CONCLUSION
The results obtained in this thesis can now be summarized as follows:

Arnold's method for stability study is based on a suitable variational formulation for fluid flows. Following Drobot and Rybarski, Mathew and Vedan, Joseph and Vedan have studied the variational formulations of barotropic and non-barotropic flows. The use of a Euclidean space $X_4$ to represent the space-time configuration space of system leads to a systematic method for deriving governing equations for fluid flows form a suitable action integral.

In Lagrangian approach the configuration space is essentially Riemannian and not Euclidean. But the curved Riemannian space flattens out more and more if we restrict ourselves to smaller and smaller region. This is the case when we consider the 4-dimensional manifold $X_4$.

In the case of Hamiltonian formulation the phase space is Euclidean. The concept of phase flow is based on the motion of a system in the phase space. This motion is, in terms of hydrodynamics, a Lagrangian description while the Liouville's theorem for phase flow is based on Eulerian equation of
continuity. On the basis of this analogy it is natural to expect a simpler theory of fields in Lagrangian and Hamiltonian formulation for hydrodynamics compared to other physical systems. It seems that the Lagrangian and Hamiltonian formulations of fluid dynamics obtained above can be justified in this sense and the evolution equations written in terms of material derivative in chapter 2 can be considered finite dimensional.

Poisson bracket formulation of field theory is not carried out in step by step correspondence with that for discrete systems. For example, Poisson bracket in field theory are defined only in terms of a pair of densities. A way for doing this is to define Poisson bracket as an integral, the integrand being variational derivatives. But Arnold uses the Poisson bracket with the gradients of the functions

\[
\frac{\partial r}{\partial t} = \{\mathbf{v}, r\},
\]

where \{A B\} is the Poisson bracket of the vector fields defined by

\[
[A B]_i = \sum (\frac{\partial A_i}{\partial x_j} B_j - \frac{\partial B_i}{\partial x_j} A_j)
\]

This can be compared to the Hamiltonian system we have obtained in chapter 2.

The equilibrium solution of the equations of non-dissipative continuum mechanics are usually found by minimizing appropriate variational integral. However, when presented with a dynamical problem one encounters systems of
evolution equations for which the Lagrangian viewpoint, even if applicable is no longer appropriate or natural to the problem. In this case, the Hamiltonian formulation of systems of evolution equations assumes the natural variational role for the system. The excessive reliance on canonical coordinates guaranteed by the Darboux theorem in finite dimensions, is no longer valid for the evolution equations. The Poisson bracket approach generalizes in this context.

The Poisson brackets of the Hamiltonian system of two dimensional incompressible inviscid flow, two dimensional barotropic flow and three dimensional adiabatic (non-barotropic) flow are given in Holm et al. (1985). Here we note that for barotropic and non-barotropic flows the Poisson brackets are defined in terms of the variable $p^\alpha$ of Drobot and Rybarski (1959). These Hamiltonian structures are used by them in the stability studies.

But it is to be noted that the Hamiltonian structure is used only for obtaining integrals of motion in the study of stability. Instead the variational formulation developed by Mathew, Joseph and Vedan can be used to get known conservation laws of motion and the corresponding infinitesimal generators can be used to define flows with given constants of motion. These are used to define concepts like equivorticity used by Arnold.
In the case of two dimensional flows it is shown by Arnold (1965) that a suitable combination of two integrals of motion, being a first integral, has all the properties of a Liapunov function in a suitable metric and may be used to establish stability in an exact nonlinear sense. In the case of three dimensional flow the conservation of vorticity does not permit the construction of a Liapunov function, instead he uses the property that a stationary flow possesses an extremum in kinetic energy with respect to the variations of velocity fields with the same prescribed vorticity. Arnold has proved this for incompressible flow. Later Grinfeld has generalized this to the case of inviscid barotropic flow in a potential field. He uses only the condition on constancy of sign to establish sufficient condition for stability of flow. The formula for the second variation of fields of equivorticity is derived and used in stability analysis.

In chapter 3, we have generalized the result of Joseph in finding the infinitesimal generator for the variational principle from which the conservation of helicity follows. The stability criterion obtained refers only to formal stability but shows that the conditions obtained by Grinfeld in his example may not be sufficient for stability.

In chapter 4 again we have obtained infinitesimal generator for variational symmetry. This leads to conservation of potential vorticity in the case of non-barotropic flows. The
stability criterion is obtained.

Our studies point to a new direction for stability studies based on Lagrangian formulation instead of the Hamiltonian formulation used by other authors. The role and applicability of Arnold's method are being widely discussed in the literature. It is interesting to note that after more than two decades Rouchon (1991) has given a mathematical proof of a remark by Arnold (1965) that nonlinear stability criterion for steady state solutions for incompressible equations is never satisfied when three dimensional rather than two dimensional perturbations are considered. A stronger mathematical foundation for Arnold's method and deeper investigation into its applicability are challenging open problems.

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