CHAPTER 4
4.1 INTRODUCTION

Unlike barotropic flows, the stability of non-barotropic flows is rarely treated in literature. One of the reasons is that till recently non-barotropic flows were not known to have sufficient conserved quantities as in the case of barotropic flows. But it has been shown by Eckart (1960), Bretherton (1970) and Mobbs (1981) that well-known conservation laws associated with vorticity for barotropic flows can be generalized to the case of non-barotropic flows by replacing velocity $\bar{u}$ in some of their quantities by $\bar{u} - \eta \vec{s}$ where $\eta$ is thermacy and $s$ the specific entropy. Further, it is to be noted that Kelvin's circulation theorem for barotropic flows is a special case of a more general one in which the closed curve is lying on the surfaces $s = \text{constant}$ (Pedlosky (1979)). It has been shown by Joseph (1993) that the basis of these conservation laws is that the flow considered is isentropic. The only available results of stability of non-barotropic flows are the stability of adiabatic flows by Dikii (1965b) and Holm et al. (1985).

In this chapter we obtain the stability criterion for non-barotropic flows based on the variational principles due to Mathew and Vedan (1989). The infinitesimal generator of
transformation group that leaves the potential vorticity invariant is used to define the structure that remains invariant under the flow.

4.2 CONSERVATION OF POTENTIAL VORTICITY

Although we have introduced a new four-vector \( S^\alpha, (\alpha = 0,1,2,3) \) in chapter 2 for non-barotropic flows, the Lagrangian contains only \( S^0 \) so that only \( S^0 \) enters into our variational principle. Following Mathew and Vedan (1989) we consider \( \eta^\alpha = 0 \), and \( \Theta^\alpha = 0 \), \( (\alpha = 0,1,2,3) \). We use generalized hydromechanical transformation with \( \xi^0 = 0 \). Let \( (\xi^1, \xi^2, \xi^3) \) be the components of the three-dimensional vector \( \vec{\xi} \) and \( (u^1, u^2, u^3) \) be the components of \( \vec{u} \). Let \( \nabla \) denote the spacial divergence operator

\[
\nabla = \left[ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right].
\]

Then we find that in equation (2.17)

\[
\eta^\alpha = 0 \quad \text{and} \quad \Theta^\alpha = 0, \; \alpha = (0,1,2,3). \tag{4.1}
\]

provided

\[
\nabla \cdot (\rho \vec{\xi}) = 0, \; \rho \vec{\xi} \cdot \nabla s = 0 \quad \text{and} \quad \frac{\partial}{\partial t} (\rho \vec{\xi}) + \nabla \times \left( \rho \vec{\xi} \times \vec{u} \right) = 0. \tag{4.2}
\]

Then

\[
\vec{\xi} = \frac{1}{\rho} \nabla f \times \nabla s, \tag{4.3}
\]
is a solution of equations (4.2), where $f$ satisfies the equation

$$\nabla \left[ Df/Dt \right] \times \nabla s = 0 ,$$

(4.4)

where $D/Dt$ is the material differentiation operator.

Equation (4.2) and its solution (4.3) have appeared in Katz and Lynden-Bell (private communication) and Friedman and Schutz (1978). Joseph (1993) has pointed out that conservation of potential vorticity follows from the above equations by comparing the derivation of Katz and Lynden-Bell. Here we give the details as follows:

Let $\vec{a} = (a_1, a_2, a_3)$ be any three vector such that $D\vec{a}/Dt = 0$. Then the equation (4.4) is satisfied if $f = f(\vec{a})$. Mathew and Vedan (1991) have proved the following theorem.

**THEOREM 4.1**

If there exists a divergence symmetry for the action integral

$$W = \int_\mathcal{T} \, dr \, L (\vec{x}, \vec{p}(\vec{x}), \vec{s}(\vec{x})) ,$$

(4.5)

depending on $r$ arbitrary functions and their derivatives up to a given order $q$, there exist exactly $r$ linearly independent
identities between the Euler-Lagrange expressions $\psi_{\alpha}$ and their derivatives, provided the symmetry corresponds to generalized hydromechanical transformations. Theorem 4.1 leads to the equation

$$\partial_{\beta} (T^\beta_{\alpha} \xi^\alpha - L, \xi^\beta) = \psi_{\alpha} \xi^\alpha .$$

The corresponding conserved density is

$$T^\alpha_{\alpha} \xi^\alpha - L, \xi^\alpha .$$

From the above choice of $\xi^\alpha$ we have $\xi^0 = 0$ and $\xi^i$ given by the equation (4.3), $i = 1, 2, 3$. Then

$$T^0_{\alpha} \xi^\alpha - L, \xi^0 = \bar{u} \cdot \nabla f \times \nabla s .$$

Thus we have

$$\int_V dV \bar{u} \cdot \nabla f \times \nabla s ,$$

is a constant.

Using Green's theorem,

$$\int dV \bar{u} \cdot \nabla f \times \nabla S = \int_{S} \bar{n} \cdot (\bar{u} \cdot \nabla s) f + \int dV f(\nabla \times \bar{u}) \cdot \nabla s \quad (4.6)$$

where $V$ is a three dimensional volume with surface $S$. 
We choose \( f \) which is non-zero only within volume \( V \). Then the first term on the right hand side vanishes. Since \( f \) is arbitrary we get

\[
\frac{\bar{\omega} \cdot \nabla_s}{\rho},
\]

is a constant, where \( \bar{\omega} \) is the vorticity. This is the law of conservation of potential vorticity. Thus we have seen that the infinitesimal generator of the transformation of the domain for which the potential vorticity is constant is \( \xi = \frac{1}{\rho} \nabla f \times \nabla s \).

Potential vorticity conservation can also be obtained directly from the infinitesimal criterion for hydromechanical invariance (Chapter 3).

The generalized form of the variational principles of Drobut and Rybarski (1959) and Mathew and Vedan (1989) is based on extending the field of dependent variables by considering the entropy flux vector \( s^\alpha \) in addition to the momentum flux vector \( p^\alpha \) in the four dimensional manifold \( \chi_4 \). These lead to the following definitions in the case of non-barotropic flows.

Let

\[
\vec{V} = \xi^\alpha \frac{\partial}{\partial x^\alpha} + \eta^\alpha \frac{\partial}{\partial p^\alpha} + \theta^\alpha \frac{\partial}{\partial x^\alpha},
\]

(4.8)

where

\[
\eta^\alpha = \partial_{\beta} \left( p^\beta \xi^\alpha - p^\alpha \xi^\beta \right),
\]

and

\[
\theta^\alpha = \partial_{\beta} \left( s^\beta \xi^\alpha - s^\alpha \xi^\beta \right).
\]

(4.9)
**DEFINITION 4.1**

\[ \bar{V} \text{ is a hydromechanical variational symmetry if} \]

\[ \bar{V}(L) + L \text{ Div } \bar{\xi} + \partial_{\alpha \beta} \frac{\partial L}{\partial p_{\beta}} \xi^\alpha + \partial_{\alpha \beta} \frac{\partial L}{\partial s_{\beta}} \xi^\alpha = 0 \] \hspace{1cm} (4.10)

or is a divergence.

**DEFINITION 4.2**

The action integral \( W \) (equation (4.5)) is said to be hydromechanically div invariant if

\[ \bar{V}(L) + L \text{ Div } \bar{\xi} + \partial_{\alpha \beta} \frac{\partial L}{\partial p_{\beta}} \xi^\alpha + \partial_{\alpha \beta} \frac{\partial L}{\partial s_{\beta}} \xi^\alpha = 0 \]

is a divergence.

**DEFINITION 4.3**

The action \( W \) is said to be hydromechanically invariant if

\[ \bar{V}(L) + L \text{ Div } \bar{\xi} + \partial_{\alpha \beta} \frac{\partial L}{\partial p_{\beta}} \xi^\alpha + \partial_{\alpha \beta} \frac{\partial L}{\partial s_{\beta}} \xi^\alpha = 0 \]

**THEOREM 4.2**

If \( \eta^\alpha \) and \( \theta^\alpha \) vanish, then the action \( W \) is hydromechanically div invariant.
\textbf{PROOF:}

We have

\[ \nabla(L) + L \text{Div} \hat{\xi} + \alpha_{\alpha} \frac{\partial L}{\partial y \beta} \hat{\xi} + \alpha_{\alpha} \frac{\partial L}{\partial s \beta} \hat{\xi} = \alpha_{\alpha}(L \hat{\xi}) \]

(4.11)

Hence the theorem.

\textbf{THEOREM 4.3}

The hydromechanical variations \( \eta^\alpha \) and \( \theta^\alpha \) vanish, if

\[ \hat{\xi} = \frac{1}{\rho} \nabla f \times \nabla s , \]

where \( f \) satisfies the equation

\[ \nabla [Df/Dt] \times \nabla s = 0 , \]

\( s \) being the entropy.

\textbf{PROOF:}

Follows from equations (4.1)-(4.4)

\textbf{THEOREM 4.4}

In the case of non-barotropic flows potential vorticity

\[ \int_V dV \frac{\tilde{\omega} \cdot \nabla s}{\rho} , \]
is a constant of motion.

**PROOF:**

Let \( \vec{\gamma} = \frac{1}{\rho} \nabla f \times \nabla s \)

Then by theorem 4.3 we have \( \eta^\alpha = 0 \) and \( \theta^\alpha = 0 \), \( \alpha = (0,1,2,3) \).

Thus \( \vec{\dot{v}} = \xi^\alpha \frac{\partial}{\partial x^\alpha} \).

By theorem 4.2 we have

\[
\vec{V}(L) + L \text{ Div } \vec{\xi} + \partial_\beta \frac{\partial L}{\partial p^\beta} \xi^\alpha + \partial_\alpha \frac{\partial L}{\partial s^\beta} \xi^\alpha = \partial_\alpha (L \xi^\alpha).
\]

Then we have the variation

\[
\Delta w = \int_\tau d\tau \frac{\partial}{\partial \beta}(L \xi^\beta).
\]

ie.,

\[
\int_\tau d\tau \left\{ \partial_\beta \left( \tau_\alpha^\beta \xi^\alpha \right) - \psi_\alpha \xi^\alpha \right\} = \int_\tau d\tau \partial_\beta (L \xi^\beta),
\]

where \( \tau_\alpha^\beta \) and \( \psi_\alpha \) are given by equations (2.19) and (2.20).

Since \( \tau \) is arbitrary,

\[
\partial_\beta \left( \tau_\alpha^\beta \xi^\alpha \right) - \psi_\alpha \xi^\alpha = \partial_\beta (L \xi^\beta).
\]
During motion \( \frac{\partial L}{\partial p^\alpha} (p_\beta \xi^\alpha - p_\alpha \xi^\beta) + \frac{\partial L}{\partial \alpha} (s_\beta \xi^\alpha - s_\alpha \xi^\beta) \) ie.,

\[
\psi_\alpha \xi^\alpha = \partial_\beta \left\{ \frac{\partial L}{\partial p^\alpha} (p_\beta \xi^\alpha - p_\alpha \xi^\beta) + \frac{\partial L}{\partial s^\alpha} (s_\beta \xi^\alpha - s_\alpha \xi^\beta) \right\}.
\]

Thus the corresponding conserved density is

\[
\int dV \, \rho \xi^i \frac{\partial L}{\partial p^i}.
\]

Substituting \( \bar{\xi} \) from equation (4.3) and \( \frac{\partial L}{\partial p^i} = \tilde{u} \), we get the above integral as

\[
\int dV \, \tilde{u} \cdot \nabla f \times \nabla s.
\]

Thus we have \( \int dV \, \tilde{u} \cdot \nabla f \times \nabla s \) is a constant.

Comparing with equation (4.5), the result follows.

It is to be noted that though vorticity conservation and Helmholtz theorem were obtained by Mathew and Vedan directly from Noether's theorem, the derivation of conservation of potential vorticity (1991) was not straightforward. Helicity
conservation was obtained by Joseph (1993) from the invariance criterion, but potential vorticity conservation was obtained by relating the equations (4.1,4.2,4.3) to a corresponding result by Katz and Lynden-Bell. Here we complete the proof first by investigating the correct relation with equations of Katz and Lynden-Bell and then show that this can be easily obtained from the invariance criterion itself. The more general case of symmetries corresponding to non-vanishing hydromechanical variations is still an open problem.

4.3 EQUI-POTENTIAL VORTICITY FLOWS

**DEFINITION 4.4**

Two fields \((\rho, \tilde{u}, \tilde{s})\) and \((\rho', \tilde{u}', \tilde{s}')\) are equi-potential vorticity fields if there exists a smooth, volume preserving mapping \(g\) of the domain \(V\) into itself such that

\[
\int_{V'} dV \frac{(\nabla \times \tilde{u}) \cdot \nabla s}{\rho} = \int_{gV'} dV \frac{(\nabla \times \tilde{u}') \cdot \nabla s'}{\rho'} \tag{4.12}
\]

The law of conservation of potential vorticity has the following form:

**THEOREM 4.5**

Let \(\tilde{x}(t)\) be the trajectory of a fluid particle and \(g\) be the flow map

\[g : \tilde{x}(0) \rightarrow \tilde{x}(t)\]

Then the fields \([\rho(\tilde{x},0), \tilde{u}(\tilde{x},0), \tilde{s}(\tilde{x},0)]\) and \([\rho(\tilde{x},t), \tilde{u}(\tilde{x},t), \tilde{s}(\tilde{x},t)]\)
are equi-potential vorticity fields.

**PROOF:**

The proof follows from the conservation of potential vorticity.

Mathew and Vedan (1989) have stated the variational principle from which the equations of motion follows. Conversely we can state the theorem as follows:

**THEOREM 4.6**

In the case of non-barotropic flows

\[ W = \int_{\tau} d\tau \, L \]

is invariant under all hydromechanical transformations, \( \xi^\alpha \) vanishing on the boundary.

The action integral \( W \) is absolutely invariant under the Galilean transformation. This shows that the system has a first integral

\[ \int_V d\nu \, E , \quad (4.13) \]

where
\[ E = \frac{1}{2\rho^0} \left[ (p^4)^2 + (p^2)^2 + (p^0)^2 \right] + \varepsilon(p^0, \mathbf{s}^0) + \rho^0 U, \quad (4.14) \]

is the total energy. Hence the total energy is a constant of motion.

As in the case of barotropic flows the equations for non-barotropic flows form a system like (3.1). The steady state corresponds to equilibrium position of the system.

Following Arnold we give a structure to the space of \( p^\alpha \) and \( s^\alpha \) as follows:

Two fields belong to the same sheet if they are equi-potential vorticity fields. That is, two fields belong to the same sheet if there exists a transformation between them which leaves the potential vorticity invariant.

By theorem 4.5 this structure is invariant under the flow. As in the case of barotropic flows we can obtain steady state flow equations from the variational principle by considering three dimensional volume instead of four dimensional space considered by Mathew and Vedan (1989). Also we have obtained the conservation of potential vorticity. Now we consider the energy integral (4.13) of steady flow.
THEOREM 4.7

\[ \Delta \int_{V} dV \, E = 0, \]

\( \xi^i \) being the variations corresponding to which \( \eta^\alpha = 0 \) and \( \sigma^\alpha = 0 \) and vanishing on the boundary of \( V \).

PROOF:

\[ \Delta \int_{V} dV \, E = \int_{V} dV \, \partial_i \left( E \, \xi^i \right) \]

\[ = \int_{V} dV \, \partial_i \left\{ \left( \frac{1}{2\rho^0} \left( \nabla^2 \rho^0 \right)^2 + \left( \nabla^2 \eta^0 \right)^2 + \left( \nabla^2 \sigma^0 \right)^2 \right) + \varepsilon \left( p^0, s^0 \right) + p^0 u \right\} \xi^i \]

\[ = 0 \]

Taking \( \vec{\xi} = \frac{1}{\rho} \nabla \times \nabla \phi \), where \( \vec{\xi} = (\xi^1, \xi^2, \xi^3) \), we get the following result:

THEOREM 4.8:

\( E \) has stationary value for steady flows compared to all close equipotential vorticity flows.

PROOF:

Proof follows from theorem 4.7 with \( \vec{\xi} \) given by (4.3).
4.4 STABILITY OF NON-BAROTROPIC FLOWS

Using the variational principle (Chapter 2) developed by Mathew and Vedan (1989) we have found out the $\xi^\alpha$ which corresponds to the invariance of potential vorticity. Among all fields $\rho^\alpha, s^\alpha$ and which correspond to a constant potential vorticity, steady flow has an extremum for the total energy. We have to find out the second variation of the energy integral (4.13) to study the stability of non-barotropic flows. If it is of definite sign the flow is stable.

Let

$$J = \int_V dV \, E,$$

where $E$ is given by equation (4.14). Then the second variation of the functional $J$ is

$$\Delta^2 J = \Delta \int_V dV \, \partial_i (E \xi^i).$$

Without writing the derivation we give the simplified form of the variation as

$$\Delta^2 J = \int_V dV \left\{ \left( \bar{\omega} \cdot \nabla \right) \delta U + \frac{i}{\rho} \left( \rho' - u_j^i u_j \right) \left( \delta \rho \right)^2 - 2u_j \delta u_j^i \delta \rho \right.$$

$$+ \frac{i}{\rho} \delta \left( \rho u_j^i \right) \delta (\rho u_j) + \rho u_j \delta^2 u_j^i + \rho \frac{\partial}{\partial x_k} \left( \frac{i}{\rho^2} \omega \delta \rho \right) \right\},$$

(4.15)
where pressure \( p = p^0 \frac{\partial \epsilon}{\partial p^0} + s^0 \frac{\partial \epsilon}{\partial s^0} - \epsilon \).

The integrand in the second variation has the same form as for the barotropic flow. But it is to be noted that the variations to be considered are different as they correspond to equi-potential vorticity flows.

As pointed out in the beginning the study of stability of non-barotropic flows is still in the initial stages. Though we are not giving any specific examples, as for barotropic flows the role of vortex stretching seems to have a significant role in destabilizing flows.

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