Chapter-1

INTRODUCTION

1.1 HELLY'S THEOREM AND AXIOMATIC CONVEXITY

The applicability and the intuitive appeal of the notion of convexity have led to a wide range of notions of "Generalized Convexity". For several of them, theorems related to Helly's, were either a motive or a by-product of the investigation. Helly's theorem, which was first published by Johan Radon in 1921 and later in 1925 by Helly himself states that "each family of convex sets in $\mathbb{R}^d$, which is finite or whose members are compact, has a nonempty intersection, provided each subfamily of at most $d+1$ sets has nonempty intersection". The formulation of Helly's theorem can be found in the famous paper of L. Danzer, B. Grunbaum and V. Klee [6], called "Helly's theorem and its Relatives". Restricting Helly's theorem to finite families of convex sets, it is clear that the theorem is formulated completely in terms of convex sets, their intersections and the dimension $d$ of the underlying space.

A convex set can be defined as the intersection of large basic convex sets (for example, half spaces in vector spaces) or by the property of being closed with...
respect to a certain family of finitary operators
(For example, n-ary operators of the form
\[ (x_1, x_2, \ldots, x_n) \rightarrow \sum_{i=1}^{n} \lambda_i x_i \text{ in } \mathbb{R}^d, \]
where the \( \lambda_i \)'s are non-negative and sum to 1). This remark
leads to the following definition.

A set \( X \), together with a collection \( \mathcal{E} \) of
distinguished subsets of \( X \), called convex sets, forms
a convexity space or aligned space, if the following
axioms are satisfied:

\( C_1: \emptyset \in \mathcal{E}, \ X \in \mathcal{E} \)

\( C_2: \mathcal{E} \) is closed under arbitrary intersections

\( C_3: \mathcal{E} \) is closed for the unions of totally ordered
subcollections. \( \mathcal{E} \) is called an alignment or convexity on \( X \).
The convex hull of a set \( S \) in \( X \) (the smallest convex set
containing \( S \)) is defined as \( \text{conv}(S) = \bigcap \{ A \in \mathcal{E} \mid S \subseteq A \} \).
Those families of sets which satisfy \( C_1 \) and \( C_2 \) are known
as Moore families or closure systems. The axioms \( C_1 \) and \( C_2 \)
were first used by F.W. Levi [31] in 1951 and later on by
Eckhoff [10], Jamison [24], Kay and Wamble [29] and
Sierksma [43]. The term 'alignment' is due to Jamison [24].
Hammer [21] has shown that for Moore families the axiom \( C_3 \)
is equivalent to the "domain finiteness" condition which states that for each $S \subseteq X$, $\text{conv}(S) = \bigcup \{\text{conv}(T) | T \subseteq S, |T| < \infty\}$. ($|T|$ denotes the cardinality of $T$). Alternative terminologies for convexity spaces are "algebraic closure systems" ([5]) and "domain finite convexity spaces" ([10], [21], [29], [43]-[45]). As mentioned earlier, the axiomatization of convexity is motivated by the fact that most combinatorial properties of ordinary convex sets in $R^d$ like Helly, Radon and Caratheodory theorems can be studied in the general context of convexity spaces.

1. **Helly property**

A convexity space $(X, C)$ has the Helly property $H_k$, if a finite family of convex sets of $X$ has an empty intersection, then this family contains at most $k$ members with an empty intersection. The Helly number of $(X, C)$ is the smallest integer $k$, such that $H_k$ holds. Helly's theorem states that the Helly number for the ordinary convexity in $R^d$ is $d+1$. For further examples, see Danzer [6], Jamison [25] and Sierksma [45].
2. Partition property

Closely related to Helly's theorem is the classical theorem of Radon published in 1921. The theorem states that each set of $d+2$ or more points in $\mathbb{R}^d$ can be expressed as the union of two disjoint sets, whose convex hulls have a common point. See Danzer et al. ([6]). Radon's theorem was generalized in 1966 by H. Tverberg ([50]). Instead of 2-partitions, he has investigated arbitrary $m$ partitions. The theorem states that each set $S$ in $\mathbb{R}^d$ with $|S| > (m-1)(d+1)+1$ can be partitioned into $m$ pairwise disjoint sets with intersecting convex hulls.

Thus, we have, that the convexity space $(X, \mathcal{G})$ has Partition property $P_{k,n}$, if $\{P_i\}_{i \in I}$ is a family of $n=|I|$ points, there exists a partition of $I$ into $k$ parts $I_1, I_2, \ldots, I_k$ such that

$$\bigcap_{1 \leq j \leq k} \text{conv}(\{P_i | i \in I_j\}) \neq \emptyset.$$  

Tverberg's theorem states that the ordinary convexity in $\mathbb{R}^d$ has property $P_k'(k-1)(d+1)+1$ and for $k=2$, we get Radon's theorem. An important problem related to Radon partitions posed by Eckhoff in analogy with Tverberg's theorem is the following:
Eckhoff's conjecture

Suppose an aligned space \((X, \mathcal{C})\) has Radon number \(r\). Does the partition inequality \(P_m \leq (m-1)(r-1)+1\) always hold? Jamison [27] has shown that the partition conjecture holds for order convexities, tree-like convexities etc.

3. Caratheodory property

The classical theorem of Caratheodory, states that, when \(A \subseteq \mathbb{R}^d\), each point of \(\text{conv} A\) is a convex combination of \(d+1\) or fewer points of \(A\). The theorem of Caratheodory was published in 1907. See Danzer [6]. A convexity space \((X, \mathcal{C})\) has the Caratheodory property \(C_k\), if \(x \in \text{conv}(A)\), then \(x \in \text{conv}(F)\), for some \(F \subseteq A\), with \(|F| \leq k\), for any \(A \subseteq X\). The Caratheodory number of \((X, \mathcal{C})\) is the smallest number such that \(C_k\) holds. Ordinary convexity in \(\mathbb{R}^d\) has Caratheodory number \(d+1\) (Caratheodory theorem).

1.2. INTERVAL CONVEXITIES

An interval \(I\) on a set \(X\) is a mapping \(I: X \times X \rightarrow 2^X\). The \(I\)-closed subsets of \(X\) are subsets \(C \subseteq X\) such that \(I(x, y) \subseteq C\), for every \(x, y \subseteq C\). The collection \(\mathcal{C}_I\) of \(I\)-closed subsets satisfies the axioms \(C_1, C_2, C_3\) of convexity.
spaces. The axiom $C_3$ is a consequence of the finitary property of convex hulls and the fact that, for a subset $A$ of $X$, $\text{conv}(A) = \bigcup_{k \in \mathbb{N}} I^k(A)$, where $I^k(A)$ is defined as $I^0(A) = A$ and $I^{k+1}(A) = I(I^k(A) \times I^k(A))$. The function $I$ is called an interval-function of the convexity space $(X, \mathcal{G}_I)$. Convexity spaces admitting an interval function are named Interval Convexity Spaces, see Calder([4]). Most of the usual convexities are interval convexities. For example, ordinary convexity in $\mathbb{R}^d$, metric convexity ($d$-convexity) in metric spaces, order convexity in partially ordered sets and geodesic convexity and minimal path convexity in graphs.

**Metric convexity**

The concept of convexity in metric spaces was introduced by Menger. It is the interval convexity generated by the metric interval $d-[x,y] = \left\{ z \in X \mid d(x,z) + d(z,y) = d(x,y) \right\}$, for points $x,y$ in the metric space $(X,d)$. For various geometric developments involving Menger's and other closely related notions of metric convexity, see Blumenthal ([2]) and Buseman ([3]).
Order convexity

The usual order convexity in a partially ordered set \((P, \preceq)\) is the interval convexity generated by the usual order interval \([x, y] = \{z \in P \mid x \preceq z \preceq y\} \text{ or } y \preceq z \preceq x\}
for points \(x, y \in P\). Order convexity generated by the order interval function has been studied by Franklin ([17]) in 1962. See also Jamison-Waldner ([27]), Jamison ([25]).

1.3 GRAPH CONVEXITIES AND CONVEX GEOMETRIES

Convexity in Graphs

The first explicit use of convexity in graphs has been made perhaps by Feldman and Hogassen. Most of their results deal with geodesic convexity. A more general point of view appeared in Sekanina ([42]) in 1975 and Müller ([33]) in 1980. A systematic approach arises in Farber-Jamison ([15]).

A graph convexity (Duchet) is a pair \((G, \mathcal{C})\) formed with a connected graph \(G\) with vertex set \(V\) and a convexity \(\mathcal{C}\) on \(V\) such that \((V, \mathcal{C})\) is a convexity space, satisfying the additional axiom,

\[ \text{GC: Every convex subset of } V \text{ induces a connected subgraph.} \]

See ([9]).
In the study of convexity in graphs, two types of convexity have played a prominent role, namely the "minimal path convexity or monophonical convexity and geodesic convexity or d-convexity".

**Minimal path convexity**

The minimal path convexity in a connected graph $G$ is the interval convexity in $V(G)$, generated by the minimal path interval $m-[x,y]$, where $m-[x,y]$ is the set of all vertices of all chordless paths from the vertex $x$ to the vertex $y$ in $G$, and a chord of a path in $G$ is an edge joining two nonconsecutive vertices in the path. See Jamison ([25]) and Duchet ([9]).

**Geodesic convexity**

Let $d-[x,y]$ denote the set of all vertices of all shortest paths between the vertices $x$ and $y$ in $G$. The convexity generated by the interval function $d-[x,y]$ is called the geodesic convexity or distance convexity in $G$. The $d$-convexity is the metric convexity associated with the usual distance function $d(x,y)$ in $G$.

Early researches on $d$-convexity in graphs were motivated by an important problem posed by Ore in 1962,
which is the following: "Characterize the geodetic graphs: that is, graphs in which every pair of vertices is joined by a unique shortest path ".

Graphs with only the trivial geodesic subgraphs have been called distance convex simple graphs by Hebbare and others. See Hebbare ([23]), Batten ([1]). Unlike m-convexity, the geodesic convexity is very general and has been intensively studied since 1981. See Jamison ([25]), Soltan ([49]) and Farber ([13]).

Convex geometries

Convex geometries were introduced independently by Edelman and Jamison in 1980. They are finite convexity spaces in which the finite Krein–Milman property holds. That is every convex set is the convex hull of its extreme points. There are numerous equivalent ways of defining a convex geometry. See Edelman–Jamison ([12]).

We have the following characterizations of graphs.

(i) The m-convexity in a graph G is a convex geometry if and only if G is chordal. A chordal graph is one in which every cycle of length at least four has a chord.
(ii) The geodesic convexity in G is a convex geometry if and only if G is a disjoint union of Ptolemaic graphs.

G is a Ptolemaic graph, if for every four vertices x, y, z, w in G, the Ptolemaic inequality
\[ d(x,y) \cdot d(z,y) \leq d(x,z) \cdot d(y,w) + d(x,w) \cdot d(z,y) \]
holds. See Farber–Jamison ([14], [15]). Major references on the abstract theory of convexity are Jamison ([24], [25]), Sierksma ([45]) and Soltan ([46]). A recent survey of various convexities in discrete structures is in Düchert ([8]).

1.4. DIGITAL AND COMPUTATIONAL CONVEXITIES

The growing field of computer science has also seen the emergence of studies dealing with convexity. This began in the early 1960's, when Freeman ([56]) investigated the representation of straight line segments on a digital grid and Bilanski ([54]), gave an algorithm for determining the vertices of a convex polyhedron. Convexity can be discussed in computer science from the following view:

(1) Digital Geometry, and (2) Computational Geometry.

1. Digital geometry

To generalize convexity and related notions such as straight line segments to the geometry of digital grids, and analyse their properties, in this framework.
Convexity in the two dimensional digital images has been studied by several authors in particular Kim ([57]), Kim and Rosenfeld ([58]) and Ronse ([59]). In contrast with Euclidean images, several non equivalent definitions can be given for digital images. The rectangular grid of two dimensions can be viewed as the set $\mathbb{Z}^2$, where $\mathbb{Z}$ is the set of integers, so that pixels can be represented by integer co-ordinates. The basic notions of $k$-adjacency, $k$-connected paths, $k$-connectedness ($k=4$ or $8$) in the geometry of rectangular digital grids can be realized in $\mathbb{Z}^2$ with the integer valued metrics (graph metrics), denoted as $d_1$ (for $k=4$) and $d_2$ (for $k=8$), defined as

$$d_1(x,y) = |x_1-y_1| + |x_2-y_2|$$

and $d_2(x,y) = \max(|x_1-y_1|, |x_2-y_2|)$,

for $x = (x_1,x_2)$ and $y = (y_1,y_2)$ in $\mathbb{Z}^2$.

We can view a $k$-connected path in the rectangular grid as a path in the graph metric space $(\mathbb{Z}^2,d)$, where $d$ is $d_1$ or $d_2$, according as $k=4$ or $k=8$ respectively. Thus the distance geometry in $\mathbb{Z}^2$, generated by the integer valued metrics $d_1$ and $d_2$ is closely related to the geometry of the digital rectangular grid of two-dimension. This is the motivation of our study of the $d_1$-convexity and $d_2$-convexity in the integer lattice.
2. Computational Geometry

One wants to evaluate the computational complexity of various operations related to convex sets, and to find optimal computer algorithms for them. Important problems are the determination of the convex hull, that of vertices, faces, volume or diameter of convex bodies, intersection of convex polyhedra, extremal distance between convex polyhedra and maximal convex subsets of non-convex sets.

The first computational question relating to convexity is the design of algorithms, for finding the convex hull of a set of points. The digital convex hull is dealt with in Yau ([62]). A related problem is the determination of the computational complexity of the construction of the convex hull of a set of points. A bibliography on digital and computational convexity is seen in ([61]). See Preparata-Shamos ([38]), for recent developments in computational geometry.

1.5 PRELIMINARIES

Let \((X, \mathcal{C})\) be any convexity space. That is, \(\mathcal{C}\) is a collection of subsets of the set \(X\), such that (i) \(\emptyset, X \in \mathcal{C}\), (ii) \(\mathcal{C}\) is closed under arbitrary intersections,
(iii) \( \mathcal{C} \) is closed for the unions of totally ordered subcollections. \( \mathcal{C} \) is called an alignment or convexity on \( X \). The convex hull of a set \( A \) in \( X \) is defined as

\[
\text{conv}(A) = \bigcap \left\{ B \in \mathcal{C} \mid A \subseteq B \right\}.
\]

**Definition 1.5.1.**

The Caratheodory number of a convexity space \((X, \mathcal{C})\) is defined as the smallest nonnegative integer \( c \), such that

\[
\text{conv}(A) = \bigcup \left\{ \text{conv}(B) \mid B \subseteq A \text{ and } |B| < c \right\}, \text{ for all } A \subseteq X.
\]

**Definition 1.5.2.**

The Helly number \( h \) of \((X, \mathcal{C})\) is defined to be the infimum of all nonnegative integers \( k \), such that the intersection of any finite collection of convex sets is nonempty, provided the intersection of each subcollection of at most \( k \) elements is nonempty. Or equivalently,

**Definition 1.5.3.**

A convexity space \((X, \mathcal{C})\) has the Helly number \( h \), if \( h \) is the smallest nonnegative integer such that \( A \subseteq X \) and \( |A| = h+1 \Rightarrow \bigcap \left\{ \text{conv}(A \setminus a) \mid a \in A \right\} \neq \emptyset \), for all \( A \subseteq X \), where \( A \setminus a \) denotes \( A \setminus \{a\} \).
Definition 1.5.4.

A convexity space \((X, \mathcal{C})\) has the Radon number \(r\), if \(r\) is the infimum of all positive integers \(k\), with the property that, each set \(A\) in \(X\) with \(|A| \geq k\), admits a partition \(A = A_1 \cup A_2\) with \(A_1 \cap A_2 = \emptyset\) and such that \(\text{conv}(A_1) \cap \text{conv}(A_2) \neq \emptyset\). Such a partition is called a Radon partition of \(A\).

Definition 1.5.5.

The generalized Radon number or Tverberg type Radon number \(P_m\) of a convexity space \((X, \mathcal{C})\) is defined as the infimum of all positive integers \(k\), with the property that, each set \(A\) in \(X\) with \(|A| \geq k\) admits an \(m\)-partition \(A = A_1 \cup \ldots \cup A_m\), into pairwise disjoint sets \(A_i\) such that \(\text{conv}(A_1) \cap \text{conv}(A_2) \cap \ldots \cap \text{conv}(A_m) \neq \emptyset\). Such an \(m\)-partition of \(A\) is called a Radon \(m\)-partition of \(A\). We need the theorem of Levi.

Theorem 1.5.6. (Levi)

Let \((X, \mathcal{C})\) be a convexity space. If the Radon number \(r\) of \((X, \mathcal{C})\) exists, then the Helly number \(h\) exists, and \(h \leq r - 1\).
Theorem 1.5.7. (Eckhoff and Jamison)

Let \((X, \mathcal{C})\) be a convexity space with Caratheodory number \(c\) and Helly number \(h\). Then the Radon number \(r\) of \((X, \mathcal{C})\) exists, and \(r \leq c(h-1)+2\).

Definition 1.5.8.

Let \((X, \mathcal{C})\) be a convexity space. A subset \(B\) of \(X\) is said to be (convexly) independent if \(b \notin \text{conv}(B \setminus b)\), for each \(b \in B\).

Definition 1.5.9.

The rank of a convexity space \((X, \mathcal{C})\) is defined as the supremum of the cardinalities of the independent sets. It is noted that the rank of a convexity space \((X, \mathcal{C})\) is an upper bound for both the Helly number \(h\) and the Caratheodory number \(c\).

\[ N = \{1, 2, 3, \ldots\} \] is the set of natural numbers and \(\mathbb{Z}\) denotes the set of integers. The graph theoretic terminology used in this thesis are as in Harary ([22]). We use induction in some of the proofs.

1.6. AN OVERVIEW OF THE MAIN RESULTS OF THIS THESIS

A rather active area in modern convexity theory is concerned with the computation of several "invariants" in general convexities. This thesis contributes mainly to this in some interval convexities, where the underlying set is a discrete set. The "invariants" that we discuss in this thesis are the Caratheodory, Helly, Radon and Tverberg type Radon numbers.
In chapter 2, we consider $\mathbb{Z}^n$ as a model. Metric convexity (d-convexity), with respect to the integer valued metrics $d_1, d_2, d_3$ are defined. For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$, the metrics $d_1, d_2$ and $d_3$ are defined respectively as:

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \quad d_2(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|$$

and $d_3(x, y)$ is the number of co-ordinates in which $x$ and $y$ differ.

The order convexity is defined with respect to the partial order $x \preceq y$ if and only if $x_i \leq y_i$ for all $i$. It is shown that every $d_1$-convex set is both order convex and $d_3$-convex. Also it is obtained that there is no finite Helly and Radon numbers for the order convexity and $d_3$-convexity. The $d_1$-convex sets has Caratheodory number 'n' and Helly number 2. Using Jamison-Eckhoff theorem, it is shown that the Radon number 'r' of the $d_1$-convexity attains the bound $n+2$, for $n=2$ and $n=3$. For $d_2$-convexity, the rank is found to be $2^n$, and the Helly number equals the rank. The Radon number for $d_2$-convexity is found to be $2^n+1$ and the Caratheodey number is $2^{n-1}$. Tverberg type Radon number is also obtained for $d_2$-convexity.

For $d_3$-convexity the Caratheodory number is $n$. 

In chapter 3, we extend the definitions of order convexity and $d_1$-convexity. The geodesic convexity is defined as the join of all convex sets that are contained in a set of convex sets. The geodesic convexity is a concept that is useful in the study of convex sets and their properties.
In chapter 3, we extend the definitions of order convexity and \(d\)-convexity in \(Z^n\) to the infinite dimensional sequential space \(Z^\infty\). The \(d\)-intervals are defined using the \(d\)-intervals in the finite dimensional submodules of \(Z^\infty\). The analogous results of Caratheodory, Helly and Radon type numbers are obtained for these convexities in \(Z^\infty\).

Chapter 4 deals with the geodesic convexity in the finite geometric structure known as "Generalized Polygons", considering it as a bipartite graph \(\Gamma\). The geodesic convexity in \(\Gamma\) is not exactly a convex geometry but finite Krein-Milman property holds for every proper \(d\)-convex subset of \(\Gamma\). It is shown that a \(d\)-convex subset \(K\) of \(\Gamma\) has the Krein-Milman property if and only if \(\text{diam}(K) < n\). Various center concepts, such as center, centroid and distance centre in \(\Gamma\) are studied. Finally, the Helly, Radon and Caratheodory type theorems for the geodesic convexity are obtained. It is shown that the \(m\)-convexity in \(\Gamma\) is the trivial convexity, consisting of the null set \(\emptyset\) and whole vertex set \(V\) of \(\Gamma\). In the last section of this chapter (4.5), we discuss an interesting result, which holds for any finite connected bipartite graph \(G\). We order the vertices of \(G\) called the "the canonical ordering of \(G\)”, as given by Mülder, and show that the "geodesic alignment" on \(G\) is the join of order alignments, with respect to all possible canonical orderings of \(G\).
In chapter 5, our discussion is mainly in the discrete plane $\mathbb{Z}^2$. Using the concept of hemispaces, it is shown that an intersection convex set $A$ of $\mathbb{Z}^2$ (an intersection convex set of $\mathbb{Z}^n$ is defined by Doignon as the intersection of a convex set in $\mathbb{R}^n$ with $\mathbb{Z}^n$) is $d_1$-convex if and only if the supporting lines of $A$ are parallel to the co-ordinate axes and $A$ is $d_2$-convex if and only if the supporting lines of $A$ have slope $\pm 1$.

Finally, a computational problem is dealt with. An algorithm for computing the $d_2$-convex hull of a finite set of points in $\mathbb{Z}^2$ is given and also the complexity of the algorithm is computed.