Chapter 2

DOMINATING SETS IN
BIPARTITE GRAPHS

Given a graph $G$, there are many ways to construct bipartite graphs using the vertex set and edge set of $G$. Some of the ways are described in the previous chapter. The problem of determining which bipartite graphs are realizable as $VE$ graph or $EV$ graph or $VV$ graph of some graph is equally interesting. This problem is discussed in the first section of this chapter. Results on $X$-domination, $Y$-domination, $X$-irredundant and strong non-split domination are also studied in this chapter.
2.1 Introduction

In the bipartite theory of graphs developed in [18, 19], many equivalent formulations of the concepts in graphs were suggested for bipartite graphs. Some of them are $X$-domination, $Y$-domination, $X$-independence and hyper independence, etc. This chapter is devoted to a detailed study of these new notions in bipartite graphs. The concept of $X$-irredundant set is introduced and studied in this chapter. Strong non-split domination in graphs is extended to bipartite graphs as strong nonsplit $X$-domination. $X$-vertex critical graphs are also studied.

2.2 Realization of a bipartite graph as $VE$, $EV$ and $VV$ of a graph

Definition 2.2.1. Let $G$ be a bipartite graph. $G$ is said to be a vertex edge bipartite realizable graph ($VE$-realizable graph) if $G = VE(H)$ for some graph $H$.

Remark 2.2.2. If $G = C_4$ and $G = VE(H)$, then $H$ is a multigraph.

Proposition 2.2.3. A bipartite graph $G \neq C_4$, $G = (X, Y)$ is a $VE$ realizable graph if and only if every vertex of $Y$ is of degree 2.
Proof: Let $G$ be a bipartite graph. Let $G \neq C_4$ and every vertex in $Y$ is of degree 2. Consider the graph $H$ with $V(H) = X$ and $E(H) = \{uv : v, u \in X, u$ and $v$ are $X-$adjacent $\}$. Since $G \neq C_4$, $H$ is a simple graph. Since every vertex in $Y$ is of degree 2, every vertex in $Y$ corresponds to an edge in $H$. It can be easily verified that $G = VE(H)$.

Conversely, Suppose $G = VE(H)$, where $H$ is a simple graph. Any edge in $H$ is incident with two vertices of $H$. Therefore, every vertex $y \in Y$ of $G$ is of degree 2.

Definition 2.2.4. Let $G$ be a bipartite graph. $G$ is said to be edge vertex bipartite realizable graph (EV realizable graph) if $G = EV(H)$ for some graph $H$.

Proposition 2.2.5. A bipartite graph $G \neq C_4$, $G = (X,Y)$ is a EV realizable graph if and only if every vertex of $X$ is of degree 2.

Proof: Let $G$ be a bipartite graph. Let $G \neq C_4$ and every vertex in $X$ is of degree 2. Consider the graph $H$ with $V(H) = Y$ and $E(H) = \{uv : v, u \in Y, u$ and $v$ are adjacent to a vertex of $X \}$. Since $G \neq C_4$, $H$ is a simple graph. Since every vertex in $X$ is of degree 2, every vertex in $X$ corresponds to an edge in $H$. It can be easily verified that $G = EV(H)$. 

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Conversely, suppose $G = EV(H)$. Then $H$ is a simple graph. Any edge in $H$ is incident with two vertices of $H$. Therefore, every vertex $x \in X$ of bipartite graph $G$ is 2.

**Definition 2.2.6.** Let $G$ be a bipartite graph. $G$ is said to be Vertex-vertex bipartite realizable graph (VV realizable graph) if $G = VV(H)$ for some graph $H$.

**Theorem 2.2.7.** [12] A graph $G$ is VV realizable if and only if the following conditions are true.

1. $G$ has an even number of vertices (and edges if we want $G$ to be duplicate of a graph without loops).

2. $G$ has no odd cycles.

3. Let $G$ be connected. If $G$ contains a cycle $C_{2n} : a_1a_2 \cdots a_{2n}a_1$ where $n > 1$ is odd then $G$ should be symmetrical about this cycle in the sense that if there is a branch at a point $a_k(k \leq n)$ of the cycle, then there should be an isomorphic branch (not necessarily distinct) at the point $a_{n+k}$ of the cycle. Further, if these branches are one and the same then this branch should be the VV-graph of some graph.

In other cases (that is, when $G$ has only cycles of the form $C_{2n}$ where $n$ is even or when $G$ is a tree) it should be possible in $G$ to remove a number of lines each joining two similar points such that the removal of these edges results in two isomorphic components of $G$. 

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4. If $G$ is disconnected, then the number of components of $G$ which are not of the types mentioned as above should be even and these should be in isomorphic pairs.

2.3 $X$-Dominating set and $Y$-Dominating set

Definition 2.3.1. [18] A Subset $D$ of $X$ is an $X$-dominating set if every vertex in $X - D$ is $X$-adjacent to at least one vertex in $D$.

Definition 2.3.2.

Let $x \in X$. $N_Y(x) = \{u \in X \mid x \text{ and } u \text{ are } X\text{-adjacent}\}$. Elements in the set $N_Y(x)$ are called $X$-neighbors of $x \in X$. Let $S \subseteq X$ and $N_Y(S) = \bigcup N_Y(x)$ where $x \in S$.

The following are equivalent definitions:

A subset $S \subseteq X$ is a $X$-dominating set if and only if

1. for every $x \in X - S$, there exists $u \in S$ and $y \in Y$ such that $x$ and $u$ are adjacent to $y \in Y$.

2. for every $x \in X - S$, $d(x, S) = 2$, where $d(x, S) = \min\{d(x, u) : v \in S\}$.


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4. for every \( x \in X - S \), \( |N_Y(x) \cap S| \geq 1 \).

5. for every \( x \in X \), \( |N_Y[x] \cap S| \geq 1 \).

**Remark 2.3.3.** Any superset of a \( X \)-dominating set is a \( X \)-dominating set.

**Definition 2.3.4.** [18] A \( X \)-dominating set \( S \) is a minimal \( X \)-dominating set if no proper subset of \( S \) is \( X \)-dominating set. The minimum cardinality of a minimal \( X \)-dominating set is called the \( X \)-domination number of \( G \) and is denoted by \( \gamma_X(G) \). The maximum cardinality of a minimal \( X \)-dominating set of \( G \) is called upper \( X \)-domination number and is denoted by \( \Gamma_X(G) \).

**Definition 2.3.5.** Let \( S \subseteq X \) and let \( u \in S \). \( v \in X - S \) is called a \( Y \)-private neighbor of \( u \) with respect to \( S \) if \( u \) is the only point in \( S \) such that \( u \) and \( v \) have common adjacent point in \( Y \).

**Definition 2.3.6.** Let \( S \subseteq X \). Let \( u \in S \). \( u \) is called an \( Y \)-isolate of \( S \) if there exists no adjacent vertex \( v \in S - \{u\} \) such that \( u \) and \( v \) have a common point in \( Y \).

**Theorem 2.3.7.** A subset \( S \subseteq X \) is a minimal \( X \)-dominating set if and only if for every \( u \in S \) one of the following conditions is satisfied:

(i) \( u \) is an \( Y \)-isolate of \( S \).

(ii) there exists a \( v \in X - S \) such that \( v \) is a \( Y \)-private neighbor of \( u \) with respect to \( S \).

**Proof:** Let \( S \) be a minimal \( X \)-dominating set of \( G \) and let \( u \in S \). Then \( S - \{u\} \) is not a \( X \)-dominating set. Hence, some vertex \( v \) in \( X - (S - \{u\}) \) is
not $X$-adjacent to any vertex in $S - \{u\}$. Then either $v = u$ in which case $u$ is $Y$-isolate of $S$ which is condition (i) or $v \in X - S$ and $v$ is not $X$-adjacent to any vertex of $X - (S - \{u\})$. That is $v$ is a $Y$-private neighbor of $u$ which is (ii).

Let us assume that $S$ is not a minimal $X$-dominating set. There exists a vertex $u \in S$ such that $S - \{u\}$ is a $X$-dominating set. Hence $u$ is $X$-adjacent to at least one vertex in $S - \{u\}$, and so condition (i) does not hold for $S$. Also every vertex in $X - S$ is $X$-adjacent to at least one vertex in $S - \{u\}$, and so condition (ii) does not hold for $u$. Thus $S$ does not satisfy (i) and (ii).

**Remark 2.3.8.** Complement of a $X$-dominating set need not be a $X$-dominating set.

For, in the following example, the set $S = \{b, c, d\}$ is a $X$-dominating set but the complement of $S$ is not a $X$-dominating set.
Theorem 2.3.9. Let $G$ be a bipartite graph with $N_Y(x) \neq \phi$ for every $x \in X$. Then the complement $X - S$ of every minimal $X$-dominating set of $G$ is a $X$-dominating set of $G$.

Proof: Let $S$ be a minimal $X$-dominating set of $G$. Assume that $u \in S$ is not $X$-adjacent to any vertex in $X - S$. By hypothesis, $N_Y(x) \neq \phi$ for every $x \in X$. Therefore, $u$ must be $X$-adjacent to at least one vertex in $S - \{u\}$. Thus, $S - \{u\}$ is a $X$-dominating set contradicting the minimality of $S$. Thus every vertex in $S$ is $X$-dominated by at least one vertex in $X - S$ and hence $X - S$ is a $X$-dominating set. ■

Corollary 2.3.10. Let $S$ be a minimal $X$-dominating set. $(X - S) \cup D$ where $D = \{x \in S/N_Y(x) = \phi\}$ is a $X$-dominating set.

$X$-domination number of certain standard graphs

Proposition 2.3.11. Let $G$ be a graph with $N_Y(x) \neq \phi$ for every $x \in X$. If there exists a vertex $y \in Y$ of degree $p$ then $\gamma_X(G) = 1$.

Proof: Let $y \in Y$ be a vertex of degree $p$. Then $y$ is adjacent to every vertex of $X$. Every vertex in $X$ is $X$-adjacent to all other vertices of $X$. Therefore $\gamma_X(G) = 1$. ■

Corollary 2.3.12. $\gamma_X(K_{m,n}) = 1$.
Proposition 2.3.13. The X-domination number of \( C_{6k} \) is \( k \).

Proof: Let the vertices of \( G \) be \( x_1, x_2, \ldots, x_{6k} \). Let \( X = \{x_1, x_3, \ldots, x_{6k-1}\} \); \( Y = \{x_2, x_4, \ldots, x_{6k}\} \). Edges are \( e_i = x_ix_{i+1} \) and \( e_0 = x_{6k}x_1 \). The graph \( H \) is obtained from \( G \) by taking \( V(H) = X \) and two vertices of \( H \) are adjacent if and only if they are \( X \)-adjacent in \( G \). Then \( \gamma_X(G) = \gamma(H) = k \) since \( H \) is a cycle on \( 3k \) vertices. Therefore we get \( \gamma_X(C_{6k}) = k \). \( \blacksquare \)

Proposition 2.3.14. \( \gamma_X(C_l) = \lceil l/6 \rceil \) for every \( l \geq 4, l \equiv 0, 2, 4(\text{mod} 6) \).

Proof: Routine.

Bounds on X-domination number

Let \( d_Y(x) \) denote the number of vertices \( X \)-adjacent to \( x \in X \). Let \( \Delta_Y \) denote \( \max \{d_Y(x) / x \in X\} \).

Proposition 2.3.15. Let \( G \) be a bipartite graph with \( p \) vertices and \( q \) edges. Then \( p - q + 1 \leq \gamma_X \leq p - \Delta_Y \).

Proof: Let \( S \) be a \( X \)-dominating set with \( \gamma_X \) vertices. Therefore, \( |X - S| = p - \gamma_X \). Then there are at least \( p - \gamma_X + 1 \) edges from \( X - S \) to \( S \). Hence \( p - \gamma_X + 1 \leq q \). Therefore, \( p - q + 1 \leq \gamma_X \).

Let \( x \in X \) be a vertex such that \( d_Y(x) = \Delta_Y \). Clearly \( X - N_Y(x) \) is a \( X \)-dominating set and therefore, \( \gamma_X \leq p - \Delta_Y \). Therefore, \( p - q + 1 \leq \gamma_X \leq p - \Delta_Y \). \( \blacksquare \)
Proposition 2.3.16. Let $G$ be a graph with $N_Y(x) \neq \phi$ for every $x \in X$. Then $\gamma_X \leq |X|/2$.

Proof: Let $S$ be a $\gamma_X$ set of $G$. Then $X - S$ is a $X$-dominating set. Therefore,

$$\gamma_X \leq |X - S| \leq |X| - \gamma_X \Rightarrow \gamma_X \leq |X|/2.$$

Proposition 2.3.17. Let $G$ be a graph with $N_Y(x) \neq \phi$ for every $x \in X$. Then $[p/(\Delta_Y + 1)] \leq \gamma_X \leq p - \beta_X$.

Proof: Let $S$ be a maximum $X$-independent set of $G$. Then every vertex in $S$ is $X$-adjacent to some vertex in $X - S$, since $N_Y(x) \neq \phi$ for every $x \in X$. Hence, $X - S$ is a $X$-dominating set with cardinality $p - \beta_X$. Therefore,

$$\gamma_X \leq p - \beta_X.$$

Let $S$ be a $X$-dominating set of cardinality $\gamma_X$ and let $S = \{v_1, v_2, \ldots v_{\gamma_X}\}$. Since every vertex in $X - S$ is $X$-adjacent to some vertex in $S$, we have

$$|X - S| \leq \sum_{i=1}^{\gamma_X} d_Y(v_i) \leq \gamma_X \Delta_Y.$$  

Hence, $p - \gamma_X \leq \gamma_X \Delta_Y$. That is, $p \leq (\Delta_Y + 1)\gamma_X$, which implies $[p/(\Delta_Y + 1)] \leq \gamma_X$. Thus, $[p/(\Delta_Y + 1)] \leq \gamma_X \leq p - \beta_X$.

Proposition 2.3.18. [11] In a bipartite graph, every maximal $X$-independent set is a minimal $X$-dominating set.

Definition 2.3.19. A subset $S$ of $X$ which is $X$-independent and $X$-dominating is called $X$-independent $X$-dominating set.
The existence of such a set is guaranteed by the above proposition.

**Definition 2.3.20.** A $X$-independent, $X$-dominating set of minimum cardinality is called $X$-independent, $X$-domination number of a graph $G$ and is denoted by $i_X(G)$.

Clearly $i_X(G) \leq \beta_X(G)$. Thus we have $\gamma_X(G) \leq i_X(G) \leq \beta_X(G) \leq \Gamma_X(G)$.

**Hyper $X$-independent set**

**Definition 2.3.21.** Let $G$ be a bipartite graph. A subset $S$ of $X$ is hyper $X$-independent if $N_Y(x) \subseteq S$ for every $x \in S$. The maximum cardinality of a hyper $X$-independent set is denoted by $\beta_{hX}(G)$.

**Proposition 2.3.22.** In a bipartite graph, every hyper independent set is hyper $X$-independent.

**Proof:** Let $S$ be a hyper independent set of $G$. Suppose $S$ is not hyper $X$-independent set. Then there exists $x \in S$ such that $N_Y(x) \subseteq S$. Let $y \in N(x)$. Let $z \in N(y)$. If $z = x$ then $z \in S$. If $z \neq x$, then $z \in N_Y(x) \subseteq S$. Therefore, $N(y) \subseteq S$, a contradiction.

Since $S$ is a hyper independent set.

**Remark 2.3.23.** The converse of the above need not be true.

$S = \{a, d\}$ is hyper $X$-independent set but not a hyper independent set since $N(e) \subseteq S$.  

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Remark 2.3.24. $\beta_X(G) \leq \beta_h(G) \leq \beta_{hX}(G)$.

$Y$-dominating set

Definition 2.3.25. Let $G = (X, Y, E)$ be a bipartite graph. A subset $S \subseteq X$ which dominates all vertices in $Y$ is called a $Y$-dominating set of $G$. The $Y$-domination number denoted by $\gamma_Y(G)$ is the minimum cardinality of a $Y$-dominating set of $G$.

Proposition 2.3.26. Let $G$ have no isolates. Every $Y$-dominating set is a $X$-dominating set.

Proof: Let $S \subseteq X$ be a $Y$-dominating set. The set of vertices in $S$ are adjacent to vertices of $Y$. Let $x \in X - S$. There exists an edge incident with $x \in X - S$ and $y \in Y$. Since $y \in Y$ is adjacent to a vertex in $S$. $S$ is an $X$-dominating set. \qed

Remark 2.3.27. An $X$-dominating set need not be a $Y$-dominating set. Consider the graph $G$, $\{b\}$ is a $X$-dominating set but not a $Y$-dominating set.
Remark 2.3.28. \( \gamma_X(G) \leq \gamma_Y(G) \leq \gamma(G) \).

Gallai type Theorem

Theorem 2.3.29. Let \( G \) be a bipartite graph. A subset \( D \) of \( X \) is \( Y \)-dominating set if and only if \( X - D \) is hyper independent set of \( G \).

Proof: Let \( D \subseteq X \) be a \( Y \)-dominating set of \( G \). Every vertex of \( Y \) is adjacent to atleast one vertex in \( D \). Clearly, \( N(y) \) is not contained in \( X - D \) for every \( y \in Y \). Therefore, \( X - D \) is hyper independent set.

Conversely, let \( D \) be a hyper independent set. \( N(y) \) is not contained in \( D \) for every \( y \in Y \). Hence, every \( y \in Y \) is adjacent to at least one vertex of \( X - D \). Hence, \( X - D \) is a \( Y \)-dominating set of \( G \). \( \blacksquare \).

Corollary 2.3.30. Let \( G \) be a bipartite graph with \( |X| = p \) then \( \gamma_Y(G) + \beta_h(G) = p \).

Proof: Let \( S \) be a \( Y \)-dominating set. Then \( X - S \) is hyper independent set.

\( \beta_h \geq p - \gamma_Y \Rightarrow \beta_h + \gamma_Y \geq p \).
Conversely, let $S$ be a maximum hyper independent set. Then $X - S$ is a $Y$-dominating set. Therefore, $\gamma_Y(G) \leq p - \beta_h(G) \Rightarrow \gamma_Y(G) + \beta_h(G) \leq p$.

Hence, $\gamma_Y(G) + \beta_h(G) = p$. $\blacksquare$

**Theorem 2.3.31.** For any graph $G$, $\gamma(G) = \gamma_t(G)$ if and only if there exists a $\gamma$ set $S$ in $G$ such that $V - S$ is hyper independent set in the graph $VV(G)$.

**Proof:** Let $S$ be a $\gamma$ set of $G$ such that $V - S$ is hyper independent set in graph $VV(G)$. Then $S$ is a $Y$-dominating set in graph $VV(G)$. In $G$, every vertex is adjacent to a vertex of $S$. Hence, $S$ is a total dominating set. Thus, $\gamma_t(G) \leq |S| = \gamma(G)$. $\gamma(G) \leq \gamma_t(G)$. Therefore, $\gamma_t(G) = \gamma(G)$.

Conversely, assume $\gamma(G) = \gamma_t(G)$. Let $S$ be a $\gamma_t(G)$-set in $G$. Then $S$ is a $Y$-dominating set in graph $VV(G)$. Therefore, $V - S$ is hyper independent set in the graph $VV(G)$. Therefore, we have a $\gamma$ set $S$ in $G$ such that $V - S$ is hyper independent set in $VV(G)$. $\blacksquare$

**Proposition 2.3.32.** For any graph $G$, every total dominating set in $G$ is a dominating set in the graph $G_2$.

**Proof:** Let $S$ be a total dominating set in $G$. Let $x \in V - S$. Then there exists $u \in S$ such that $x$ and $u$ are adjacent in $G$. Since $S$ is a total dominating set in $G$, there exists $v \in S$ such that $u$ and $v$ are adjacent in
G. Hence, in $G2$, $v$ and $x$ are adjacent. Therefore, $S$ is a dominating set in graph $G2$.

**Theorem 2.3.33.** For any graph $G$, $\gamma(G2) = \gamma_1(G)$ if and only if there exists a $\gamma(G2)$ set $S$ such that $V - S$ is hyper independent set in graph $V(V(G))$.

**Theorem 2.3.34.** For any graph $G$, $\gamma_1(G) = \alpha_1(G)$ if and only if there exists a $\gamma_1$ set $S$ such that $E - S$ is hyper independent set in the graph $E(V(G))$.

**Proof:** Let $S$ be a $\gamma_1$ set of $G$ such that $E - S$ is hyper independent set in the graph $E(V(G))$. Then $S$ is a $Y$-dominating set in graph $E(V(G))$. In $G$, that is $S$ is a set of edges which cover all vertices in $G$. Therefore, $S$ is an edge covering set. Hence, $\alpha_1 \leq |S| = \gamma_1(G)$. But $\gamma_1(G) \leq \alpha_1(G)$. Therefore, $\gamma_1(G) = \alpha_1(G)$.

Conversely, let $\gamma_1(G) = \alpha_1(G)$. Let $S$ be a $\alpha_1$ set of $G$. Then $S$ is a $Y$-dominating set in graph $E(V(G))$. Then $E - S$ is hyper independent set of $E(V(G))$ and every edge covering set is an edge dominating set. Therefore, there exists an edge dominating set $S$ such that $E - S$ is hyper independent set in the graph $E(V(G))$.

**Proposition 2.3.35.** For any graph $G$, let $S \subseteq X(G)$ be a distance 2-dominating set of the subdivision graph of $G$. Then $S$ is a dominating set of $G$.  

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Proof: Let $S = \{u_1, u_2, \ldots, u_r\} \subseteq V(G)$ be a distance 2-dominating set of $S(G)$. Let $e_i \in E(G)$ and let $e_i = xy$. Suppose $x, y \notin S$. Then the subdivided vertex $x^1$ of $xy$ cannot be distance 2-dominated by $S$. Therefore, $x$ or $y \in S$. That is, $S$ is a $Y$-dominating set of the graph $VE(G)$. Therefore, every edge in $G$ is incident with at least one vertex of $S$. Hence, in $G$ every vertex of $V - S$ is adjacent to at least one vertex of $S$. Hence, $S$ is a dominating set of $G$.

Corollary 2.3.36. For any graph $G$, $\gamma(G) \leq \gamma_2(S(G))$.

Proof: Let $S$ be a minimum distance 2-dominating set of $S(G)$ such that $S \subseteq V(G)$. Then $S$ is a dominating set of $G$.

Proposition 2.3.37. Any $Y$-dominating set of $VE(G)$ is a distance 2-dominating set of $S(G)$ contained in $V(G)$ and conversely.

Proof: Let $S$ be a $Y$-dominating set of $VE(G)$. Then $S \subseteq V(G)$. If $S = V(G)$, then clearly, $S$ is a dominating set of $S(G)$ and hence a distance 2-dominating set of $S(G)$. Let $S \subseteq V(G)$. Let $x \in V(G) - S$. Let $e = xy$. Since $S$ is a $Y$-dominating set of $VE(G)$, $y \in S$. Therefore, in $S(G)$, $d(x, y) = 2$. Hence $y \in S$ distance 2-dominates $x$. Let $w^1 \in V(S(G))$ where $w^1$ is the subdivided vertex of an edge $e = uv$. Since, $S$ is a $Y$-dominating set of
$G$, $u$ or $v \in S$. Therefore, $S$ dominates $W^1$. Therefore, $S$ is a distance 2-dominating set of $S(G)$.

Conversely, let $e = uv \in E(G)$. Suppose $u, v \notin T$. Let $u^1$ be the subdivided vertex of $uv$, there exists $x \in T$ such that $d(x, u^1) \leq 2$. Since $x \in V(G)$, $d(x, u^1) = 1$. Therefore, $x = u$ or $v$, a contradiction. Therefore, $u$ or $v \in T$. Therefore, $T$ is a $Y$-dominating set of $VE(G)$.

Theorem 2.3.38. For any graph $G$, $\gamma_2(G) = \gamma(G)$ if and only if there exists a $\gamma_2$ set $S$ in $G$ such that $V - S$ is hyper independent set in graph $VV^+(G)$

Proof: Let $S$ be a $\gamma_2$ set in $G$ such that $V - S$ is hyper independent set in the graph $VV^+(G)$. Then, $S$ is a $Y$-dominating set in the graph $VV^+(G)$. Therefore, $S$ is a dominating set in $G$. Therefore, $\gamma(G) \leq |S| = \gamma_2(G)$.

Hence, $\gamma_2(G) = \gamma(G)$.

Conversely, let $S$ be a minimum dominating set in $G$. Then $S$ is a $Y$-dominating set in $VV^+(G)$. Hence, $V - S$ is a hyper independent set in $VV^+(G)$. But every $\gamma$ set of $G$ is a $\gamma_2$ set of $G$.

Therefore, there exists a $\gamma_2$ set $S$ of $G$ such that $V - S$ is hyper independent set in the graph $VV^+(G)$.

Theorem 2.3.39. Let $G$ be a bipartite graph. A subset $D$ of $X$ is $X$-
dominating set if and only if $X - D$ is hyper $X$-independent set.

**Proof:** Let $D$ be a $X$-dominating set. For every $x \in X - D$, there exists $x_1$ in $D$ such that $x$ and $x_1$ are $X$-adjacent. Hence, $N_Y(x)$ is not a subset of $X - D$ for every $x \in X - D$. Therefore, $X - D$ is a hyper $X$-independent set.

Conversely, let $S$ be a hyper $X$-independent set. Then $N_Y(x)$ is not a subset of $S$ for every $x \in S$. Equivalently for every $x \in S$, there exists $x_1$ in $X - S$ such that $x$ and $x_1$ are $X$-adjacent. Hence, $X - S$ is a $X$-dominating set.

**Corollary 2.3.40.** Let $G$ be a bipartite graph with $|X| = p$. Then $\gamma_X(G) + \beta_{hX}(G) = p$.

**Proof:** Let $D$ be a $\gamma_X$ set of $G$. Then $X - D$ is hyper $X$-independent set. Therefore, $\beta_{hX} \geq p - \gamma_X \Rightarrow \gamma_X + \beta_{hX} \geq p$.

Conversely, if $S$ is a $\beta_{hX}$ set of $G$, then $X - S$ is a $X$-dominating set. Therefore, $\gamma_X \leq p - \beta_{hX} \Rightarrow \gamma_X + \beta_{hX} \leq p$. Hence, $\gamma_X + \beta_{hX} = p$.

**Proposition 2.3.41.** For any graph $G$, every dominating set in graph $G^2$ is a dominating set in $G^2$.

**Proof:** Let $S$ be a dominating set of $G^2$. For every $x \in V - S$, there exists $y \in S$ such that $x$ and $y$ are adjacent in $G^2$. Therefore, in $G$, $x$ and $y$ have
a common vertex. Thus, \( d(x, S) \leq 2 \) for every \( x \in V - S \) in \( G \). Hence, \( S \) is a dominating set in \( G^2 \).

**Corollary 2.3.42.** \( \gamma(G^2) \leq \gamma(G2) \).

**Theorem 2.3.43.** For any graph \( G \), \( \gamma(G^2) = \gamma(G2) \) if and only if there exists a \( \gamma(G^2) \) set \( S \) such that \( V - S \) is hyper \( X \)-independent set in \( VV(G) \).

**Proof:** Let \( S \) be a \( \gamma(G^2) \) set such that \( V - S \) is a hyper \( X \)-independent set of \( VV(G) \). Then \( S \) is a \( X \)-dominating set in \( VV(G) \). Let \( X,Y \) be the bipartition of \( VV(G) \). For every \( x \in X - S \), there exists \( u \in S \) such that \( x \) and \( u \) are \( X \)-adjacent. In \( G \), \( x \) and \( u \) are incident at a common vertex. Hence, in \( G2 \), \( x \) and \( u \) are adjacent. Therefore, for every \( x \in V - S \) there exists \( u \in S \) such that \( x \) and \( u \) are adjacent. Thus, \( S \) is a dominating set in \( G2 \). Therefore, \( \gamma(G2) \leq |S| = \gamma(G^2) \). Hence, \( \gamma(G2) = \gamma(G^2) \).

Conversely, let \( S \) be a \( \gamma- \) set of \( G2 \). Then, \( S \) is a \( X \)-dominating set in the graph \( VV(G) \). Therefore, \( V - S \) is hyper \( X \)-independent set. Every dominating set in \( G2 \) is a dominating set in \( G^2 \) and \( \gamma(G2) = \gamma(G^2) \). Hence, there exists a \( \gamma(G^2) \) set \( S \), such that \( V - S \) is hyper \( X \)-independent set in \( VV(G) \).

**Remark 2.3.44.** For any graph \( G \), every dominating set in graph \( G^2 \) is a
distance 2-dominating set in $G$.

**Theorem 2.3.45.** Any dominating set of $G^2$ is a $X$-dominating set in $VV^+(G)$ and conversely.

**Proof:** Let $S$ be a dominating set of $G^2$. Then, for any $y \in V - S$, there exists $x \in S$ such that $d(x, y) \leq 2$. Therefore, $S$ is a $X$-dominating set in $VV^+(G)$.

Conversely, let $S$ be a $X$-dominating set of $VV^+(G)$. Then for any $y \in V - S$, there exists $x \in S$ such that $d(x, y) \leq 2$. Therefore, $S$ is a dominating set of $G^2$.

**Corollary 2.3.46.** If $S$ is a dominating set of $G^2$, then $V - S$ is a hyper $X$-independent set in $VV^+(G)$ and conversely.

**Corollary 2.3.47.** $\gamma(G^2) = \beta_{hX}(VV^+(G))$.

**Sufficient Conditions of graphs for which $\gamma_X(G) = \frac{\gamma(G)}{2}$**

**Theorem 2.3.48.** Let $G$ be a bipartite graph with $|X| = p; |Y| = q; p \geq q \geq 2$ such that $d(G) = d(G)$. Then $\gamma_X(G) = \frac{\gamma(G)}{2}$.

**Proof:** Let $G$ be a bipartite graph with $d(G) = d(G)$. If there exists at least one vertex $y \in Y$ of degree $p$, then $\gamma_X(G) = 1$. Also, by theorem in ??, there
exists a vertex \( x \in X \) which is adjacent to all vertices in \( Y \). Hence, \( \{x, y\} \) forms a minimal dominating set of \( G \). Therefore \( \gamma_X(G) = \frac{\gamma(G)}{2} \).

If such a vertex \( y \in Y \) does not exist then by \( p \leq 2q - 1 \). Let \( M_i = \{x \in X / y_i \text{ does not belong to } N_G(x)\} \) \( 1 \leq i \leq q \). Then in the proof of theorem in ?? we obtain the following: \( M_i \neq \phi \) for \( 1 \leq i \leq q \) and \( M_i \) are pairwise disjoint subsets of \( X \) and since \( p \leq 2q - 1 \), there exist some \( j \in \{1, 2, 3, \cdots q\} \) such that \( |M_j| = 1 \). Let \( M_j = \{x\} \).

**Claim:** \( M_j \) is a \( X \)-dominating set.

Let \( x_1 \in X - \{x\} \). By theorem in ??, the degree of \( x_1 \) is at least \( q - 1 \). There exists some \( y_k \in Y, y_k \neq y_j \) such that \( x_1 \) is adjacent to \( y_k \). But \( x \) is adjacent to \( y_k \). Therefore \( \{x\} \) is a \( X \)-dominating set. \( \{x, y_j\} \) forms a dominating set of \( G \). Hence, \( \gamma_X(G) = \gamma(G)/2 \).

**Theorem 2.3.49.** Let \( G \) be a graph with no isolated vertex, \( |X| = p; |Y| = 2 \) and \( p \geq 2 \). If \( d_i(G) = d_i(G) \) then \( \gamma_X(G) = \frac{\gamma(G)}{2} \).

**Proof:** By theorem in ??, there exists exactly one vertex \( y_1 \) of degree \( p \). Hence, \( \gamma_X(G) = 1 \). Let \( y_2 \in Y \). Since \( y_2 \) is not isolated, there exists \( x \in X \) such that \( x \) and \( y_2 \) are adjacent. \( x \) is already adjacent to \( y_1 \). Therefore \( \gamma_Y(G) = 1 \). Hence, \( \gamma_X(G) = \gamma_Y(G) = 1 \).
\{y_1, y_2\} forms a minimum dominating set of \( G \). Therefore \( \gamma_X(G) = \gamma(G)/2 \).

Characterization of graphs for which \( \gamma_X(G) = \gamma_Y(G) \)

**Proposition 2.3.50.** Let \( G \) be a graph with no isolated vertex and let \( |X| = p; |Y| = q \). Suppose \( X \) has a vertex of degree \( q \). Then \( \gamma_X(G) = \gamma_Y(G) \).

**Proof:** Let \( x \in X \) be a vertex of degree \( q \). Then \( \gamma_Y(G) = 1 \). Since the vertices are not isolated, there is an edge between \( x_1 \in X - \{x\} \) and \( y_1 \in Y \). Already \( xy_1 \in E \). Hence, \( \{x\} \) is a \( X \)-dominating set. Therefore \( \gamma_X(G) = 1 \).

Hence, \( \gamma_X(D = \gamma_Y(G) \).

**Proposition 2.3.51.** Let \( G \) be a bipartite graph without isolated vertex with \( d(G) = d(\bar{G}) \) then \( \gamma_X(G) = \gamma_Y(G) \).

**Proof:** Let \( G \) be a bipartite graph with \( d(G) = d(\bar{G}) \). Using [1], number of vertices of \( X \) of degree \( q \) \( \geq \) number of vertices of \( Y \) of degree \( p \). Hence, \( \gamma_X(G) = \gamma_Y(G) \).

**Proposition 2.3.52.** If \( G \) is a graph without isolated vertices with \( |X| = p; |Y| = 2 \) such that \( d_L(G) = d_l(\bar{G}) \) then \( \gamma_X(G) = \gamma_Y(G) \).

**Proof:** Let \( G \) be a graph with \( d_L(G) = d_l(\bar{G}) \). [1] there exists a vertex of \( Y \) of degree \( p \). Therefore, \( \gamma_X(G) = \gamma_Y(G) \).
**Theorem 2.3.53.** Let $G = (X, Y, E)$ be a bipartite graph. $\gamma_X(G) = \gamma_Y(G)$ if and only if there exists a $\gamma_X$ set $S$ such that $X - S$ is hyper independent set.

**Proof:** Let $S$ be a $\gamma_X$ set such that $X - S$ is hyper independent set. Then $S$ is a $Y$-dominating set of $G$. Therefore, $\gamma_Y(G) \leq |S| = \gamma_X(G) \Rightarrow \gamma_Y(G) \leq \gamma_X(G)$. But $\gamma_X(G) \leq \gamma_Y(G)$. Therefore, $\gamma_X(G) = \gamma_Y(G)$.

Conversely, let us assume $\gamma_X(G) = \gamma_Y(G)$. Let $S$ be a $\gamma_Y$ set of $G$. Therefore, $X - S$ is hyper independent set of $G$. But every $Y$-dominating set is a $X$-dominating set and $\gamma_X(G) = \gamma_Y(G)$, it follows that $S$ is a minimum $X$-dominating set of $G$. \hfill \Box

**Proposition 2.3.54.** Let $G$ be a bipartite graph without isolated vertices with $p \geq q \geq 2$. Let $d(G) = d(G)$ and let $p \geq 2q - 1$. Then, $\gamma_X(G) = \gamma_Y(G)$.

**Proof:** By a Theorem 1.1.26 (proved in [1]) there exists a vertex in $Y$ of degree $p$ and a vertex in $X$ of degree $q$. Hence, $\gamma_X(G) = 1 = \gamma_Y(G)$. \hfill \Box

**Remark 2.3.55.** Suppose $p \leq 2q - 1$. If $q = 2$, the graph satisfying the hypothesis and with no vertex in $X$ of degree $q$ are
In both these graphs \( \gamma_X(G) = 2 = \gamma_Y(G) \).

Let \( q = 3 \) and \( p = 5 \). Consider the graph \( G \):

In this graph, \( d(G) = d(\overline{G}) \), \( G \) and \( \overline{G} \) have no isolated vertices but \( \gamma_X(G) = 1 \neq \gamma_Y(G) = 2 \).

Characterization of bipartite graphs for which \( \gamma_X(G) = \gamma(G) \)

**Theorem 2.3.56.** Let \( G \) be a bipartite graph. Then \( \gamma_X(G) = \gamma(G) \) if and only if there exists a \( \gamma_X \) set \( S \) such that

(i) \( |S| = |N(S)| \)

(ii) \( N(S) = Y \).

**Proof:** Assume there exists a \( \gamma_X \) set \( S \) such that (i) \( |S| = |N(S)| \) (ii) \( N(S) = Y \). \( \gamma_X(G) \leq \gamma(G) \) always. Clearly \( Y \) is a dominating set of \( G \). \( \gamma(G) \leq |Y| = |N(S)| = |S| = \gamma_X(G) \). Hence \( \gamma(G) = \gamma_X(G) \).

\( \gamma_X(G) = \gamma(G) \). \( \gamma_X(G) = \gamma_Y(G) \) there exists a \( \gamma_X \) set \( S \) such that \( X - S \) is hyper independent in \( G \). Then, \( S \) is a \( Y \)-dominating set of \( G \). Therefore,
Then $S$ is a $\gamma_X(G)$ set by the condition of minimality of $S$, $S$ is an $Y$-isolate, any two vertices in $S$ are not $X$-adjacent. Therefore, $|S| = |N(S)|$.

**Theorem 2.3.57.** In a bipartite graph $G$, $\gamma_X(G) = \gamma(G)$ if and only if $G$ is a galaxy.

**Proof:** If $G$ is a galaxy then $\gamma_X(G) = \gamma(G)$. Conversely, if $\gamma_X(G) = \gamma(G)$ then there exists a $\gamma_X$ set $S$ such that (i) $|S| = |N(S)|$ (ii) $N(S) = Y$. Every vertex in $Y$ is adjacent to exactly one vertex of $S$. If $d(y) \geq 2$ for any $y \in Y$, then it is adjacent to a vertex of $X - S$. Any two vertices in $X - S$ is not $X$-adjacent through different vertices of $Y$. For if so, it contradicts $\gamma_X(G) = \gamma(G)$. Hence, $G$ is a galaxy.

### 2.4 Bipartite theory of complement of a graph

**Bipartite theory of Domination in the Graph $\overline{G^2}$ and $\overline{G^2}$**

**Definition 2.4.1.** [15] Let $G = (X, Y, E)$ be a bipartite graph. We define complement of $G$ denoted by $\overline{G_B} = (X, Y, E^c)$ as follows:

(i) No two vertices in $X$ are adjacent.

(ii) No two vertices in $Y$ are adjacent.
(iii) $x \in X$ and $y \in Y$ are adjacent in $\overline{G}_B$ if and only if $x \in X$ and $y \in Y$ are not adjacent in $G$.

**Theorem 2.4.2.** In a connected graph $G = (V, E)$, $\gamma_X(\overline{VV(G)}) = \gamma(\overline{G^2})$.

**Proof:** Let $S$ be a $\gamma_X$ set of $\overline{VV(G)} = (X, Y, E^1)$. For every $u \in X - S$, there exists $v \in S$ such that $u$ and $v$ are adjacent to $y \in Y$. In $VV(G)$, $u \in X - S$ and $v \in S$ are not adjacent to $y \in Y$.

**Case (a):** $y = u$ or $v$.

Assume $y = u$. In $VV(G)$, $u \in X - S$ and $v \in S$ are not adjacent to $u \in Y$. Therefore, in $G$, $u \in V - S$ and $v \in S$ are not adjacent. Hence, in $\overline{G}$, $u \in V - S$ and $v \in S$ are adjacent. In $\overline{G^2}$, $u \in V - S$ and $v \in S$ are adjacent. Therefore, $S$ is a dominating set in $\overline{G^2}$.

Assume $y = v$. In $VV(G)$, $v \in X - S$ and $u \in S$ are not adjacent to $v \in Y$. Therefore, in $G$, $v \in V - S$ and $u \in S$ are not adjacent. Hence, in $\overline{G}$, $v \in V - S$ and $u \in S$ are adjacent. In $\overline{G^2}$, $v \in V - S$ and $u \in S$ are adjacent. Therefore, $S$ is a dominating set in $\overline{G^2}$.

**Case (b):** $y \neq u$ and $y \neq v$.

In $VV(G)$, $u \in X - S$ and $v \in S$ are not adjacent to $y \in Y$. Therefore, in $G$, $u \in V - S$ and $v \in S$ are not adjacent to $y \in V$. Hence, in $\overline{G}$, $u \in V - S$
and \( v \in S \) are adjacent to \( y \in V \). That is in \( \overline{G^2} \), \( u \in V - S \) and \( v \in S \) are adjacent. Therefore, for every \( u \in V - S \), there exists \( v \in S \) such that \( u \) and \( v \) are adjacent in \( \overline{G^2} \). \( \gamma(G^2) \leq |S| = \gamma_X(VV(G)) \).

Conversely, Let \( D \) be a \( \gamma \) set of \( \overline{G^2} \). Then for every \( u \in V - D \), there exists \( v \in D \) such that \( d(u,v) \leq 2 \) in \( \overline{G} \).

**Case (a):** \( d(u,v) = 1 \) in \( \overline{G} \).

Since \( d(u,v) = 1 \) in \( \overline{G} \), \( u \in V - D \) and \( v \in D \) are not adjacent in \( G \). Hence, \( u \in X \) and \( v \in Y \) are not adjacent in \( VV(G) \). Thus, \( u \in X \) and \( v \in Y \) are adjacent in \( \overline{VV(G)} \). That is \( u \in X - D \) and \( v \in Y \) are adjacent in \( \overline{VV(G)} \).

Thus, \( v \in Y \) of \( \overline{VV(G)} \) is adjacent to \( v \in D \subseteq X \) of \( \overline{VV(G)} \). Hence, \( u \in X - D \) and \( v \in D \) are adjacent to a common vertex in \( Y \). Thus \( D \) is a \( X \)-dominating set in \( \overline{VV(G)} \).

**Case (b):** \( d(u,v) = 2 \) in \( \overline{G} \).

\( u \) and \( v \) are adjacent in \( G \) and not adjacent to any vertex \( w \) in \( G \). In \( VV(G) \), \( u \in X - D \) and \( v \in D \) are not adjacent to any vertex \( w \in Y \). In \( \overline{VV(G)} \), \( u \in X - D \) and \( v \in D \) are adjacent to a vertex \( w \in Y \). \( D \) is a \( X \)-dominating set in \( \overline{VV(G)} \).

In either case, we get \( D \) is a \( X \)-dominating set in \( \overline{VV(G)} \). Therefore,
\( \gamma_X(\overline{VV(G)}) \leq G^2 \). Hence, \( \gamma_X(\overline{VV(G)}) = \gamma(G^2) \).

**Theorem 2.4.3.** In a connected graph \( G = (V, E) \), \( \gamma_X(\overline{VV^+(G)}) = \gamma(G^2) \).

**Proof:** Let \( S \) be a \( \gamma_X \) set in \( \overline{VV^+(G)} = (X, Y, F) \). For every \( u \in X - S \) there exists \( v \in S \) such that \( u \) and \( v \) are adjacent to a vertex in \( y \in Y \). Let \( y \neq u \) and \( y \neq v \). Then \( u \in X - S \) and \( v \in S \) are not adjacent to \( y \in Y \) in \( VV^+(G) \). Hence, in the graph \( G \), \( u \in V - S \) and \( v \in S \) are not adjacent to \( y \in V \). That is, \( u \in V - S \) and \( v \in S \) are adjacent to a vertex \( y \in V \) in \( \overline{G} \). Therefore, \( u \in V - S \) and \( v \in S \) are adjacent in graph \( \overline{G^2} \). Hence, \( S \) is a dominating set in \( \overline{G^2} \). That is, \( \gamma(\overline{G^2}) \leq |S| = \gamma_X(\overline{VV^+(G)}) \).

Conversely, let \( D \) be a \( \gamma \) set of \( \overline{G^2} \). For every \( u \in V - D \), there exists \( v \in D \) such that \( u \) and \( v \) are adjacent in \( \overline{G^2} \). That is, \( u \) and \( v \) have a common neighbor \( w \) in \( \overline{G} \). Therefore, \( u \) and \( v \) are not adjacent to \( w \) in \( G \). That is, \( u \in X - D \) and \( v \in D \) are not adjacent to \( w \in Y \) in \( VV^+(G) \). So, \( u \in X - D \) and \( v \in D \) are adjacent to \( w \in Y \) in \( \overline{VV^+(G)} \). Therefore, \( D \) is a \( X \)-dominating set in \( \overline{VV^+(G)} \). Hence, \( \gamma_X(\overline{VV^+(G)}) \leq |D| = \gamma(\overline{G^2}) \). That is, \( \gamma_X(\overline{VV^+(G)}) = \gamma(G^2) \).

**Theorem 2.4.4.** In a connected graph \( G = (V, E) \), \( \gamma_Y(\overline{VV_B(G)}) = \gamma(\overline{G}) \).

**Proof:** Let \( S \) be a \( \gamma_Y \) set of the graph \( \overline{VV_B(G)} = (X, Y, E^1) \). For every
y \in Y$, there exists $x \in S$ such that $x$ and $y$ are adjacent in $\overline{V V_B(G)}$. Then $x$ and $y$ are not adjacent in $V V(G)$. That is, in $G$, $x$ and $y$ in $V$ are not adjacent. In $\overline{G}$, $x$ and $y$ are adjacent. Therefore, for every $y \in V(\overline{G}) - S$, there exists $x \in S$ such that $x$ and $y$ are adjacent. Therefore, $S$ is a dominating set in $\overline{G}$. $\gamma(\overline{G}) \leq |S| = \gamma_Y(\overline{V V_B(G)})$.

Conversely, let $D$ be a $\gamma(\overline{G})$ set. For every $x \in V(\overline{G}) - D$ there exists $u \in D$ such that $x$ and $u$ are adjacent in $\overline{G}$. That is, $x$ and $u$ are not adjacent in $G$. Therefore, $x$ and $u$ are not adjacent. Hence, $x$ and $u$ are adjacent in $\overline{V V_B(G)}$.

Therefore, for every $x \in Y$ there exists $u \in D$ such that $x$ and $u$ are adjacent. Therefore, $D$ is a $Y$-dominating set in $\overline{V V_B(G)}$. $\gamma_Y(\overline{V V_B(G)}) \leq |D| = \gamma(\overline{G})$.

Hence, $\gamma_Y(\overline{V V_B(G)}) = \gamma(\overline{G})$.

**Theorem 2.4.5.** In a connected graph $G = (V, E)$, $\gamma_Y(\overline{V V_B^+(G)}) = \gamma_t(\overline{G})$.

**Proof:** Let $S$ be a $\gamma_Y$ set in $\overline{V V_B^+(G)} = (X, Y, E^t)$. Every $y \in Y$ is adjacent to an element of $S$ in $\overline{V V^+}$. Therefore, $y \in Y$ is not adjacent to $x \in S$ in $V V^+(G)$. That is, in $G$, $y \in V$ is not adjacent to $x \in S$ in $V V^+(G)$. Therefore, $S$ is a $\gamma_t$ set in $\overline{G}$. Hence, $\gamma_t(\overline{G}) \leq |S| = \gamma_Y(\overline{V V_B^+(G)})$.

Conversely, let $D$ be a $\gamma_t$ set of $\overline{G}$. For every $y \in V$, there exists
$x \in D$ such that $x$ and $y$ are adjacent. Then, $x$ and $y$ are not adjacent in $G$. Therefore, $x$ and $y$ are not adjacent in $\overline{VV^+(G)}$. Hence, $x$ and $y$ are adjacent in $\overline{VV^+_B(G)}$. For every $y \in Y$, there exists $x \in D$ such that $x$ and $y$ are adjacent in $\overline{VV^+_B(G)}$. Therefore, $D$ is a $Y$-dominating set in $\overline{VV^+_B(G)}$. Therefore, $\gamma_Y(\overline{VV^+_B(G)}) = \gamma_G(G)$. ■

Bipartite theory of Mixed Domination

**Theorem 2.4.6.** For a graph $G$, $\gamma_Y(VNe(G)) = \gamma_G(G)$. 

**Proof:** Let $S \subseteq V$ be a $\gamma_Y(VNe)$ set. Elements of $S$ are adjacent to $N[e] \ \forall e \in E(G)$. That is in $G$ elements of $S$ weakly dominates edges of $G$. Therefore, $S$ is a vertex-edge weak dominating set. Thus, $\gamma_G(G) \leq |S| = \gamma_Y(VNe)$. 

Conversely, let $S$ be a vertex edge weak dominating set of $G$. Then elements of $S$ weakly dominate all edges of $G$. Equivalently, for every edge $e \in E(G)$, there exists $v \in S$ such that $v \in N[e]$. In the graph $VNe(G)$, $N[e]$ is adjacent to at least one vertex of $S$. Therefore, $S$ is a $Y$-dominating set in $VNe(G)$. That is, $\gamma_Y(VNe) \leq |S| = \gamma_G(G)$. 

Hence, $\gamma_Y(VNe(G)) = \gamma_G(G)$. ■

**Theorem 2.4.7.** Every distance $2$-dominating set in $G$ is a $X$-dominating
set in $VNe(G)$.

**Proof:** Let $S$ be a minimum distance 2-dominating set of $G$. Then $\forall u \in V - S, \exists v \in S$, such that $u$ and $v$ are at a distance $\leq 2$.

Case (i): $d(u, v) = 1$

$u$ and $v$ are incident to a common edge $e$. In graph $VNe$, $u$ and $v$ are incident to a vertex $N[e]$. Hence, $S$ is a $X$-dominating set in $VNe(G)$.

Case (ii): $d(u, v) = 2$

Let $u - v$ path be $ue_1we_2v$. $N[e_1] = N[u] \cup N[w]$ and $N[e_2] = N[w] \cup N[v]$. In graph $VNe(G)$, $u$ and $v$ are incident with both $N[e_1]$ and $N[e_2]$. Hence, $S$ is a $X$-dominating set in $VNe$.

**Remark 2.4.8.** The converse of the above result is not true. In $G$, $S = \{a\}$ is not a distance 2-dominating set but in $VNe$ the set $S = \{a\}$ is a $X$-dominating set.

![Diagram](image)
Theorem 2.4.9. For any graph $G$, $\gamma_Y(EN^+) = S\gamma_{10}(G)$.

Proof: Let $D$ be a $\gamma_Y(EN^+)$ set. In $EN^+(G)$, every $N[v]$ for every $v \in V(G)$ is adjacent to a vertex in $D$. In $G$, $D$ strongly dominates all vertices of $G$.

Therefore, $S\gamma_{10}(G) \leq |D| = \gamma_Y(EN^+)$. 

Conversely, let $D$ be a $S\gamma_{10}$ set. Edges in $D$ strongly dominates all vertices of $G$. $\langle N[v] \rangle$ contains at least one edge of $D$ for every $v \in V(G)$. Therefore, $D$ is a $Y$-dominating set in $EN^+(G)$. Therefore, $\gamma_Y(EN^+) \leq |D| = S\gamma_{10}(G)$.

Hence, $\gamma_Y(EN^+) = S\gamma_{10}(G)$. \hfill \blacksquare$

2.5 Well $X$-Dominated and $X$-Excellent bipartite graphs

Definition 2.5.1. A bipartite graph $G$ is called well $X$-dominated if all minimal $X$-dominating sets have the same cardinality.
Proposition 2.5.2. Let $G = (X, Y, E)$ be a bipartite graph with $|X| = p$ and $|Y| = q$. If there exists a vertex $y \in Y$ of degree $p$ then $G$ is well $X$-dominated.

Proof: If there exists a vertex $y \in Y$ of degree $p$, then every vertex in $x \in X$ is $X$-adjacent to other vertices of $X$. Therefore, every vertex in $X$ is a $\gamma_X$-set of $G$. Hence, $G$ is well $X$-dominated.

Definition 2.5.3. A bipartite graph $G$ is called an $X$-excellent graph if every vertex in $X$ is in a minimum $X$-dominating set.

Observation 2.5.4. $C_{2n}$ and $K_{m,n}$ are $\gamma_X$-excellent.

Note: A vertex $u \in X$ is called end vertex if $|N_Y(u)| = 1$ and the vertex $X$-adjacent to $u$ is called a support vertex.

Observation 2.5.5. If $G \neq K_{2,1}$ then there exists a $\gamma_X$-set containing all the support vertices of $G$.

Observation 2.5.6. For any $\gamma_X$-excellent graph $G$, every end vertex is in some $\gamma_X$-set and no end vertex is in every $\gamma_X$-set of $G$.

Proof: Any $\gamma_X$-set contains either an end vertex of $G$ or its support. ■

Observation 2.5.7. Consider a support vertex that is $X$-adjacent to two or more end vertices. In this case the support vertex must be in every $\gamma_X$-set. As a result, the end vertices will be in no $\gamma_X$-set. Hence, a graph with any support vertex $X$-adjacent to more than one end vertex is not $\gamma_X$-excellent.
Notation: Let $H$ be a bipartite graph, $H = (X, Y, E)$, $|X| = p$. The X-corona of $H$ is the bipartite graph $G = (X^1, Y^1, E^1)$ where $X^1 = X \cup \{u_1, u_2, \ldots, u_p\}$ and $Y^1 = Y \cup \{v_1, v_2, \ldots, v_p\}$ and every vertex in $X$ is X-adjacent to a unique $u_i$ through $v_i, 1 \leq i \leq p$.

Let $G_1$ be the family of graphs $G = (X, Y, E)$ satisfying the following: $(|X| - 2)$ vertices of $X$ are X-adjacent to exactly two vertices of $X$ and the remaining vertices are X-adjacent to exactly one vertex of $X$ and all the X-adjacency are through different elements of $Y$.

**Theorem 2.5.8.** Every bipartite graph is an induced subgraph of a $\gamma_X$-excellent graph.

**Proof:** Consider any bipartite graph $H$ and let $G$ be the X-corona of a $H$. Every vertex in $X(H)$ is now a support vertex in $G$. Therefore, $X(H)$ is a $\gamma_X$-set of $G$. Also, the set of end vertices in $G$ is a $\gamma_X$-set. Hence, every vertex in $X(G)$ is in some $\gamma_X$-set and so $G$ is an X-excellent graph. Since, $H$ is an induced subgraph of $G$, every graph is an induced subgraph of some X-excellent graph.  

The following result can be easily proved.

**Proposition 2.5.9.** A bipartite graph $G = (X, Y)$ on even number of ver-
tices such that every vertex in $X$ is $X$-adjacent to exactly two vertices in $X$ and all $X$-adjacency are through different vertices is a cycle.

**Proposition 2.5.10.** Let $G$ be a connected bipartite graph belonging to $G_1$. Then $G$ is $X$-excellent if and only if $G = K_{2,1}$ or $|X| \equiv 1 \pmod{3}$.

**Proof:** Let $G \in G_1$. Suppose $G = K_{2,1}$. Then clearly $G$ is $X$-excellent. Suppose $|X| \equiv 1 \pmod{3}$. Let $|X| = 4n + 1$. Then as $G$ is connected $G$ is a path on $8n + 1$ vertices and hence, $G$ is $X$-excellent.

Suppose $G \neq K_{2,1}$ and $|X| \equiv 2 \pmod{3}$. Let $|X| = 3n + 2$. Then $G$ is a path on $6n + 3$ vertices. Let $X(G) = \{u_1, u_2, \cdots, u_{3n+2}\}$ and $Y(G) = \{v_1, v_2, \cdots, v_{3n+1}\}$. Then $\gamma_X(G) = n + 1$ and $u_3, u_6, u_9, \cdots, u_{3n}$ do not belong to any $\gamma_X$- set of $G$. Therefore, $G$ is not $X$-excellent.

Let $|X| \equiv 0 \pmod{3}$. Let $|X| = 3n$. Then $G$ is a path on $6n - 1$ vertices. Let $X(G) = \{u_1, u_2, \cdots, u_{3n}\}$ and $Y(G) = \{v_1, v_2, \cdots, v_{3n-1}\}$. Clearly, $\gamma_X(G) = n$ and $\{u_2, u_5, \cdots, u_{3n-1}\}$ is the unique $\gamma_X$- set of $G$. Hence, $G$ is not $X$-excellent.

**Definition 2.5.11.** A map $\phi : X \to X$ of a bipartite graph $G = (X, Y, E)$ is called an $X$-automorphism if $\phi$ is one to one and onto and whenever $u, v \in X$ are $X$-adjacent then $\phi(u)$ and $\phi(v)$ are $X$-adjacent and vice versa.
Definition 2.5.12. A bipartite graph \( G = (X, Y, E) \) is \( X \)-transitive if for any \( u, v \in X \), there exists an \( X \)-automorphism \( \phi \) such that \( \phi(u) = v \).

Remark 2.5.13. \( X \)-automorphism of \( VE(G) \) is an automorphism of \( G \).

Theorem 2.5.14. Every \( X \)-transitive bipartite graph is \( X \)-excellent.

Proof: Let \( G = (X, Y, E) \) be a \( X \)-transitive bipartite graph. Let \( u \in X \).

Let \( S \) be a minimum \( X \)-dominating set of \( G \). Suppose \( u \notin S \). Let \( v \in S \).

As \( G \) is \( X \)-transitive, there exists an \( X \)-automorphism \( \phi \) such that \( \phi(u) = v \).

Let \( S = \phi(S) = \{ \phi(x) : x \in S \} \).

Claim: \( S^1 \) is a minimum \( X \)-dominating set of \( G \).

Since \( |S^1| = |S|, |S^1| = \gamma_X(G) \). Let \( x \in X - S^1 \). Let \( y \in X \) be such that \( \phi^1(y) = x \). As \( x \notin S^1, y \notin S \). Therefore, there exists \( w \in S \) such that \( w \) is \( X \)-adjacent to \( y \). Therefore, \( \phi(y) \) is \( X \)-adjacent to \( \phi(w) \). That is, \( x \) is \( X \)-adjacent to \( \phi(w) \in S^1 \). Therefore, \( S^1 \) is a \( X \)-dominating set. Therefore, \( S^1 \) is a minimum \( X \)-dominating set containing \( u \).

Therefore, \( G \) is \( X \)-excellent.

\[ \square \]

Corollary 2.5.15. Every transitive graph is excellent.

Corollary 2.5.16. If \( G \) is bipartite graph and \( G \) is \( VE- \) realization of a transitive graph, then \( G \) is \( X \)-excellent. For: let \( G = VE(H) \), where \( H \) is transitive. Then \( VE(H) \) is \( X \)-transitive and hence \( X \)-excellent.
Remark 2.5.17. $P_{3n+1}$ (n ≥ 1) is not transitive, but is excellent. Therefore, $VE(P_{3n+1})$ is not X-transitive but X-excellent.

2.6 X-Irredundant set

Definition 2.6.1. Let $G = (X, Y, E)$ be a bipartite graph. Let $S \subseteq X$. Let $u \in S$. A vertex $v$ is a private X-neighbor of $u$ with respect to $S$ if $u$ is the only point of $S$, X-adjacent to $v$.

Definition 2.6.2. A set $S$ is X-irredundant set if every $u \in S$ has a private X-neighbor. The X-irredundance number of a graph $G$ is the minimum cardinality of a maximal X-irredundant set of $G$ and is denoted by $ir_X(G)$.

The upper X-irredundance number of a graph $G$ is the maximum cardinality of a maximal X-irredundant set of $G$ and is denoted by $IR_X(G)$.

Theorem 2.6.3. A X-dominating set $S$ is a minimal X-dominating set if and only if it is X-dominating and X-irredundant.

Proof: Let $S$ be a X-dominating set. Then $S$ is a minimal X-dominating set if and only if for every $u \in S$ there exists $v \in X - (S - \{u\})$ which is not X-dominated by $S - \{u\}$. Equivalently, $S$ is a minimal X-dominating set if and only if $\forall u \in S$, $u$ has at least one private X-neighbor with respect to $S$. Thus $S$ is minimal X-dominating set if and only if it is X-irredundant.

Conversely, Let $S$ is both X-dominating and X-irredundant.
Claim: S is a minimal $X$-dominating set.

If $S$ is not a minimal $X$-dominating set, then there exists $v \in S$ for which $S - \{v\}$ is $X$-dominating. Since $S$ is $X$-irredundant, $v$ has a private $X$-neighbor of with respect to $S$ say $u$ ($u$ may be equal to $v$). By definition, $u$ is not $X$-adjacent to any vertex in $S - \{v\}$. $S - \{v\}$ is not a $X$-dominating set, a contradiction. Hence, $S$ is a minimal $X$-dominating set.

By the above theorem, any minimal $X$-dominating set is an $X$-irredundant set. Therefore, $X$-irredundant sets exist.

**Theorem 2.6.4.** Every minimal $X$-dominating set is a maximal $X$-irredundant set.

**Proof:** Every minimal $X$-dominating set $S$ is $X$-irredundant set.

Claim: $S$ is a maximal $X$-irredundant set.

Suppose $S$ is not a maximal $X$-irredundant set. Then there exists a vertex $u \in X - S$ for which $S \cup \{u\}$ is $X$-irredundant. Therefore, there exists at least one vertex $x$ which is a private $X$-neighbor of $u$ with respect to $S \cup \{u\}$. Hence, no vertex in $S$ is $X$-adjacent to $x$. Thus $S$ is not $X$-dominating set, a contradiction. Hence, $S$ is maximal $X$-irredundant set.

**Remark 2.6.5.** Clearly $ir_X(G) \leq \gamma_X(G)$ and $\Gamma_X(G) \leq IR_X(G)$. Thus we
have the $X$-dominating sequence $ir_X(G) \leq \gamma_X(G) \leq i_X(G) \leq \beta_X(G) \leq \Gamma_X(G) \leq IR_X(G)$.

**Theorem 2.6.6.** For any graph $G$, (a) $ir_X(VE(G)) = ir(G)$ (b) $ir_X(EV(G)) = ir^1(G)$

**Proof:** Let $S$ be a $ir_X$ set of $VE(G) = (X, Y, E^1)$. Every $v$ has a private $X$-neighbor $u$. Equivalently, $v$ is $X$-adjacent to $u$ and no other vertex in $S$ is $X$-adjacent to $u$. In $G$, $v \in S$ is the only vertex adjacent to $u$ and no other vertex in $S$ is adjacent to $u$. Therefore, $S$ is an irredundant set of $G$. $ir(G) \leq |S| = ir_X(VE(G))$.

Let $U$ be an $ir-$ set of $G$. For every vertex $v \in U$, $pn[v, U] \neq \emptyset$. Every vertex $v \in U$ has at least one private neighbor with respect to $u$. In $VE(G)$, that is every vertex $v \in U$ has at least one private $X$-neighbor. Therefore, $U$ is an $X$-irredundant set. Hence, $ir_X(VE(G)) \leq |U| = ir(G)$. Hence, $ir_X(VE(G)) = ir(G)$.

(b) Let $S$ be an $ir_X$ set of $EV(G) = (X, Y, E^1)$. Every $e$ has a private $X$-neighbor $f$. Equivalently, $e$ is $X$-adjacent to $f$ and no other vertex in $S$ is $X$-adjacent to $f$. In $G$, $e \in S$ is the only edge adjacent to $f$ and no other edge in $S$ is adjacent to $f$. Therefore, $S$ is an edge irredundant set of $G$. 

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Hence, \( ir^1(G) \leq |S| = ir_X(EV(G)) \).

Let \( U \) be a \( ir^1 \)-set of \( G \). For every edge \( e \in U \), \( pn[e, U] \neq \phi \). Hence, every edge \( e \in U \) has at least one private neighbor. That is, in \( EV(G) \), every vertex \( e \in U \) has at least one private \( X \)-neighbor. Therefore, \( U \) is an \( X \)-irredundant set in \( EV(G) \). Thus, \( ir_X(EV(G)) \leq |U| = ir^1(G) \). Hence, \( ir_X(EV(G)) = ir^1(G) \). ■

2.7 Strong nonsplit \( X \)-Domination number of a bipartite graph

Let \( G = (X, Y, E) \) be bipartite graph.

Definition 2.7.1. A \( X \)-dominating set of \( G \) is said to be a strong nonsplit \( X \)-dominating set of \( G \) if every vertex in \( X - D \) is \( X \)-adjacent to all other vertices in \( X - D \). (That is, \( X - D \) is \( X \)-complete). The strong nonsplit \( X \)-domination number of a graph \( G \), denoted by \( \gamma_{nsX}(G) \) is the minimum cardinality of a strong nonsplit \( X \)-dominating set.

Remark 2.7.2. Let \( G \) be a bipartite graph with at least one non \( Y \)-isolate say \( x \). Then \( X - \{x\} \) is a strong nonsplit \( X \)-dominating set of \( G \). Therefore, every bipartite graph with at least one non \( Y \)-isolate has a strong nonsplit \( X \)-dominating set of \( G \).
Notation: Let $S_p$ be a bipartite graph $(X, Y, E)$, $|X| = p$; $|Y| = p - 1$ with a vertex in $X$, $X$-adjacent to all other vertices of $X$ through different $y \in Y$ and all vertices in $X - \{x\}$ are end vertices.

**Theorem 2.7.3.** (a) $\gamma_{s_{n\in X}}(K_{m,n}) = 1$ (b) $\gamma_{s_{n\in X}}(C_{2n}) = |X| - 2$ if $n \neq 2$ and $is = 1$ if $n = 2, 3$ (c) $\gamma_{s_{n\in X}}(S_p) = p - 1$.

**Theorem 2.7.4.** Let $G$ be a bipartite graph with $p \geq 3$ and there exists vertices $x_1, x_2, x_3$ which are mutually $X$-adjacent. Then $\gamma_{s_{n\in X}}(G) \leq p - 2$.

**Proof:** By hypothesis, there exists vertices $x_1, x_2, x_3$ which are mutually $X$-adjacent. Then $X - \{x_2, x_1\}$ is a strong non-split $X$-dominating set of $G$.

$\gamma_{s_{n\in X}}(G) \leq p - 2.$

**Theorem 2.7.5.** A strong non-split $X$-dominating set $D$ of $G$ is minimal if and only if for all $v \in D$, one of the following conditions hold

(i) $u$ is an $Y$-isolate of $D$.

(ii) there exists a $u \in X - D$ such that $u$ is $Y$-private neighbor of $v$.

(iii) there exists a vertex $w \in X - D$ such that $w$ is not $X$-adjacent to $v$.

**Proof:** Let $D$ be a minimal strong non-split $X$-dominating set.

Let $v \in D$, then $D - \{v\}$ is not a strong non-split $X$-dominating set. Either there exists $w \in X - (D - \{v\})$ which is not $X$-adjacent to $v \in D$ or vertices in $X - (D - \{v\})$ are not $X$-complete.
Case (i) there exists \( w \in X - (D - \{v\}) \) which is not \( X \)-adjacent to \( v \in D \) then either \( v = w \) in which case \( v \) is an \( Y \)-isolate of \( D \) which is (i) or \( w \in X - D \). If \( w \) is not \( X \)-adjacent with any vertex in \( D \) then \( w \) is a \( Y \)-private neighbor of \( v \) which is (ii).

Case (ii) vertices in \( X - (D - \{v\}) \) are not \( X \)-complete. Equivalently there is a vertex \( w \in X - D \) which is not \( X \)-adjacent to \( v \) which is (iii).

Conversely, let for some \( v \in V \) some of the three conditions hold. Then \( D - \{u\} \) is a \( X \)-dominating set of \( D \) such that \( (X - D) \cup \{v\} \) is \( X \)-complete. Therefore, \( D - \{v\} \) is a strong nonsplit \( X \)-dominating set of \( G \). That is \( D \) is not a minimal strong nonsplit \( X \)-dominating set of \( G \).

Theorem 2.7.6. Let \( G \) be a graph with \( \Delta_Y(G) \leq p - 2 \). Let \( D \) be a strong nonsplit \( X \)-dominating set of \( G \) such that \( (D) \) is a \( X \)-clique and \( |D| \leq \delta_Y(G) \). Then (i) \( D \) is a minimal nonsplit \( X \)-dominating set. (ii) \( X - D \) is also a minimal strong nonsplit \( X \)-dominating set of \( G \).

Proof: Since \( \Delta_Y(G) \leq p - 2 \), for every \( v \in D \), there exists \( w \in X - D \) such that \( v \) and \( w \) are not \( X \)-adjacent. Hence, \( D \) is a minimal nonsplit \( X \)-dominating set. Since \( |D| \leq \delta_Y(G) \), every vertex in \( D \) is \( X \)-adjacent to some vertex in \( X - D \). Since \( (D) \) is a \( X \)-clique, \( X - D \) is a strong nonsplit
X-dominating set of G. Also by the above theorem, X−D is also minimal. □

**Theorem 2.7.7.** For any connected graph G, \( \beta_X(G) \leq \gamma_{s_{xn}}(G) \) and the bound is sharp.

**Proof:** Let D be a \( \gamma_{s_{xn}} \) set of G. Then any two vertices in X−D are X-adjacent. Moreover every vertex in X−D is X-adjacent to a vertex of D. Therefore, \( \beta_X \leq |D| = \gamma_{s_{xn}}(G) \). The bound is attained in \( K_{m,n} \). □

**Definition 2.7.8.** A strong nonsplit X-dominating set D is said to be an X-independent strong nonsplit X-dominating set if D is X-independent.

**Theorem 2.7.9.** If a connected graph G has an X-independent strong nonsplit X-dominating set and \( d(u,v) \leq 6 \) \( \forall u,v \in X \).

**Proof:** Let D be an X-independent strong nonsplit X-dominating set of G.

Let \( u,v \in X(G) \).

**Case (i):** \( u,v \in X−D \).

Then u and v are X-adjacent. Hence, \( d(u,v) = 2 \leq 6 \).

**Case (ii):** \( u \in D \) and \( v \in X−D \).

Since D is X-independent, \( N_Y(u) \in X−D \). Let \( w \in N_Y(u) \). If \( w = v \) then \( d(u,v) = 2 \). Let \( w \neq v \) then \( d(u,v) \leq d(u,w) + d(w,v) = 4 \leq 6 \).

**Case (iii):** Let \( u,v \in D \).
Since $G$ is connected, there exists two vertices $w_1, w_2 \in X - D$ such that $u$ is $X$-adjacent to $w_1$ and $v$ is $X$-adjacent to $w_2$. If $w_1 = w_2$ then $d(u, v) \leq d(u, w_1) + d(w_1, v) \leq 4$. If $w_1 \neq w_2$ then as $w_1, w_2 \in X - D, W_1$ and $w_2$ are $X$-adjacent. Therefore, $d(u, v) \leq d(u, w_1) + d(w_1, w_2) + d(w_2, v) = 6$. ■

**Corollary 2.7.10.** If $\gamma_X(G) = \gamma_{snaX}(G)$, then $d(u, v) \leq 6 \forall u, v \in X(G)$.

**Theorem 2.7.11.** For any bipartite graph $G$, $p - \omega_X(G) \leq \gamma_{snaX}(G) \leq p - \omega_X + 1$.

**Proof:** Let $D$ be a $\gamma_{snaX}$-set. Then $X - D$ is a $X$-clique. Therefore, $\omega_X(G) \geq |X - D| = p - \gamma_{snaX}$. Therefore, $\gamma_{snaX}(G) \geq p - \omega_X(G)$.

Let $S$ be a $X$-clique set of order $\omega_X(G)$. Then $(X - S) \cup \{w\}, w \in S$ is a strong nonsplit $X$-dominating set. Hence, $\gamma_{snaX}(G) \leq |X - S| + 1 = p - \omega_X + 1$. ■

**Theorem 2.7.12.** Let $G$ be a connected bipartite graph with $\omega_X(G) \geq \delta_Y(G)$. Then $\gamma_{snaX}(G) \leq p - \delta_Y(G)$ and the bound is attained if and only if one of the following conditions is satisfied (i) $\omega_X(G) = \delta_Y(G)$ (ii) $\omega_X(G) = \delta_Y(G) + 1$ and every $\omega_X$-set $S$ of $X$ contains a vertex not $X$-adjacent to any vertex of $X - S$.

**Proof:** Suppose $\omega_X(G) \geq \delta_Y(G) + 1$. Then, $\gamma_{snaX}(G) \leq p - \omega_X + 1 \leq p - \delta_Y - 1 + 1 = p - \delta_Y$. Let $\omega_X(G) = \delta_Y$. Let $S$ be a $\omega_X$-set of $G$ with
\( |S| = \omega_X(G) \). Since \( |S| = \delta_Y(G) \) every vertex in \( S \) is \( X \)-adjacent to at least one vertex in \( X - S \). That is, \( X - S \) is a \( X \)-dominating set and hence a the nonsplit \( X \)-dominating set. Therefore, \( \gamma_{\text{smax}}(G) \leq p - \omega_X(G) \leq p - \delta_Y(G) \).

Already, \( p - \omega_X(G) \leq \gamma_{\text{smax}}(G) \). Therefore, \( \gamma_{\text{smax}}(G) = p - \delta_Y(G) \).

Assume condition (ii). That is, \( \omega_X(G) = \delta_Y(G) + 1 \) and every \( \omega_X \)-set \( S \) contains a vertex not \( X \)-adjacent to any vertex of \( X - S \). Let \( w \in S \) be the vertex not \( X \)-adjacent to any vertex of \( X - S \). Then \( (X - S) \cup \{w\} \) is a nonsplit \( X \)-dominating set. Therefore, \( \gamma_{\text{smax}}(G) \leq p - \omega_X(G) + 1 = p - \delta_Y - 1 + 1 = p - \delta_Y(G) \). That is \( \gamma_{\text{smax}}(G) \leq p - \delta_Y(G) \), Since every \( \omega_X \) set of cardinality \( \delta_Y + 1 \) contains a vertex not \( X \)-adjacent to any vertex of \( X - S \). Therefore, \( \gamma_{\text{smax}}(G) \geq p - \delta_Y \). Hence, \( \gamma_{\text{smax}}(G) = p - \delta_Y \).

Conversely, let \( \gamma_{\text{smax}}(G) = p - \delta_Y \). Then, \( \omega_X = \delta_Y \) or \( \omega_X = \delta_Y + 1 \).

Suppose there exists a \( \omega_X \)-set \( S \) with \( |S| = \delta_Y + 1 \) such that every vertex in \( S \) is \( X \)-adjacent with some vertex in \( X - S \). Then \( X - S \) is a strong nonsplit \( X \)-dominating set of \( G \). Hence, \( \gamma_{\text{smax}} \leq p - \delta_Y - 1 \), a contradiction. Hence, one of the given conditions is satisfied.
2.8 $X$-vertex critical graphs

Definition 2.8.1. A bipartite graph $G$ is called $X$-vertex critical if $\gamma_X(G - x) \geq \gamma_X(G)$.

We define $V_{\gamma_X}(G) = \{v \in X : v \text{ belongs to every } \gamma_X \text{ set of } G\}$

Theorem 2.8.2. A bipartite graph $G$ has a unique minimum $X$-dominating set if and only if the set $V_{\gamma_X}(G)$ is a $X$-dominating set of $G$.

Proof: Let $D$ be unique minimum $X$-dominating set then $V_{\gamma_X}(G) = D$, which is $X$-dominating set of $G$. $V_{\gamma_X}(G)$ is $X$-dominating set of $G$. Let $D$ be a minimum $X$-dominating set of $G$. Then, $V_{\gamma_X}(G) \subseteq D$. Therefore, $V_{\gamma_X}(G) = D$. \qed

Definition 2.8.3. Let $S \subseteq X$ be set of vertices. Let $u \in S$. $v$ is $X$-private neighborhood of $u$ denoted by $pn_X[u, S]$ is defined as $pn_X[u, S] = N_Y[v] \cap S = \{u\}$.

Theorem 2.8.4. Let $G$ be a bipartite graph and let $G$ have a unique $\gamma_X$-set $D$. Then every vertex in $D$ has atleast two $X$-private neighbors.

Proof: Let $D$ be a unique $\gamma_X$-set. Let $x \in D$. Then either $x$ itself is a $X$-private neighbor of $x$ in $D$ or there exists a $x_1 \in X$ which is a $X$-private neighbour of $x \in D$. Let $D_1 = (D - \{x\}) \cup \{x\}$. Then, $D_1$ is not a $X$-
dominating set. Therefore, there exists a $X$-private neighbour of $x$ say $x_2$ in $X - D_1$ and $x_2 \neq x_1$. Therefore, $x$ has at least two $X$-private neighbours. ■

**Theorem 2.8.5.** Let $D$ be a $\gamma_X$-set of a graph $G$. If $\gamma_X(G - x) > \gamma_X(G)$ $\forall x \in D$, then $D$ is unique $\gamma_X$-set of $G$.

**Proof:** Let $D$ be $\gamma_X$-set of $G$ such that $\gamma_X(G - x) > \gamma_X(G)$ $\forall x \in D$. Suppose there is a $\gamma_X$-set $D^1$ of $G$ different from $D$. There exists $x \in D - D^1$ and $D^1$ $X$-dominates $X - \{x\}$, a contradiction, since $|D^1| \geq \gamma_X(G - x) > \gamma_X(G) = |D|$. Hence, $D$ is unique $\gamma_X$-set of $G$. ■

**Remark 2.8.6.** Let $D$ be a $\gamma_X$-set of $G$ for which every vertex in $D$ has at least two $X$-private neighbours need not imply $\gamma_X(G - x) > \gamma_X(G)$ $\forall x \in D$.

![Graph G](image1)

![Graph G - x1](image2)
Proof: $D = \{x_1, x_2\}$ is a $\gamma_X$-set in $G$. $x_1$ has got two private X-neighbours namely $x_1$ and $x_3$ and $x_2$ has two X-private neighbours namely $x_2$ and $x_4$. $\gamma_X(G - x) = \gamma_X(G)$.

Theorem 2.8.7. Let $G$ be a bipartite graph without X-isolates. Then there exists a $\gamma_X$-set $D$ of $G$ such that every vertex $v \in D$ has X-private outside $D$.

Proof: Let $D$ be a $\gamma_X$- of $G$. Let $u \in D$. Then $u$ is either a X-isolate of $D$ or has a X-private neighbour in $X - D$.

Suppose $u$ is an X-isolate of $D$. Since $u$ is not an X-isolate of $G$, there exists a vertex $v \in X - D$ such that $u$ and $v$ are $X$-adjacent. Let $D_1 = (D - \{u\}) \cup \{v\}$. $D_1$ is a $\gamma_X$- set of $G$ in which $v$ has a X-private neighbour namely $u$.

Repeating this process, after a finite stage, we get a $\gamma_X$-set $D^*$ of $G$ such that every vertex $v \in D^*$ has a X-private outside $D^*$.

We define the sets $V^+_X$ and $V^0_X$ as follows:

$V^+_X = \{x \in X : \gamma_X(G - x) > \gamma_X(G)\}$ and $V^0_X = \{x \in X : \gamma_X(G - x) = \gamma_X(G)\}$.

Theorem 2.8.8. A vertex $v \in V^+_X$ if and only if $N_Y(v) \neq \emptyset$ and $v$ is in every
\( \gamma_X \)-set of \( G \) and no subset \( S \subseteq X - N_Y[v] \) with cardinality \( \gamma_X(G) \) dominates \( G - \{v\} \).

**Proof:** Suppose \( v \in V_X^+ \). It is clear \( N_Y(v) \neq \emptyset \). For: suppose \( N_Y(v) = \emptyset \), \( v \) is an \( X \)-isolate. Therefore, \( \gamma_X(G - v) = \gamma_X(G) \), a contradiction. Let \( D \) be a \( \gamma_X \)-set which does not contain \( v \). Then \( D \) \( X \)-dominates \( G - \{v\} \), a contradiction, since, \( \gamma_X(G - v) > \gamma_X(G) \). Hence, \( v \) is in all \( \gamma_X \) set of \( G \). Since \( \gamma_X(G - x) > \gamma_X(G) \), the other condition follows.

Conversely, let \( v \in X \) satisfy the conditions \( N_Y(v) \neq \emptyset \), \( v \) is in every \( \gamma_X \)-set of \( G \) and no subset \( S \subseteq X - N_Y[v] \) with cardinality \( \gamma_X(G) \) \( X \)-dominates \( G - v \).

Suppose \( \gamma_X(G - x) \leq \gamma_X(G) \). Let \( D_1 \) be a minimum \( X \)-dominating set of \( G - \{v\} \). Let \( V' \in N_Y(v) \). If \( v' \in D^1 \), then \( D_1 \) is a \( X \)-dominating set of \( G \) with cardinality less than or equal to \( \gamma_X(G) \). Therefore, \( |D_1| = \gamma_X(G) \) and \( D_1 \) does not contain \( D \), a contradiction. Therefore, \( N_Y(v) \cap D_1 = \emptyset \). Hence, \( D_1 \subseteq X - N_Y[v] \). If \( |D_1| = \gamma_X(G) \), \( |D_1 \cup \{v\}| < \gamma_X(G) \), a contradiction, since \( D_1 \cup \{v\} \) is a \( X \)-dominating set of \( G \). Therefore, \( |D_1| = \gamma_X(G) - 1 \).

Let \( v_1 \in N_Y(v) \). \( v_1 \notin D_1 \). Then \( D_1 \cup \{v_1\} \) is a \( X \)-dominating set of \( G \) with cardinality \( \gamma_X(G) \) and \( v \notin D \cup \{v_1\} \), a contradiction. Therefore,
\[ \gamma_X(G - v) > \gamma_X(G). \]

**Theorem 2.8.9.** Suppose \( v \in V_X^+ \). Then for any \( \gamma_X \)-set \( S \) of \( G \), \( pn_X[u, S] \) contains at least two non \( X \)-adjacent vertices.

**Proof:** Since \( u \in V_X^+ \), \( u \) belongs to every \( \gamma_X \)-set of \( G \). Therefore, \( u \in S \).

Suppose \( pn_X[u, S] = \{ u \} \). Then \( (S - \{ u \}) \cup \{ v \} \) where \( v \in N_Y(u) \) is a \( \gamma_X \)-set of \( G \) not containing \( u \), a contradiction. Therefore, \( pn_X[u, S] \) contains a vertex say \( w \) in \( X - S \). Suppose any two vertices of \( pn_X[u, S] \) are \( X \)-adjacent. Then \( (S - \{ u \}) \cup \{ w \} \) for any \( w \in pn_X[u, S] \) is a \( \gamma_X \)-set of \( G \) not containing \( u \), a contradiction.

We define the set \( V_X^- = \{ v \in X : \gamma_X(G - v) < \gamma_X(G) \} \).

**Theorem 2.8.10.** A vertex \( v \) is in \( V_X^- \) if and only if \( pn_X[v, S] = \{ v \} \) for some \( \gamma_X \)-set \( S \) containing \( v \).

**Proof:** Let \( v \in V_X^- \) and \( D \) be a \( \gamma_X \)-set of \( G - v \). Then \( S = D \cup \{ v \} \) is a \( \gamma_X \)-set of \( G \). If \( D \) contains \( N_Y(v) \), then \( D \) is \( X \)-dominating set of \( G \) contradicting our assumption \( \gamma_X(G - v) < \gamma_X(G) \). Hence, \( pn_X[v, S] = \{ v \} \).

Conversely if \( pn_X[v, S] = \{ v \} \) for some \( \gamma_X \) set \( S \) containing \( v \), then \( S - \{ v \} \) dominates \( G - v \) hence, \( v \in V_X^- \).

**Theorem 2.8.11.** If \( u \in V_X^+ \) and \( v \in V_X^- \) then \( x \) and \( y \) are not \( X \)-adjacent.
Proof: Suppose $u$ and $v$ are $X$-adjacent. Let $S$ be a $X$-dominating set of $G-v$ with cardinality $\gamma_X(G)-1$. If $S$ contains $u$ then $S$ is a $X$-dominating set of $G$ (since $u$ and $v$ are $X$-adjacent) a contradiction, since $|S| = \gamma_X(G) - 1$.

Therefore, $S$ does not contain $u$. $S \cup \{v\}$ is a $\gamma_X-$set of $G$ not containing $u$, a contradiction, since $u$ belongs to every $\gamma_X-$set of $G$. Therefore, $u$ and $v$ are not $X$-adjacent.

Theorem 2.8.12. For any bipartite graph $G$, $|V_X^0| \geq 2|V_X^+|$.

Proof: Let $u \in V_X^+$. Let $S$ be any $\gamma_X-$set of $G$. Then $u \in S$ and $pn_X[u, S]$ contains atleast two non $X$-adjacent vertices, say $w_1, w_2 \in X - S$. Clearly, $w_1, w_2 \notin V_X^-$. Also $w_1, w_2 \notin V_X^-$ since $u \in V_X^+$ and $w_1, w_2$ are $X$-adjacent to $u$ (using $u \in V_X^+$ and $v \in V_X^-$ then $u$ and $v$ are not $X$-adjacent). Therefore, $w_1, w_2 \in V_X^0$. Hence, every vertex in $V_X^+$ has atleast two private $X$-neighbours in $V_X^0$. Therefore, $|V_X^0| \geq 2|V_X^+|$.

Theorem 2.8.13. $\gamma_X(G) \neq \gamma_X(G) \forall v \in X$ if and only if $X = V_X^-$.

Proof: If $X = V_X^-$, then $\gamma_X(G) \neq \gamma_X(G) \forall v \in X$.

Suppose $\gamma_X(G) \neq \gamma_X(G) \forall v \in X$. Then $V_X^0 = \phi$. Suppose $V_X^+ \neq \phi$. Then, as $|V_X^0| \geq 2|V_X^+|$, we get that $V_X^0 \neq \phi$, a contradiction. Therefore, $V_X^+ = \phi$.

Therefore, $X = V_X^-$. 

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Theorem 2.8.14. If $v \in V_X^-$ and $v$ is not an $X$-isolate then there exists a $\gamma_X$-set $S$ of $G$ such that $v \notin S$.

Proof: Let $v \in V_X^-$ and $u$ be not $X$-isolate. Since $v \in V_X^-$, there exists a $\gamma_X$-set $S$ of $G$ such that $pn_X[v,S] = \{v\}$. Therefore, $S - \{v\}$ does not contain any $X$-neighbour of $v$. Therefore, $S - \{v\} \subseteq X - N_Y[v]$. Also $S - \{v\}$ $X$-dominates $G - v$. Therefore, $(S - \{v\}) \cup \{u\}$ is a $\gamma_X$-set of $G$ for any $u \in N_Y(v)$ and $(S - \{v\}) \cup \{u\}$ does not contain $v$. ■

Definition 2.8.15. A bipartite graph $G$ is in CVR if $\gamma_X(G - v) \neq \gamma_X(G)$ $\forall v \in X$.

Theorem 2.8.16. A bipartite graph $G$ is in CVR if and only if $\forall v \in X$ there exists a $\gamma_X$-set $S$ of $G$ such that $pn_X[v,S] = \{v\}$.

Proof: We know that, $\gamma_X(G) \neq \gamma_X(G) \forall v \in X$ if and only if $X = V_X^-$. Therefore, $\forall v \in X$, there exists a $\gamma_X$-set $S$ of $G$ such that $pn_X[v,S] = \{v\}$. ■