (β,B) is a Bohr compactification of S and R is any closed congruence on B, then the quotient space B/R is a semigroup compactification of S and conversely any semigroup compactification (α,A) of S is topologically isomorphic to B/R for some closed congruence R on B.

It is known that the family of congruences on a semigroup is a complete lattice [HCO]. Here we consider Fr(B), the family of semigroup compactifications of S. We define a partial order in Fr(B), and prove that the equivalence classes form a complete lattice. This is the content of Section 3.

Kenneth D. Magill, Jr. [M], has shown that if is a locally compact Hausdorff space, then their lattices of compactifications K(X) and A(X) are isomorphic and G(X) and B(X) isomorphic. But this would...