Chapter 5

GALLAI TYPE THEOREMS

The aim of this chapter is to study $X$-domatic partitions of a bipartite graph and derive Gallai type theorems involving $X$-domatic partition and coloring.

5.1 Introduction

$X$-domatic partition of a bipartite graph is an extension of the concept of domatic partition in graphs. Various Gallai type theorems involving $Y$-domination number and chromatic number are derived. Nordhaus-Gaddum type results are derived. $X$-indominable graphs are defined and studied.
5.2 X-Domatic partition of a bipartite graph

Definition 5.2.1. A X-domatic partition of $G = (X,Y,E)$ is a partition of $X$, all of whose elements are X-dominating sets in $G$. The X-domatic number of $G$ is the maximum number of classes of a X-domatic partition of $G$. The X-domatic number of a graph $G$ is denoted by $d_X(G)$.

Since $\pi = \{X\}$ is a X-domination partition, existence of a X-domatic partition is guaranteed in any bipartite graph.

Remark 5.2.2. $1 \leq d_X(G) \leq p$.

Theorem 5.2.3. In a bipartite graph $G$, $d_X(G) = p$ if and only if every vertex in $X$ is of X-degree $p - 1$.

Proof: Let $d_X(G) = p$. Every vertex in $X$ is a X-dominating set. Therefore, X-degree of every vertex in $X$ is $p - 1$. Conversely, if every vertex in $X$ is of X-degree $p - 1$, then every vertex $x \in X$ is X-adjacent to all other vertices in $X$. Therefore, $d_X(G) = p$.

Theorem 5.2.4. In a bipartite graph $G$, $d_X(G_B) = p$ if and only if for every $u,v \in X$, there exists $y \in Y$ such that $y$ is not adjacent to $u,v$ in $G$.

Proof: Let $d_X(G_B) = p$. Every vertex in $G_B$ is X-adjacent to all other vertices. There exists, $y \in Y$ such that $u$ and $v$ are adjacent to $y \in Y$ in $G_B$. Then, $y \in Y$ is not adjacent to $u$ and $v$ in $G$.  

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Conversely, suppose that for every \( u, v \in X \), there exists \( y \in Y \) such that \( y \) is not adjacent to \( u \) and \( v \) in \( G \). Then \( y \) is adjacent to \( u \) and \( v \) in \( \overline{G} \). Hence, \( u \) and \( v \) are \( X \)-adjacent in \( \overline{G} \). Hence, \( d_X(\overline{G}) = p \).

\[ \text{Theorem 5.2.5.} \quad \text{In a bipartite graph } G, \quad d_X(G) + d_X(\overline{G}) = 2p \text{ if and only if} \]

1. Every vertex in \( X \) is of \( X \)-degree \( p - 1 \).
2. There exists \( y \in Y \) such that for every \( u, v \in X \), \( y \) is not adjacent to \( u \) and \( v \) in \( G \).

\[ \text{Proof:} \quad \text{Assume the two conditions. By the above two theorems, we get} \]
\[ d_X(G) = p \quad \text{and} \quad d_X(\overline{G}) = p. \quad \text{Hence,} \quad d_X(G) + d_X(\overline{G}) = 2p. \]

Conversely, assume \( d_X(G) + d_X(\overline{G}) = 2p \). This is possible only when \( d_X(G) = p \) and \( d_X(\overline{G}) = p \). By the above two theorems we get the two conditions.

\[ \text{Remark 5.2.6.} \quad \text{Let } \Lambda \text{ be the family of graphs such that every vertex in } X \text{ is of } X \text{-degree } p - 1 \text{ and for any two vertices } u \text{ and } v \text{ in } X, \text{ there exists } y \in Y \text{ such that } u \text{ and } v \text{ are not adjacent to } y. \]

\[ \text{Theorem 5.2.7.} \quad d_X(G) + d_X(\overline{G}) = 2p \text{ if and only if } G \in \Lambda. \]

\[ \text{Remark 5.2.8.} \quad \text{Let } \Lambda_1 \text{ be the family of graphs in which a vertex in } Y \text{ is of degree } |X| \text{ and there exists a unique } \gamma_Y - \text{ set of } G \text{ of cardinality 2.} \]
Theorem 5.2.9. Let $G$ be a graph with $|X| \geq 3$ other than $K_{2,1}$. Then $d_X(G) + d_X(G_B) = 2p - 1$ if and only if $G \in \Lambda_1$.

**Proof:** Let $G \in \Lambda_1$. Clearly, $d_X(G) = p$, $d_X(G_B) = p - 1$. Hence, $d_X(G) + d_X(G_B) = 2p - 1$.

Let $d_X(G) + d_X(G_B) = 2p - 1$. The possibility is $d_X(G) = p$ and $d_X(G_B) = p - 1$ or $d_X(G) = p - 1$ and $d_X(G_B) = p$.

**Case (i):** $d_X(G) = p$ and $d_X(G_B) = p - 1$.

Every vertex in $X$ is of $X$-degree $p - 1$. If every vertex in $X$ is $X$-adjacent to other vertices through different $y \in Y$, then $d_X(G_B) < p - 1$, a contradiction.

Therefore, there exists a vertex $y \in Y$ of full degree.

**Claim:** $\gamma_Y(G) = 2$.

If $\gamma_Y(G) = 1$, then a vertex in $X$ is of full degree. Therefore, $d_X(\overline{G_B}) = 1 \neq p - 1$, a contradiction. Suppose $\gamma_Y(G) \geq 3$. Then any subset of $X$ of cardinality 2 is not a $Y$-dominating set of $G$. Therefore, if $u, v \in X$, then there exists $y \in Y(G)$ such that $u$ and $v$ are not adjacent with $y$. Hence $u$ and $v$ are $X$-adjacent in $\overline{G_B}$. Therefore, $X$ is $X$-complete in $G_B$. Therefore, $d_X(G_B) = p \neq p - 1$, a contradiction. Therefore, $\gamma_Y(G) = 2$. **Claim:** The existence of $\gamma_Y$- set is unique in $G$.

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Let $S_1$ and $S_2$ be two minimum $Y$-dominating sets of $G$. Let $S_1 = \{u_1, u_2\}$ and $S_2 = \{u_3, u_4\}$. Suppose $u_1$ and $u_2$ are $X$-adjacent in $G_B$. Then there exists $y \in Y(G_B)$ such that $u_1$ and $u_2$ are adjacent with $y$ in $G_B$. Therefore, $u_1$ and $u_2$ are not adjacent with $y \in G$. Therefore, $y$ is not $Y$-dominated by $u_1$ and $u_2$, a contradiction. Therefore, $u_1$ and $u_2$ are adjacent in $G_B$.

Similarly, $u_3$ and $u_4$ are not $X$-adjacent in $G_B$. If $u_1, u_2, u_3$ and $u_4$ are all distinct, then $\{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}$ are all non $X$-dominating sets in $G_B$. Therefore, $d_X(G_B) \leq p - 2$, a contradiction. Let $S_1$ and $S_2$ have a common vertex. Let $S_1 = \{u_1, u_2\}$ and $S_2 = \{u_3, u_4\}$. Then $\{u_1\}, \{u_2\}, \{u_3\}$ are all non $X$-dominating set in $G_B$. Therefore, $d_X(G_B) \leq p - 2$, a contradiction.

Therefore, $\gamma_Y$-sets of cardinality 2 in $G$ is unique.

**Theorem 5.2.10.** In a bipartite graph $G$, $\gamma_X(G) + d_X(G) \leq p + 1$.

**Proof:** If $\gamma_X(G) = 1$ then $\gamma_X(G) + d_X(G) \leq p + 1$. If $\gamma_X(G) = k (K \geq 2)$ then $d_X(G) \leq p - k + 1$.

Therefore, $\gamma_X(G) + d_X(G) \leq (k + p - k + 1) = p + 1$.

**Lemma 5.2.11.** In a bipartite graph $G$, $d_X(G) \leq 1 + \delta_Y(G)$.

**Proof:** Let $x \in X$ be a vertex with minimum $X$-degree. Suppose $d_X >
1 + \delta_Y(G). Since \( G \) has at least \( d_X(G) - 1 \) \( X \)-neighbours in \( X \), \( d_Y(x) \geq d_X(G) - 1 > \delta_Y(G) \), a contradiction. Therefore, \( d_X(G) \leq 1 + \delta_Y(G) \).

**Theorem 5.2.12.** If \( G \) is a connected unicyclic graph then \( d_X(G) \leq 3 \).

**Proof:** Since \( G \) is unicyclic, \( G \) contains a unique even cycle \( C_{2n} \), where \( n \) is even. Let \( G = C_{2n} \). We note that \( d_X(C_6) = 3 \) and \( d_X(C_4) = 2 \). Also, \( \gamma_X(C_l) = \left\lceil \frac{l}{6} \right\rceil \) for all \( l \geq 4 \) and \( l \equiv 0, 2, 4 (mod 6) \). Therefore, \( \gamma_X(C_l) \geq \frac{l}{6} \).

Hence, \( d_X(G) \leq \frac{p}{\gamma_X} = \frac{p}{l/6} = 3 \).

In case \( G \neq C_p \), then \( G \) has a pendant vertex and therefore \( d_X(G) \leq 1 + \delta_Y(G) = 2 \).

**Corollary 5.2.13.** For a connected unicyclic graph \( G \), \( d_X(G) = 3 \) if and only if \( G = C_{2n} \) where \( |X| \equiv 0(mod 3) \).

**Proof:** Assume \( d_X(G) = 3 \). When \( |X| \leq 2 \), \( G \) will have a vertex of full \( X \)-degree implying \( d_X(G) = 2 \), a contradiction. Hence \( |X| \geq 3 \). \( G \) cannot have a pendant \( X \)-vertex since in that case \( d_X(G) \leq 2 \). Hence, \( G = C_{2n} \). Since, \( \gamma_X(C_l) = \left\lceil \frac{l}{6} \right\rceil \geq 4 \) and \( l \equiv 0, 2, 4 (mod 6) \) and \( d_X(G) = 3 \), \( |X| \) is a multiple of \( 3 \).

Conversely, if \( |X| \equiv 0(mod 3) \) and \( G = C_{2n} \) where \( X = \{x_1, x_2, \ldots, x_{3n}\} \) and \( Y = \{y_1, y_2, \ldots, y_{3n}\} \). The sets \( X_1 = \{x_1, x_4, \ldots\} \), \( X_2 = \{x_2, x_5, x_8, \ldots\} \)
and \(X_3 = \{x_3, x_6, \cdots \}\) form three \(\gamma_X\)-sets and hence, \(d_X(G) = 3\).

**Corollary 5.2.14.** If \(G\) is a connected unicyclic graph with \(|X| \geq 3\), then \(d_X(G) = 2\) if and only if \(G \neq C_{2n}\) where \(|X| \equiv 0(\text{mod}3)\).

**Proof:** Let \(d_X(G) = 2\) and \(|X| \geq 3\). By the above corollary \(G \neq C_{2n}\) where \(|X| \equiv 0(\text{mod}3)\). Conversely, let \(G \neq C_{2n}\) where \(|X| \equiv 0(\text{mod}3)\). Since \(G\) is unicyclic, \(d_X(G) \leq 3\). \(d_X(G)\) cannot be three by the above corollary and \(d_X(G)\) cannot be one. Therefore, \(d_X(G) = 2\).

5.2.1 X-Domatic partition of Graph \(VE(G)\) and Chromatic number of \(G\)

**Proposition 5.2.15.** \(d_X(VE(K_n)) = n\)

**Proof:** Let \(K_n\) be the complete graph on \(n\) vertices. In the graph \(VE(G)\), every vertex is \(X\)-adjacent to all other vertices of \(X\). That is, every vertex is a \(X\)-dominating set. Therefore, \(d_X(VE(K_n)) = n\).

**Theorem 5.2.16.** For a connected graph \(G\), \(d(\overline{G}) = d_X(VE(G))\).

**Proof:** Let \(V_1, V_2, V_3, \cdots, V_n\) be the domatic partition of \(G\). Each \(V_i\) is a dominating set in \(G\). Therefore, \(V_i\) is a \(X\)-dominating set in the bipartite
graph \( VE(G) \). Hence, \( V_1, V_2, V_3, \ldots, V_n \) is a \( X \)-domatic partition of \( VE(G) \).

Hence, \( d_X(VE(G)) \geq d(G) \).

Let \( X_1, X_2, X_3, \ldots, X_n \) be \( X \)-domatic partition of \( VE(G) \). Each \( X_i \) is a \( X \)-dominating set in \( VE(G) \). Therefore, \( X_i \) is a dominating set in the graph \( G \). Hence, \( X_1, X_2, X_3, \ldots, X_n \) is a domatic partition of \( G \). Hence, \( d(G) \geq d_X(VE(G)) \). Therefore, \( d(G)=d_X(VE(G)) \).

\[ \text{Corollary 5.2.17. } d_X(VE(G)) = n \text{ if and only if } G \cong K_n. \]

\[ \text{Theorem 5.2.18. For any connected graph } G \text{ of order } n, \ d_X(VE(G)) + \chi(G) \leq 2n \text{ and equality holds if and only if } G \cong K_n. \]

**Proof:** \( d_X(VE(G)) + \chi(G) \leq n + \Delta + 1 \leq n + n - 1 + 1 = 2n \). If \( d_X(VE(G)) + \chi(G) = 2n \). Then the possible case is \( d_X(VE(K_n)) = n \) and \( \chi(G) = n \). Since, \( \chi(G) = n, G \cong K_n \). The converse is obvious.

\[ \text{Theorem 5.2.19. There exists no connected graph } G \text{ of order } n \text{ with } d_X(VE(G)) + \chi(G) = 2n - 1. \]

**Proof:** Assume \( d_X(VE(G)) + \chi(G) = 2n - 1 \). This is possible only if \( d_X(VE(G)) = n - 1 \) and \( \chi(G) = n \) or \( d_X(VE(G)) = n \) and \( \chi(G) = n - 1 \).

**Case (a):** \( d_X(VE(G)) = n \) and \( \chi(G) = n - 1 \). Then \( G \cong K_n \) and so \( \chi(G) = n \), a contradiction.
Case (b): $d_X(VE(G)) = n - 1$ and $\chi(G) = n$.

Since, $\chi(G) = n$, $G = K_n$ and hence $d_X(VE(K_n)) = n$, a contradiction. So no graphs exist.

We define certain families of graphs

$\mathcal{G}_1$ is the family of graphs on $n$ vertices which contains a clique $K = K_{n-1}$ and the other vertex is adjacent to $(n - 2)$ vertices of $G$.

$\mathcal{G}_2$ is the family of graphs on $n$ vertices which contains a clique $K = K_{n-2}$ and the other vertices are adjacent to all the $(n - 2)$ vertices of $K$.

$\mathcal{G}_3$ is the family of graphs on $n$ vertices which contains a clique $K = K_{n-1}$ and the other vertex is adjacent to $(n - 3)$ vertices of $G$.

$\mathcal{G}_4$ is the family of graphs on $n$ vertices which contain a clique $K = K_{n-1}$ and the other vertex is adjacent to $(n - 4)$ vertices of $G$.

$\mathcal{G}_5$ is the family of graphs on $n$ vertices which contain a clique $K = K_{n-2}$ and the other vertices are adjacent to the same $(n - 3)$ vertices of $G$.

$\mathcal{G}_6$ is the family of graphs which contain a clique $K = K_{n-3}$ and the remaining three vertices are independent and adjacent to all the vertices of $K$.

**Theorem 5.2.20.** For a connected graph $G$, $d_X(VE(G)) + \chi(G) = 2n - 2$ if and only if $G$ belongs to $\mathcal{G}_1$ or $\mathcal{G}_2$. 

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\textbf{Proof}: If} $G$ \text{belongs to $G_1$ or $G_2$ then $d_X(VE(G)) + \chi(G) = 2n - 2$ is obvious.

Conversely, assume that $d_X(VE(G)) + \chi(G) = 2n - 2$. This is possible only if (a) $d_X(VE(G)) = n$ and $\chi(G) = n - 2$ or (b) $d_X(VE(G)) = n - 1$ and $\chi(G) = n - 1$ or (c) $d_X(VE(G)) = n - 2$ and $\chi(G) = n$.

\textbf{Case (a):} $d_X(VE(G)) = n$ \textbf{and} $\chi(G) = n - 2$

Since $d(G) = n$ and hence $G \cong K_n$. Therefore, $\chi(G) = n$, a contradiction. A similar argument shows that (c) is not possible. Therefore, $d_X(VE(G)) = n - 1$ and $\chi(G) = n - 1$.

\textbf{Case (b):} $d_X(VE(G)) = n - 1$ \textbf{and} $\chi(G) = n - 1$

Since $\chi(G) = n - 1$, $G$ contains a clique $K = K_{n-1}$ or a clique $K = K_{n-2}$ and two other vertices which are not adjacent but adjacent to all the $(n-2)$ vertices of $K_{n-2}$. Let $x$ be the vertex in $V(G) - V(H)$.

\textbf{Case 1:} Let $G$ contain a clique $K = K_{n-1}$.

Since $G$ is connected, $x$ should be adjacent to at least one vertex of $K$. If $x$ is adjacent to $(n-2)$ vertices of $K$, then $d_X(VE(G)) = n - 1$. So, $G$ belongs to $G_1$.

If $x$ is adjacent to $\alpha$ vertices $(\alpha < n - 2)$ of $K$, we get $d_X(VE(G)) = \alpha + 1$. Hence, $\alpha = \alpha + 2$. Therefore, $\chi(G) = 2n - 2 - (\alpha + 1) = 2\alpha + 4 - 2 - \alpha - 1$.
\[ \alpha + 1 < n - 1, \text{ a contradiction.} \]

**Case 2:** Let \( G \) contain a clique \( K = K_{n-2} \) and two other vertices which are not adjacent but adjacent to all the \( n-2 \) vertices of \( K_{n-2} \). Then, \( G \in G_2 \).

Therefore, \( d_X(VE(G)) = n - 2 \) and \( \chi(G) = n \). Since \( \chi(G) = n \), \( G \cong K_n \).

But \( d_X(VE(G)) = n \neq n - 2 \), a contradiction. \( \square \)

**Theorem 5.2.21.** For a connected graph \( G \), \( d_X(VE(G)) + \chi(G) = 2n - 3 \) if and only if \( G \) belongs to \( G_3 \).

**Proof:** If \( G \) belongs to \( G_3 \), then clearly \( d_X(VE(G)) + \chi(G) = 2n - 3 \).

Conversely, assume that \( d_X(VE(G)) + \chi(G) = 2n - 3 \). This is possible only if a) \( d_X(VE(G)) = n - 3 \) and \( \chi(G) = n \) or b) \( d_X(VE(G)) = n - 2 \) and \( \chi(G) = n - 1 \) or c) \( d_X(VE(G)) = n - 1 \) and \( \chi(G) = n - 2 \) or d) \( d_X(VE(G)) = n \) and \( \chi(G) = n - 3 \).

**Case (a):** \( d_X(VE(G)) = n - 3 \) and \( \chi(G) = n \)

Since \( \chi(G) = n \), \( G \cong K_n \). But \( d_X(VE(K_n)) = n \neq n - 3 \) a contradiction.

**Case (b):** \( d_X(VE(G)) = n - 2 \) and \( \chi(G) = n - 1 \)

Since \( \chi(G) = n - 1 \), \( G \) contains a clique \( K = K_{n-1} \) or \( K_n - 2 \) and the remaining two vertices are not adjacent but adjacent to all the \( n - 2 \) vertices of \( K_{n-2} \).

**Case 1:** Let \( G \) contain a clique \( K = K_{n-1} \). Let \( x \) be a vertex of \( G \) other
than the vertices of $K$. Since $G$ is connected, $x$ is adjacent to at least one vertex of $K$. If $x$ is adjacent to $\alpha$ vertices of $K$. $(1 \leq \alpha < n - 1)$, then, $d_x(VE(G)) = \alpha + 1$. $n - 2 = \alpha + 1 \Rightarrow n = \alpha + 3$. $G$ contains a clique $K = K_{n-1}$ and the other vertex is adjacent to $n - 3$ vertices of $K$. Thus, $G$ belongs to $G_3$.

Case 2: Let $G$ contain a clique $K = K_{n-2}$ and the remaining two vertices are not adjacent but adjacent to all the $n - 2$ vertices of $K_{n-2}$. In this case, $d_x(VE(G)) = n - 1$, a contradiction.

**Case (c):** $d_x(VE(G)) = n - 1$ and $\chi(G) = n - 2$

Since $d_x(VE(G)) = n - 1$, $G$ contains a clique of order $n - 2$ and two independent vertices which are adjacent to all the $n - 2$ vertices of $K_{n-2}$. In this case $\chi(G) = n - 1$, a contradiction.

**Case (d):** $d_x(VE(G)) = n$ and $\chi(G) = n - 3$

Therefore, $d(G) = n$. Hence $G \cong K_n$. Which implies that $\chi(G) = n$, a contradiction. ■

**Theorem 5.2.22.** For a connected graph $G$, $d_x(VE(G)) + \chi(G) = 2n - 4$ if and only if $G$ belongs to $G_4$ or $G_6$ or isomorphic to $C_4$ or $P_4$.

**Proof:** If $d_x(VE(G)) + \chi(G) = 2n - 4$ then one of the following is possible
a) $d_X(VE(G)) = n - 4$ and $\chi(G) = n$ or (b) $d_X(VE(G)) = n - 3$ and $\chi(G) = n - 1$ or (c) $d_X(VE(G)) = n - 2$ and $\chi(G) = n - 2$ or (d) $d_X(VE(G)) = n - 1$ and $\chi(G) = n - 3$ or (e) $d_X(VE(G)) = n$ and $\chi(G) = n - 4$.

Case (a) and (e) are not possible. Since, if $\chi(G) = n$ or $d_X(VE(G)) = n$ then $G \cong K_n$.

Case (b): $d_X(VE(G)) = n - 3$ and $\chi(G) = n - 1$

Since $\chi(G) = n - 1$, $G$ contains a clique $K = K_{n-1}$ or a clique $K = K_{n-2}$ and the remaining two vertices are independent and adjacent to all the $n - 2$ vertices of the clique $K$.

Subcase (i): $G$ contains a clique $K = K_{n-1}$. Let $x \in V(G) - V(K)$.

Since $G$ is connected, $x$ is adjacent to at least one vertex of $K$. Let $x$ be adjacent to $\alpha$ vertices of $K$ ($1 \leq \alpha \leq n - 1$). Then $d_X(VE(G)) = \alpha + 1$ but $d_X(VE(G)) = n - 3$. Therefore, $\alpha + 1 = n - 3$. That is $n = \alpha + 4$. Therefore, $G \in \mathcal{G}_4$.

Subcase (ii): $G$ contains a clique $K = K_{n-1}$ or a clique $K = K_{n-2}$ and the remaining two vertices are independent and adjacent to all the $n - 2$ vertices of the clique $K$. In this case $\chi(G) = n - 1$ and $d_X(VE(G)) = n - 1$, a contradiction.
Case (c): $d_X(VE(G)) = n - 2$ and $\chi(G) = n - 2$.

Since $\chi(G) = n - 2$, either $G$ contains a clique $K = K_{n-2}$ and the remaining two vertices are (i) either independent and adjacent to at most $n - 3$ vertices of $K$ or (ii) adjacent and the set of neighbours of these two vertices in $K$ are distinct OR $G$ contains a clique $K = K_{n-3}$ and the remaining three vertices are independent and adjacent to all the vertices of $K$.

Suppose (i) holds. Let $x_1, x_2$ be the vertices in $V(G) - V(K)$. Suppose $x_1$ and $x_2$ are adjacent to the same set of vertices of cardinality $\alpha$ in $K$. Then $\alpha \leq n - 3$. In this case $d_X(VE(G)) = \alpha + 1 = n - 2$. Therefore, $\alpha = n - 3$.

Therefore, $G \in \mathcal{G}_5$.

When (ii) holds then the vertices $x_1, x_2$ are adjacent to at most $n - 4$ vertices of $K$. Proceeding as in (i) $\alpha = n - 4$. Therefore, $x_1$ and $x_2$ are adjacent to $n - 3$ vertices of $G$. Hence, $G \in \mathcal{G}_5$.

Suppose $G$ contains a clique $K = K_{n-3}$ and the remaining three vertices are independent and adjacent to all the vertices of $K$. Then, $G \in \mathcal{G}_6$.

Case (d): $d_X(VE(G)) = n - 1$ and $\chi(G) = n - 3$.

Let $\{ V_1, V_2, V_3, ..., V_{n-1} \}$ be a X-domatic partition of $VE(G)$. Let $t$ of these sets are singletons ($0 \leq t \leq n - 1$). Therefore, $n = \sum_{i=1}^{n-1} |V_i| \geq t + (n - 1 - t)2$. 

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Hence, \( n \geq 2n - t - 2 \). Therefore, \( t \geq n - 2 \). If \( t = n \) or \( n - 1 \) then \( d_X(VE(G)) = n \), a contradiction. Therefore, \( t = n - 2 \). Therefore, there are \( n - 2 \) full degree vertices in \( G \). Therefore, the vertex set of \( G \) contains a clique \( K_{n-2} \) and the remaining two points are independent and adjacent to all the vertices of the clique. Therefore, \( \chi(G) = n - 1 \), a contradiction.

Conversely, if \( G \) belongs to \( \mathcal{G}_4 \) or \( \mathcal{G}_5 \) or \( \mathcal{G}_6 \) or isomorphic to \( C_4 \) or \( P_4 \), it can be verified that \( d_X(VE(G)) + \chi(G) = 2n - 4 \).

\[ \square \]

### 5.2.2 X-Domatic Partition of graph \( V V^+ \) and Chromatic number of \( G \)

In what follows, \( G \) is a connected bipartite graph.

**Proposition 5.2.23.** \( d_X(VV^+(K_n)) = n \).

**Proof:** Let \( K_n \) be a complete graph on \( n \) vertices. In \( V V^+(G) \), every vertex is \( X \)-adjacent to other vertices and hence, \( d_X(VV^+(K_n)) = n \).

\[ \square \]

**Theorem 5.2.24.** For a connected graph \( G \), \( d(G^2) = d_X(VV^+(G)) \)

**Proof:** Let \( V_1, V_2, V_3, \ldots, V_n \) be domatic partition of \( G^2 \). Each \( V_i \) is a dominating set in \( G^2 \). Therefore, \( V_i \) is a \( X \)-dominating set in the bipartite graph
$VV^+(G)$. Hence, $V_1, V_2, V_3, \ldots, V_n$ is a $X$-domatic partition of $VV^+(G)$.

Hence, $d_X(VV^+(G)) \geq d(G^2)$.

Let $X_1, X_2, X_3, \ldots, X_n$ be $X$-domatic partition of $VV^+(G)$. Each $X_i$ is a $X$-dominating set in $VV^+(G)$. Therefore, $X_i$ is a dominating set in the graph $G^2$. Hence, $X_1, X_2, X_3, \ldots, X_n$ is a domatic partition of $G^2$. Hence, $d(G^2) \geq d_X(VV^+(G))$. Therefore, $d(G^2)=d_X(VV^+(G))$.

**Theorem 5.2.25.** For any connected graph $G$, $d_X(VV^+(G))+\chi(G) \leq 2n$ and equality holds if and only if $G \cong K_n$.

**Proof:** $d_X(VV^+(G))+\chi(G) \leq n + \Delta + 1 \leq n + n - 1 + 1 = \leq 2n$. If $d_X(VV^+(G))+\chi(G) = 2n$, then the possible case is $d_X(VV^+(G)) = n$ and $\chi(G) = n$. Since, $\chi(G) = n$, $G \cong K_n$ and $d_X(VV^+) = n$. The converse of the above is obvious.

$G_7$ be the family of graphs on $n$ vertices which contain a clique $K_{n-2}$ and the remaining two vertices are such that every vertex in $K_{n-2}$ is adjacent to at least one of the two vertices.

**Theorem 5.2.26.** For any connected graph $G$, $d_X(VV^+(G))+\chi(G) = 2n-1$ if and only if $G \in G_7$.

**Proof:** If $G \in G_7$ then $\chi(G) = n - 1$ and $d_X(VV^+(G)) = n - 1$ and hence,
\[ d_X(VV^+(G)) + \chi(G) = 2n - 1. \]

Assume \( d_X(VV^+) + \chi(G) = 2n - 1 \). This is possible only if \( d_X(VV^+) = n \) and \( \chi(G) = n - 1 \) or \( d_X(VV^+) = n - 1 \) and \( \chi(G) = n \).

**Case (i):** \( d_X(VV^+(G)) = n \) and \( \chi(G) = n - 1 \)

Let \( \chi(G) = n - 1 \). Let \( \{V_1, V_2, V_3, ..., V_\chi\} \) be a chromatic partition of \( G \). Let \( t \) of these \( V_i \)'s \( 1 \leq i \leq \chi \) be singletons. Then, \( |V| = n = \sum_{i=1}^{\chi-1} |V_i| \geq 2n - 2 - t \).

Therefore, \( t \geq n - 2 \). If \( t = n - 1 \), then these \( n - 1 \) singleton classes together constitute \( V \), a contradiction. Therefore, \( t = n - 2 \). Since in any chromatic partition, there is an edge between any two colour classes, the \( n - 2 \) singleton form a clique \( K = K_{n-2} \). Let \( u_1, u_2, \ldots, u_{n-2} \) form \( K_{n-2} \). The remaining two vertices say \( u_{n-1}, u_n \) of \( V(G) \) are independent and any \( u_i, 1 \leq i \leq n - 2 \) is adjacent to either \( u_{n-1} \) or \( u_n \). If \( u_{n-1} \) is not adjacent to some \( u_i \) and \( u_n \) is not adjacent to some \( u_j \) \( i \neq j, 1 \leq j \leq n - 2 \) then \( \chi(G) = n - 2 \), a contradiction.

Therefore, if \( u_{n-1} \) is not adjacent to some \( u_i \), \( u_n \) is adjacent with every \( u_i \) \( 1 \leq i \leq n - 2 \). In this case \( G^2 = K_n \) and hence \( d_X(VV^+(G)) = d_X(G^2) = n \).

**Case (ii):** \( d_X(VV^+) = n - 1 \) and \( \chi(G) = n \)

In this case \( G \cong K_n \) and hence \( d_X(VV^+(G)) = n \), a contradiction. \( \blacksquare \)

We define the family of graphs
$G_8$ is the family of graphs containing $K_{n-2} - \{uy\}$ and the remaining two vertices say $x, v$ are adjacent. One of them is adjacent to $y$ and not to any other vertex of $K_{n-2} - \{uy\}$. The other vertex is adjacent to at least one vertex of $K_{n-4}$ and not to all of them. It is also adjacent to $u$.

$G_9$ is the family of graphs containing $K_{n-2} - \{uy\}$ and the remaining two vertices say $x, v$ are adjacent. One of them is adjacent to $y$ and not to any vertex adjacent to $u$ and not to any vertex of $K_{n-4}$.

$G_{10}$ is the family of graphs containing $K_{n-2} - \{uy\}$ and the remaining two vertices say $x, v$ such that either $x$ and $v$ are adjacent to same vertex in $K_{n-4}$ or $x$ and $v$ are adjacent to different vertices in $K_{n-4}$ and $x$ and $v$ are adjacent. Also, $x$ and $v$ are not adjacent to all of the vertices of $K_{n-4}$.

$G_{11}$ is the family of graphs containing $K_{n-2} - \{uy\}$ and the remaining two vertices say $x, v$ are such that $v$ and $x$ are adjacent to same vertex in $K_{n-4}$ and $v$ is adjacent to $y$.

$G_{12}$ is the family of graphs containing $K_{n-2}$ and the remaining vertices are independent and they are adjacent to the same vertex in $V(K_{n-2}) - \{u, y\}$.

$G_{13}$ is the family of graphs containing a clique $K_{n-3}$ and the remaining 3
vertices form a clique and each of them is adjacent to at least one vertex of $K_{n-3}$ but not to all of them.

**Theorem 5.2.27.** For any connected graph $G$, $d_X(VV^+) + \chi(G) = 2n - 2$ if and only if $G \in G_8, G_9, G_{10}, G_{11}, G_{12}$ or $G_{13}$.

**Proof:** If $G \in G_8, G_9, G_{10}, G_{11}, G_{12}$ or $G_{13}$ then $d_X(VV^+) + \chi(G) = 2n - 2$ is obvious.

Conversely, assume that $d_X(VV^+) + \chi(G) = 2n - 2$.

**Case (i):** $\chi(G) = n$ and $d_X(VV^+(G)) = n - 2$.

Since $\chi(G) = n$, $G \cong K_n$ and $d_X(VV^+(G)) = n$, a contradiction.

**Case (ii):** $\chi(G) = n - 1$ and $d_X(VV^+(G)) = n - 1$.

Since $\chi(G) = n - 1$, proceeding as in the theorem 5.2.26, we get that $d_X(VV^+(G)) = n$, a contradiction.

**Case (iii):** $\chi(G) = n - 2$ and $d_X(VV^+(G)) = n$.

Since $\chi(G) = n - 2$, the number of singleton classes in any chromatic partition is greater than or equal to $n - 4$. (For: if there are $t$ singleton classes, then $n = \sum_{i=1}^{n-2} |V_i| \geq t + (n - 2 - t)2$) If there are $n - 2$ singleton's then $|V| = n = n - 2$, a contradiction. Therefore, there are either $n - 3$
singletons and a class of 3 independent vertices or \( n - 4 \) singletons and two independent classes containing together 4 vertices. If one of these two classes is a singleton, then we get \( n - 3 \) singleton and a class of 3 independent vertices. Therefore, we have two cases.

**Case (i):** \( n - 4 \) singleton class and two classes of two elements each.

Let \( \pi = \{\{u_1, u_2, \ldots, u_{n-4}\}, \{x, y\}, \{u, v\}\} \) be a chromatic partition of \( G \).

Arguing as in the previous case, if \( x \) is not adjacent to any \( u_i \), \( 1 \leq i \leq n - 4 \). \( y \) is adjacent to every \( u_i \). Similarly one of \( u, v \) say \( u \) is adjacent to every \( u_i \) \( 1 \leq i \leq n - 4 \). There is an edge between the classes \( \{x, y\} \) and \( \{u, v\}\).

**Subcase (i):** \( x \) is adjacent to \( u \).

Therefore, \( x \) is adjacent to every \( u_i \) in \( G^2 \), \( 1 \leq i \leq n - 4 \). Since, \( u \) and \( y \) are adjacent to every \( u_i \) in \( G \), \( u \) and \( y \) are adjacent to every \( u_i \) in \( G^2 \).

**Subsub Case (i):** \( v \) is not adjacent to any \( u_i \), \( 1 \leq i \leq n - 4 \). Therefore, \( v \) is adjacent to \( x \) or \( y \) in \( G \).

**Subsubsub Case (i):** Let \( v \) be adjacent to \( x \) in \( G \).

**Case (a):** \( x \) is adjacent to some \( u_i \), \( 1 \leq i \leq n - 4 \). Therefore, \( x \) is adjacent to \( y \) in \( G^2 \). Since \( u \) is adjacent to \( x \) and \( x \) is adjacent to \( v \), \( u \) is adjacent to \( v \) in \( G^2 \). Since \( y \) is adjacent to \( u_i \) and \( u \) is adjacent to \( u_i \) in \( G \). \( u \) and \( y \)
are adjacent in $G^2$. Since, $d(G^2) = n$, $G^2 = K_n$ and hence $y$ and $v$ must be adjacent in $G$. Therefore, $G \in \mathcal{G}_8$.

Case (b): $x$ is not adjacent to any $u_i$ in $G$, $1 \leq i \leq n - 4$. $u$ and $v$ are adjacent in $G^2$. Since $d(G^2) = n$, $G^2 = K_n$. Therefore, $x$ is adjacent to some $u_i$ in $G$, a contradiction.

Subsubsub Case (ii): Let $v$ be adjacent to $y$ in $G$. Then $x$ is adjacent to every $u_i$, $1 \leq i \leq n - 4$ in $G^2$ (since $x$ is adjacent to $u$ in $G$), $v$ is adjacent to every $u_i$, $1 \leq i \leq n - 4$ in $G^2$ (since $v$ is adjacent to $y$ in $G$). Also, $y$ is adjacent to $u$ in $G^2$. Since $d(G^2) = n$, $G^2 = K_n$. Therefore, $x$ and $v$ are adjacent in $G$. Therefore, $G \in \mathcal{G}_9$.

Subsub Case (ii): $v$ is adjacent to some $u_i$, $1 \leq i \leq n - 4$. Therefore, $v$ is adjacent to every $u_i$, $1 \leq i \leq n - 4$, $y$ and $u$ in $G^2$. Since $d(G^2) = n$, $G^2 = K_n$. Since $x$ is adjacent to the same $u_i$ in $G$ to which $v$ is adjacent or $x$ is adjacent to same $u_j$, $j \neq i$ in $G$ and $x$ and $v$ are adjacent. Therefore, $G \in \mathcal{G}_{10}$.

Subcase(ii): $x$ is not adjacent to both $u$ and $v$. Therefore, $y$ is adjacent to $u$ or $v$. Therefore, $x$ is adjacent to some $u_i$ in $G$.

Case A: $y$ is adjacent to $v$. 138
Subcase (i): \( v \) is adjacent to some \( u_i \) in \( G \). Therefore, \( x \) is adjacent to 
\( u,y,u_j \) (\( 1 \leq j \leq n-4 \)) in \( G^2 \). \( v \) is adjacent to \( u,y,u_j \) (\( 1 \leq j \leq n-4 \)) in \( G^2 \). \( y \) is adjacent to \( u \) in \( G^2 \). Since \( d(G^2) = n \); \( G^2 = K_n \). Therefore, \( x \) is adjacent to same \( u_i \) in \( G \) to which \( v \) is adjacent. Therefore, \( G \in \mathcal{G}_{11} \).

Subcase (ii): \( v \) is not adjacent to any \( u_i \), \( 1 \leq i \leq n-4 \).

\( x \) is adjacent to \( u,y,u_j \) (\( 1 \leq j \leq n-4 \)) in \( G^2 \). \( v \) is adjacent to \( y,u_j \) (\( 1 \leq j \leq n-4 \)) in \( G^2 \). \( y \) is adjacent to \( u \) in \( G^2 \). Since \( d(G^2) = n \), \( G^2 = K_n \). Therefore, \( v \) is adjacent to some \( u_i \) \( 1 \leq i \leq n-4 \), a contradiction.

Case B: \( y \) is adjacent to \( u \).

Subcase (i): \( v \) is adjacent to some \( u_i \) in \( G \).

Therefore, \( x \) and \( v \) are not adjacent in \( G^2 \). Since \( G^2 = K_n \), \( x \) and \( v \) must be adjacent to the same vertex \( u_i \) \( 1 \leq i \leq n-4 \) in \( G \). Therefore, \( G \in \mathcal{G}_{12} \).

Subcase (ii): \( v \) is not adjacent to any \( u_i \) \( 1 \leq i \leq n-4 \) in \( G \).

\( x \) is adjacent to \( u,y,u_j \) (\( 1 \leq j \leq n-4 \)) in \( G^2 \). Then \( v \) is an isolate in \( G \). Hence, \( v \) is adjacent to \( y \). Therefore, \( y \) is adjacent to \( u \) and \( v \). Since \( G^2 = K_n \), \( x \) and \( v \) must be adjacent in \( G \) or \( v \) is adjacent to some \( u_i \) in \( G \), a contradiction. Case (ii): Let \( n - 3 \) vertices of \( G \) form singleton classes and the remaining three vertices form a single class in any chromatic partition of
In this case, if \( \{ \{ u_1 \}, \{ u_2 \}, \{ u_3 \}, \ldots, \{ u_{n-3} \}, \{ u_{n-2}, u_{n-1}, u_n \} \} \) is a chromatic partition of \( G \), then as \( G \) has no isolates, each of \( u_{n-2}, u_{n-1}, u_n \) is adjacent to some \( u_i \), \( 1 \leq i \leq n - 3 \) (not necessarily the same). Therefore, \( G^2 = K_n - \{ u_{n-2}u_{n-1}, u_{n-1}u_n, u_{n-2}u_n \} \). Since, \( d(G^2) = n \), \( u_{n-2}u_{n-1}, u_{n-1}u_n \) and \( u_{n-2}u_n \) are edges in \( G \). Therefore, \( G \in \mathcal{G}_{13} \). 

\( \mathcal{G}_{14} \) is the family of graphs containing \( K_{n-3} \) and the remaining vertices are independent. Also two of the remaining vertices do not have a common neighbour in \( K_{n-3} \) and both of them have a common neighbour with the remaining vertices in \( K_{n-3} \).

\( \mathcal{G}_{15} \) is the family of graphs containing a clique \( K_{n-4} \) with remaining vertices \( x, y, u \) and \( v \) satisfying the following: \( x \) and \( y \) are independent. \( u \) and \( v \) are independent. \( x \) is adjacent to \( u \) and \( y \) is adjacent to \( v \). \( v \) is adjacent to some \( u_i \), \( 1 \leq i \leq n - 4 \). \( x \) is not adjacent to \( u, v \) but adjacent to some \( u_j \neq u_i \) in \( G \).

\( \mathcal{G}_{16} \) is the family of graphs containing a clique \( K_{n-6} \). The remaining six vertices \( x, y, u, v, w, z \) form three independent classes \( \{ x, Y \}, \{ u, v \}, \{ w, z \} \) of two elements. There exists at least one edge between each of these inde-
ependent classes and $y, u$ and $w$ are adjacent to every vertex in $K_{n-6}$ (i.e. out of the six remaining vertices three of them are adjacent to every vertex in $K_{n-6}$) and the three other are adjacent to the same vertex in $K_{n-6}$.

$G_{17}$ is the family of graphs containing a clique $K_{n-6}$. The remaining six vertices $x, y, u, v, w$ and $z$ form three independent classes $\{x, Y\}, \{u, v\}, \{w, z\}$ of two elements. There exists at least one edge between each of these independent classes and $y, u$ and $w$ are adjacent to every vertex in $K_{n-6}$ (i.e. out of the six remaining vertices three of them are adjacent to every vertex in $K_{n-6}$). Out of three others two are adjacent to the same vertex in $K_{n-6}$ and the remaining is not adjacent to that vertex in $K_{n-6}$ but adjacent to the two vertices.

$G_{18}$ is the family of graphs containing a clique $K_{n-6}$. The remaining six vertices $x, y, u, v, w$ and $z$ form three independent classes $\{x, Y\}, \{u, v\}, \{w, z\}$ of two elements. There exists at least one edge between each of these independent classes and $y, u$ and $w$ are adjacent to every vertex in $K_{n-6}$ (i.e. out of the six remaining vertices three of them are adjacent to every vertex in $K_{n-6}$). The three other vertices must form a clique in $G$ and each of them is adjacent to at least one vertex in $K_{n-6}$ and none of them are adjacent to
the same vertex in $K_{n-6}$ in $G$.

$G_{19}$ is the family of graphs containing a clique $K_{n-6}$. The remaining six vertices $x, y, u, v, w$ and $z$ form three independent classes $\{x, y\}, \{u, v\}, \{w, z\}$ of two elements. There exists at least one edge between each of these independent classes and $y, u$ and $w$ are adjacent to every vertex in $K_{n-6}$ (i.e., out of the six remaining vertices three of them are adjacent to every vertex in $K_{n-6}$). Of the remaining three vertices one of them say $x$ is not adjacent to any $u_i$, $1 \leq i \leq n-6$, the other two $v$ and $z$ are adjacent to some $u_j$. $u$ is adjacent to $x$ and $y$ and $v$ is adjacent to $w$ and $z$.

$G_{20}$ is the family of graphs containing a clique $K_{n-6}$. The remaining six vertices $x, y, u, v, w$ and $z$ form three independent classes $\{x, Y\}, \{u, v\}, \{w, z\}$ of two elements. There exists at least one edge between each of these independent classes and $y, u$ and $w$ are adjacent to every vertex in $K_{n-6}$ (i.e., out of the six remaining vertices three of them are adjacent to every vertex in $K_{n-6}$). Of the remaining three vertices two of them say $x, v$ are not adjacent to any $u_i$, $1 \leq i \leq n-6$, the other vertex say $z$ is adjacent to some $u_j$. $u$ is adjacent to $x$ and $v$ is adjacent to $x, y, w$ and $z$.

$G_{21}$ is the family of graphs containing a clique $K_{n-5}$ and the remaining
five vertices say \(x, y, u, v\) and \(w\) forming two independent sets \(\{x, y\}\) and \(\{u, v, w\}\). Two of them (one from each class) is adjacent to every \(u_i\) in \(K_{n-5}\) and the remaining one adjacent to the same vertex in \(K_{n-5}\).

\(G_{22}\) is the family of graphs containing a clique \(K_{n-5}\) and the remaining five vertices say \(x, y, u, v\) and \(w\) forming two independent sets \(\{x, y\}\) and \(\{u, v, w\}\). Two of them (one from each class) is adjacent to every \(u_i\) in \(K_{n-5}\). Two of the remaining are adjacent to the same vertex say \(u_i\) in \(K_{n-5}\) and the third is adjacent to same vertex in \(K_{n-5}\) different from \(u_i\). Further \(x\) is adjacent to \(u, v\) and \(w\).

\(G_{23}\) is the family of graphs containing a clique \(K_{n-5}\) and the remaining five vertices say \(x, y, u, v\) and \(w\) forming two independent sets \(\{x, y\}\) and \(\{u, v, w\}\). Two of them (one from each class) is adjacent to every \(u_i\) in \(K_{n-5}\). All the three vertices \(x, v\) and \(w\) are adjacent to different vertices in \(K_{n-5}\), \(x\) is adjacent to \(u, v\) and \(w\).

\(G_{24}\) is the family of graphs containing a clique \(K_{n-5}\) and the remaining five vertices say \(x, y, u, v\) and \(w\) forming two independent sets \(\{x, y\}\) and \(\{u, v, w\}\). Two of them (one from each class) is adjacent to every \(u_i\) in \(K_{n-5}\). Of the three vertices \(x, v\) and \(w\), one of them say \(x\) is not adjacent to any
vertex in $K_{n-5}$ and the other two are adjacent to some vertex in $K_{n-5}$. Also, $x$ is adjacent to $u, v$ and $w$ and $y$ is adjacent to $v$.

$\mathcal{G}_{25}$ is the family of graphs containing a clique $K_{n-5}$ and the remaining five vertices say $x, y, u, v$ and $w$ forming two independent sets $\{x, y\}$ and $\{u, v, w\}$. Two of them (one from each class) is adjacent to every $u_i$ in $K_{n-5}$. Of the three $x, v$ and $w$, one of them say $x$ is not adjacent to any vertex in $K_{n-5}$ and the other two are adjacent to some vertex in $K_{n-5}$. Also $x$ is adjacent to $u, v$ and $w$ and $y$ is adjacent to $v$ and $w$.

$\mathcal{G}_{26}$ is the family of graphs containing a clique $K_{n-4}$ and the remaining vertices say $x, y, u, v$ forming a single independent set. One of the four vertices say $y$ is adjacent to every vertex in $K_{n-4}$ and other three are adjacent to same vertex in $K_{n-4}$.

$\mathcal{G}_{27}$ is the family of graphs containing a clique $K_{n-4}$ and the remaining vertices say $x, y, u, v$ forming a single independent set. One of the four vertices say $y$ is adjacent to every vertex in $K_{n-4}$ and every two of $x, u$ and $v$ have a common neighbour in $K_{n-4}$.

**Theorem 5.2.28.** For any connected graph $G$, $d_X(VV') + \chi(G) = 2n - 3$ if and only if $G \in \mathcal{G}_{14}, \mathcal{G}_{15}, \mathcal{G}_{16}, \mathcal{G}_{17}, \mathcal{G}_{18}, \mathcal{G}_{19}, \mathcal{G}_{20}, \mathcal{G}_{21}, \mathcal{G}_{22}, \mathcal{G}_{23}, \mathcal{G}_{24}, \mathcal{G}_{25}, \mathcal{G}_{26}$ or
\(G_{27}\).

**Proof:** If \(G \in G_{14}, G_{15}, G_{16}, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}, G_{22}, G_{23}, G_{24}, G_{25}, G_{26}\) or \(G_{27}\), then \(d_X(VV^+) + \chi(G) = 2n - 2\) is obvious.

Conversely, assume that \(d_X(VV^+) + \chi(G) = 2n - 3\).

**Case (A):** \(\chi(G) = n\) and \(d_X(VV^+(G)) = n - 3\).

In this case \(G - K_n\) and hence \(G^2 = K_n\) and \(d_X(VV^+(G)) = n\), a contradiction.

**Case (B):** \(\chi(G) = n - 1\) and \(d_X(VV^+(G)) = n - 2\).

Since, \(\chi(G) = n - 1\), proceeding as in the theorem 5.2.27, we get \(d_X(VV^+(G)) = n\), a contradiction.

**Case (C):** Since \(\chi(G) = n - 2\), either \(G\) contains a clique \(K_{n-3}\) and the remaining vertices say \(x_1, x_2\) and \(x_3\) are independent with each of them being adjacent to some vertex of the clique OR \(G\) contains \(K_{n-4}\) and the remaining four vertices form two independent sets with an edge between them.

**Case (I):** \(G\) contains a clique \(K_{n-3}\) and the remaining vertices say \(x_1, x_2\) and \(x_3\) are independent with each of them being adjacent to some vertex of the clique. Since \(d_X(VV^+(G)) = n - 1\), we have \(d(G^2) = n - 1\). The vertices \(x_1, x_2\) and \(x_3\) must form two dominating sets in \(G^2\). This is possible only if one of them is adjacent to the other two in \(G^2\). That is, one of them say \(x_1\).
is adjacent to some $u$ in $K_{n-3}$ to which $x_3$ say is adjacent and $x_1$ is adjacent to some $v$ in $K_{n-3}$, $u \neq v$ to which $x_2$ is adjacent. Therefore, $G \in \mathcal{G}_{14}$.

**Case (II):** $G$ contains a clique $K_{n-4}$ and the remaining four vertices form two independent sets with an edge between them. Let $V(K_{n-4}) = \{u_1, u_2, \ldots, u_{n-4}\}$ and the remaining vertices $x$ is adjacent to $u$. We proceed as in Case(ii) of the theorem 5.2.27 $x$ is adjacent to $u$.

**Subcase (i):** Therefore, $x$ is adjacent to every $u_i$ in $G^2$, $1 \leq i \leq n-4$. Since $u$ and $y$ are adjacent to every $u_i$ in $G$, $u$ and $y$ are adjacent to every $u_i$ in $G^2$. $v$ is not adjacent to any $u_i$, $1 \leq i \leq n-4$ in $G$. Therefore, $v$ is adjacent to $x$ or $y$ in $G$.

**Subsubsubcase (i):** Let $v$ be adjacent to $x$ in $G$.

**Case (a):** $x$ is adjacent to some $u_i$, $1 \leq i \leq n-4$. Therefore, $x$ is adjacent to every $u_i$, $1 \leq i \leq n-4$ in $G^2$. Also $x$ is adjacent to $y$ in $G^2$. Since $u$ is adjacent to $x$ and $x$ is adjacent to $v$, $u$ is adjacent to $v$ in $G^2$. $y$ is also adjacent to $u$ in $G^2$. $d(G^2) = n - 1$ (since $d_{\chi}(VV^+(G)) = n - 1$. If $v$ is not adjacent to $y$, then $d(G^2) \neq n - 1$. If $v$ is not adjacent to $y$, then $d(G^2) \neq n - 1$. If $v$ is adjacent to $y$ then $d(G^2) = n \neq n - 1$. Therefore, this
case does not exists.

Case (b): $x$ is not adjacent to any $u_i$ in $G$. In this case, $x$ being adjacent to $u$, $x$ is adjacent to every $u_i$ in $G^2$. Since $x$ is adjacent to both $u$ and $v$, $u$ and $v$ are adjacent in $G^2$. $y$ is adjacent to $u$ in $G^2$. $y$ is not adjacent to $x$ and $v$ in $G^2$. Therefore, $d_X(G^2) \neq n - 1$. If $y$ and $v$ are adjacent in $G$, then $x$ and $y$ are adjacent in $G^2$ and $v$ is adjacent to every $u_i$, $1 \leq i \leq n - 4$ in $G^2$. Therefore, $G^2 = K_n$, a contradiction. Therefore, this case does not exists.

Subsubsubcase (ii): Let $v$ be adjacent to $y$ in $G$.

Then $v$ is adjacent to every $u_i \in G^2$. Also $x$, $u$ and $y$ are adjacent to every $u_i$ in $G^2$. Further $y$ and $u$ are adjacent in $G^2$. Therefore, $d(G^2) \neq n - 1$. If $x$ and $v$ are adjacent in $G$ then $G^2 = K_n$ and hence $d(G^2) = n \neq n - 1$.

Therefore, $x$ and $v$ are not adjacent in $G$. Therefore, $d(G^2) \neq n - 1$ and hence this case does not arise. Subsubsubcase (iii): $v$ is adjacent to some $u_i$, $1 \leq i \leq n - 4$ in $G$. Therefore, $v$ is adjacent to every $u_i$ in $G^2$. Therefore, $v$ is adjacent to $y$ and $u$ in $G^2$. Also $u$ and $y$ are adjacent in $G^2$. But $x$ and $y$ are not adjacent in $G^2$. Therefore, $d(G^2) \neq n - 1$.

If $x$ and $v$ are adjacent in $G$, then $G^2 = K_n$ and hence $d(G^2) = n \neq n - 1$, a contradiction. If $x$ is adjacent to some $u_i$ in $G$, then $x$ is adjacent to $v$ and
\( y \) in \( G^2 \). Therefore, \( G^2 = k_n \). Therefore, \( d(G^2) = n \neq n - 1 \), a contradiction.

This case does not arise.

**Subcase (ii):** \( x \) is not adjacent to both \( u \) and \( v \) in \( G \). Therefore, \( y \) is adjacent to \( u \) or \( v \) in \( G \). Therefore, \( x \) is adjacent to some \( u_i \) \( 1 \leq i \leq n - 4 \) in \( G \).

**Case (a):** \( y \) is adjacent to \( v \) in \( G \).

**Subcase (i):** \( v \) is adjacent to \( u_i \) in \( G \) such that \( u_i \) is not adjacent to \( x \) in \( G \).

Then \( x \) is adjacent to \( u, y, u_j, 1 \leq j \leq n - 4 \) in \( G^2 \). \( y \) is adjacent to \( u \). \( x \) and \( v \) are not adjacent in \( G^2 \). In this case \( d(G^2) = n - 1 \). Since there are \( n - 2 \) full degree vertices in \( G^2 \) and the remaining two vertices form an independent dominating set in \( G^2 \). Therefore, \( G \in G_{15} \). If \( x \) and \( v \) are adjacent to some vertex \( u_i \in G \), then \( G^2 = K_n \), a contradiction.

**Subcase (ii):** \( v \) is not adjacent to any \( u_i \) in \( G \).

In this case, \( xv, uv \notin (G^2) \). Since \( d(G^2) \neq n - 1 \). If \( y \) is adjacent to \( u \) in \( G \), then \( x \) and \( v \) are the only non adjacent vertices in \( G^2 \). Therefore, \( d(G^2) = n - 1 \). Therefore, \( G \in G_{15} \).

**Case (b):** \( y \) is adjacent to \( u \) in \( G \).

**Subcase (i):** \( v \) is adjacent to some \( u_i \) in \( G \), then \( G^2 = K_n \), a contradiction.
Therefore, $v$ is adjacent to some $u_i$ with which $x$ is not adjacent in $G$. Then $d(G^2) = n - 1$. Therefore, $G \in G_{15}$.

**Subcase (ii):** $v$ is not adjacent to any $u_i$ in $G$.

Since $v$ is not an isolate in $G$, $v$ is adjacent to $x$ or $y$ in $G$. But $v$ is not adjacent to $x$ in $G$. Therefore, $v$ is adjacent to $y$ in $G$. Therefore, $y$ is adjacent to both $u$ and $v$ in $G$. Therefore, in $G^2$, $x$ and $v$ are not adjacent and hence $d(G^2) = n - 1$. Therefore, $G \in G_{15}$.

**Case (D):** Let $\chi(G) = n - 3$ and $d_{X}(VV^{+}(G)) = n$.

Since $d(G^2) = n$. $G^2$ is complete. Let $t$ be the number of singletons in any chromatic partition of $G$. Therefore, $n = |V| \geq t + 2(n - 3 - t)$. Therefore, $t \geq n - 6$. If $t = n - 6$, then the remaining 6 vertices are to be grouped into three classes of two elements each. (Since $\chi(G) = n - 3$). If $t = n - 5$, then the remaining five vertices are to be grouped in two classes of 3 and 2 elements.

If $t = n - 4$, then the remaining four vertices are to be grouped into one class. $t \geq n - 4$ (since $\chi(G) = n - 3$).

**Case (i):** $G$ contains a clique $K_{n-6}$ and the remaining six vertices form three independent classes of two elements each. Also there is an edge from every
element of the clique to each of the three independent classes. Further, there is at least one edge between each of the three independent classes.

One vertex from each of the three independent classes is adjacent to every vertex of the clique $K_{n-6}$. Let $\{u_1, u_2, \ldots, u_{n-6}\}$ be the vertex set of $K_{n-6}$ and $\{x, y\}, \{u, v\}$ and $\{w, z\}$ be the independent classes. Let $yu$ and $w$ be adjacent to every vertex of $K_{n-6}$.

Subcase (i): Let $x, v$ and $z$ be adjacent to the same vertex in $K_{n-6}$ in $G$.

Then it can be easily verified that $G^2 = K_n$ and hence $d_X(VV^+(G)) = n$.

Therefore, $G \in \mathcal{G}_{16}$.

Subcase (ii): In $G$, two of the three vertices $x, v$ and $z$ are adjacent to the same vertex of $K_{n-6}$ and the remaining is not adjacent to that vertex.

Let $x$ and $v$ be adjacent to some $u_i, 1 \leq i \leq n-6$ and $z$ is adjacent to $u_j, j \neq i$. In this case, $G = K_n$ if and only if $vz$ and $xz \in E(G)$. That is, those two vertices of $x, y$ and $z$ which do not have a common neighbour $u_i$, $1 \leq i \leq n - 6$ must be adjacent in $G$. Therefore, $G \in \mathcal{G}_{17}$.

Subcase (iii): In $G$, all of the three vertices $x, v$ and $z$ are adjacent to different vertices of $K_{n-6}$. Arguing as in subcase(ii), $d(G^2) = n$ if and only if all the three edges $x, v$ and $z$ must be in $G$. Therefore, $G \in \mathcal{G}_{18}$.  

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Subcase (iv): One of $x, y, z$ is not adjacent to any of $u_i$, $1 \leq i \leq n - 6$ and the other two vertices are adjacent to same vertex in $K_{n-6}$.

In this case, $G^2 = K_n$ if and only if $v$ is adjacent to $w$ and $z$ and $u$ is adjacent to $x, y$ in $G$. Therefore, $G \in \mathcal{G}_{19}$.

Subcase (v): one of $x, y, z$ is not adjacent to any of $u_i$, $1 \leq i \leq n - 6$ and the other two vertices are not adjacent to same vertex in $K_{n-6}$ but adjacent to some vertices in $K_{n-6}$. In this case, $G^2 = K_n$ iff $v$ is adjacent to $w$ and $z$ and $u$ is adjacent to $x$ and $y$ in $G$. Therefore $G \in \mathcal{G}_{19}$.

Subcase (vi): Two of $x, y, z$ are not adjacent to any of $u_i$, $1 \leq i \leq n - 6$ and the other vertex is adjacent to some vertex in $K_{n-6}$.

In this case, $G^2 = K_n$ if and only if $v$ is adjacent to $x, y, w$ and $z$ and $u$ is adjacent to $x$ in $G$. Therefore, $G \in \mathcal{G}_{20}$.

Subcase (vii): None of them are adjacent to any of $u_i$, $1 \leq i \leq n - 6$.

In this case, $G^2 = K_n$ if and only if $v$ is adjacent to $x, y, w$ and $z$ and $u$ is adjacent to $x$ in $G$. Therefore, $G \in \mathcal{G}_{20}$.

Case (ii): $G$ contains a clique $K_{n-5}$ and the remaining five vertices form two independent set of 2 and 3 elements.

Let $V(K_{n-5}) = \{u_1, u_2, \ldots, y_{n-5}\}$. Let $x, y, u, v$ and $w$ be the remaining
vertices with \( \{x, y\} \) and \( \{u, v, w\} \) being independent. There exists an edge between these two classes say \( xv \). Let \( y \) and \( u \) be adjacent to every vertex in \( K_{n-5} \).

**Subcase (i):** The vertices \( x, v \) and \( w \) are adjacent to the same vertex in \( K_{n-5} \). Then \( G^2 = K_n \) and hence, \( G \in G_{21} \).

**Subcase (ii):** The vertices \( x, v \) are adjacent to the same vertex in \( K_{n-5} \) and \( w \) is adjacent to different vertex in \( K_{n-5} \). Also \( x \) is adjacent to \( u \) in \( G \). Then \( G^2 = K_n \) if and only if \( xv, xw \in E(G) \). Therefore, \( G \in G_{22} \). **Subcase (iii):**

All the three vertices \( x, v \) and \( w \) are adjacent to different vertices in \( K_{n-5} \). Also \( x \) is adjacent to \( u \) in \( G \). Then \( G^2 = K_n \) if and only if \( xv, xw \in E(G) \). Therefore, \( G \in G_{23} \). **Subcase (iv):** Of the three vertices \( x, v, w \) one of them say \( x \) is not adjacent to any \( u_i \) and the other two are adjacent with some vertex in \( K_{n-5} \) (same or different). In this case, \( G^2 = K_n \) if and only if \( v \) is adjacent to \( x \) and \( y \) and \( w \) is adjacent to \( x \). Therefore, \( G \in G_{24} \).

**Subcase (v):** Of the three vertices \( x, v, w \) two of them say \( x \) and \( v \) are not adjacent to any \( u_i \) and the remaining is adjacent with some vertex in \( K_{n-5} \).

In this case, \( G^2 = K_n \) if and only if \( v \) is adjacent to \( x \) and \( y \) and \( w \) is adjacent to \( x \). Therefore \( G \in G_{24} \).
Subcase (vi): None of the three vertices $x, v, w$ is adjacent to any $u_i$ in $K_{n-5}$.

In this case, $G^2 = K_n$ if and only if $v$ is adjacent to $x$ and $y$ and $w$ is adjacent to $x$ and $y$. Therefore, $G \in \mathcal{G}_{25}$.

Case (iii): $G$ contains a clique $K_{n-4}$ and the remaining four vertices $x, y, v$ and $u$ are independent and form a single colour class. Let $y$ be adjacent to every vertex in $K_{n-4}$.

Subcase (i): $x, u$ and $v$ are adjacent to same vertex in $K_{n-4}$. Then $G^2 = K_n$. Therefore, $G \in \mathcal{G}_{26}$.

Subcase (ii): Let $x$ be adjacent to some $u_i$ and $u$ and $v$ be adjacent to different vertices say $u_j$. In this case $G^2 = K_n$ if and only if $x$ and $u$ have a common adjacent vertex in $K_{n-4}$ and $x$ and $v$ have a common adjacent vertex in $K_{n-4}$ and $x$ and $v$ have a common adjacent vertex in $K_{n-4}$. Therefore, $G \in \mathcal{G}_{27}$.

Note that since $x, u$ and $v$ are not isolate vertex, each of them must be adjacent to at least one $u_i$ in $K_{n-4}$. Hence, if all the three vertices $x, u$ and $v$ are adjacent to different vertices in $K_{n-4}$, then $G^2 = K_n$ if and only if every two of them have a common neighbour and hence $G \in \mathcal{G}_{27}$. ■

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5.3 \(Y\)-Domination number of \(VV(G)\) and Chromatic number of \(G\)

Proposition 5.3.1. \( \gamma_Y(VV(K_n)) = 2 \).

Proof: \( K_n \) is a complete graph on \( n \) vertices. Let \( v \in V \). In graph \( VV(K_n) \), \( v \in V \) is adjacent to vertices other than \( v \). Some \( u \in V \) is adjacent to \( v \).

Hence, \( \gamma_Y(VV(K_n)) = 2 \).

Theorem 5.3.2. For any connected graph \( G \), \( \gamma_Y(VV(G)) + \chi(G) \leq 2n \) and equality holds if and only if \( G \cong K_2 \).

Proof: \( \gamma_Y(VV(G)) + \chi(G) \leq n + \Delta + 1 = n + n - 1 + 1 = 2n \). Let \( \gamma_Y(VV(G)) + \chi(G) = 2n \). Then the only possible case is \( \gamma_Y(VV(G)) = n \) and \( \chi(G) = n \). Since \( \chi(G) = n \), \( G = K_n \). But for the graph \( VV(K_n) \), \( \gamma_Y(VV(K_n)) = 2 \), so \( n = 2 \) and hence, \( G \cong K_2 \).

Converse is obvious.

Theorem 5.3.3. For any connected graph \( G \), \( \gamma_Y(VV(G)) + \chi(G) = 2n - 1 \) if and only if \( G \cong K_3 \).

Proof: Assume \( \gamma_Y(VV(G)) + \chi(G) = 2n - 1 \). This is possible only if \( \gamma_Y(VV(G)) = n \) and \( \chi(G) = n - 1 \) or \( \gamma_Y(VV(G)) = n - 1 \) and \( \chi(G) = n \).
Case (i): Let $\gamma_Y(VV(G)) = n$ and $\chi(G) = n - 1$.

Since $\chi(G) = n - 1$, $G$ contains a clique $K = K_{n-1}$. Let $u$ be a vertex other than the vertices of $K_{n-1}$. Since $G$ is connected, $u$ is adjacent to $v$ for some $v \in V(K)$. Then $\{u, v\}$ is a $Y$-dominating set in $VV(G)$. Hence, $\gamma_Y(VV(G)) = 2$. Since, $\gamma_Y(VV(G)) = n$, we have $n = 2$. Therefore, $K = K_1$.

Let $u$ be adjacent to $K$. Hence, $G = K_2$. But $\chi(K_2) = 2$, a contradiction.

Hence no such graph exists.

Case (ii): $\gamma_Y(VV(K_n)) = n - 1$ and $\chi(G) = n$.

Since $\chi(G) = n$, $G = K_n$, $\gamma_Y(VV(K_n)) = 2$. Therefore, $n - 1 = 2$. Hence, $n = 3$. Therefore, $G = K_3$.

Converse of the above is obvious. 

Theorem 5.3.4. For any connected graph $G$, $\gamma_Y(VV(G)) + \chi(G) = 2n - 2$ if and only if $G \cong P_3$ or $K_4$.

Proof: Assume $\gamma_Y(VV(G)) + \chi(G) = 2n - 2$. This is possible only if $\gamma_Y(VV(G)) = n$ and $\chi(G) = n - 2$ or $\gamma_Y(VV(G)) = n - 1$ and $\chi(G) = n - 1$ or $\gamma_Y(VV(G)) = n - 2$ and $\chi(G) = n$.

Case (i): Let $\gamma_Y(VV(G)) = n - 2$ and $\chi(G) = n$.

Since $\chi(G) = n$, $G = K_n$. But $\gamma_Y(VV(K_n)) = 2$. Hence, $n - 2 = 2$. 

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Therefore, \( n = 4 \). Hence, \( G = K_4 \).

**Case (ii):** Let \( \gamma_Y(VV(G)) = n - 1 \) and \( \chi(G) = n - 1 \).

Since \( \chi(G) = n - 1 \). \( G \) contains a clique \( K = K_{n-1} \) and the remaining vertex is adjacent to some vertex of \( K_{n-1} \) and the remaining vertex is adjacent to some vertex of \( K_{n-1} \) and not all of them. Then \( \gamma_t(G) = \gamma_Y(VV(G)) = 2 \).

Therefore, \( n - 1 = 2 \). Hence \( n = 3 \). Therefore, \( G = P_3 \). **Case (iii):** Let \( \gamma_Y(VV(G)) = n \) and \( \chi(G) = n - 2 \).

\( \gamma_Y(VV(G)) = \gamma_t(G) = n \). Therefore, \( G = mk_2 \). Since \( \chi(G) = 2; n - 2 = 2; n = 4 \). Therefore, \( G = 2K_2 \).

Converse of the above is obvious.

**Theorem 5.3.5.** For any connected graph \( G \), \( \gamma_Y(VV(G)) + \chi(G) = 2n - 3 \) if and only if \( G \cong K_5 \) or any one of the following graphs \( G_1 \) or \( G_2 \).

![Diagram](image)

**Proof:** Assume \( \gamma_Y(VV(G)) + \chi(G) = 2n - 3 \). This is possible only if
\[ \gamma_r(VV(G)) = n \text{ and } \chi(G) = n - 3 \text{ or } \gamma_r(VV(G)) = n - 1 \text{ and } \chi(G) = n - 2 \]
or \[ \gamma_r(VV(G)) = n - 2 \text{ and } \chi(G) = n - 1 \text{ or } \gamma_r(VV(G)) = n - 3 \text{ and } \chi(G) = n. \]

**Proof:** \[ \gamma_r(VV(G)) = n - 3 \text{ and } \chi(G) = n. \]

Since \( \chi(G) = n \), \( G = K_n \). But \( \gamma_r(VV(K_n)) = 2 \) and \( \gamma_r(VV(K_n)) = n - 3 \) and therefore \( n = 5 \). Hence, \( G \cong K_5 \).

**Case (ii):** \[ \gamma_r(VV(G)) = n - 2 \text{ and } \chi(G) = n - 1. \]

Since \( \chi(G) = n - 1 \), \( G \) contains a clique \( K_{n-1} \) and the remaining vertex is adjacent to some vertex of \( K_{n-1} \) and not to all of them. Then \( \gamma_i(G) = \gamma_r(VV(G)) = 2. \) Which implies, \( n = 4 \). \( G \) is \( K_3 \) with a pendent vertex or \( K_4 - \{e\} \).

**Case (iii):** \[ \gamma_r(VV(G)) = n - 1 \text{ and } \chi(G) = n - 2. \]

\( \gamma_i(G) = n - 1. \) Since \( \chi(G) = n - 2 \), \( G \) contains a clique \( K_{n-3} \) and the remaining three vertices are independent and each of them is adjacent to at least one vertex in \( K_{n-3} \) or \( G \) contains a clique \( K_{n-4} \) and the remaining four vertices form two independent set of 2 elements each.

**Case A:** \( G \) contains a clique \( K_{n-3} \) and the remaining three vertices are independent and each of them is adjacent to at least one vertex in \( K_{n-3} \).
Subcase (i): If all the three vertices are adjacent to different vertices in $K_{n-3}$, $\gamma(G) = 3 = n - 1$. Therefore, $n = 4$. Therefore, $\chi(G) = 2$. Since $G$ contains $K_{n-3} = K_1$. Since the remaining vertices are adjacent to different vertices, this case does not arise.

Subcase (ii): Two of the three vertices are adjacent to same vertex in $K_{n-3}$ and the remaining vertex adjacent to a different vertex of $K_{n-3}$. In this case, $\gamma(G) = 2 = n - 1$. Therefore, $n = 3$. Since, $\chi(G) = 1$. Therefore, $G = K_3$. But $G$ has no isolates. Therefore, this case does not arise.

Subcase (iii): If all the three vertices are adjacent to the same vertex in $K_{n-3}$, then $\gamma(G) = 2 = n - 1$. Therefore, $n = 3$. Hence, $\chi(G) = 1$. Therefore, this case does not arise.

Case (b): $G$ contains a clique $K_{n-4}$ and the remaining four vertices form two independent set of two elements each.

Let the four vertices other than those in $K_{n-4}$ be $x, y, u$ and $v$. Let $y$ and $u$ be adjacent to every vertex in $K_{n-4}$. $y$ and $v$ are adjacent (similar argument if $y$ and $u$ are adjacent). $x$ is adjacent to some $u_i$ in $K_{n-4}$. $x$ and $u$ are adjacent or $x$ and $v$ are adjacent. In all the cases, $\gamma(G) = 2 = n - 1$. Therefore, $n = 3$. Hence, $\chi(G) = n - 2 = 1$. Therefore, $G = \overline{K_3}$, a contradiction. □
Case (iv): \( \chi(G) = n - 3 \) and \( \gamma_Y(VV(G)) = n \).

Since \( \chi(G) = n - 3 \), \( G \) contains a clique \( K = K_{n-3} \). Let \( S = \{x, y, z\} \) be the vertices of \( G \) other than the vertices of \( K \). Then, \( \langle S \rangle = K_3, \overline{K_3}, P_3, K_2 \cup K_1 \).

**Subcase (i):** \( \langle S \rangle = K_3 \).

Since \( G \) is connected, a vertex in \( \langle S \rangle \) is adjacent to at least one vertex in \( K \).

\( \gamma(VV(G)) = 2 = n \). Hence, we get a contradiction to \( \chi(G) = n - 3 \).

**Subcase (ii):** \( \langle S \rangle = \overline{K_3} \).

If vertices of \( \overline{K_3} \) are adjacent to one vertex or different vertices of \( K \), we get \( \gamma_Y(VV(G)) = 3 \), a contradiction to \( \chi(G) = n - 3 \). Hence, no such graph exists.

**Subcase (iii):** \( \langle S \rangle = P_3 \).

If either \( x \) or \( z \) is adjacent to a vertex of \( K \). We get \( \gamma_Y(VV(G)) = 2 \), a contradiction to \( \chi(G) = n - 3 \). If \( y \) is adjacent to a vertex of \( K \), we get \( \gamma_Y(VV(G)) = 2 \), a contradiction to \( \chi(G) = n - 3 \).

**Subcase (iv):** \( \langle S \rangle = K_2 \cup K_1 \).

Since \( G \) is connected, vertices in \( K_1 \) and \( K_2 \) are adjacent to vertices of \( K \).

If vertex \( z \in V(K_1) \) is adjacent to \( u \in V(K) \). The vertex \( x \in V(K_2) \) is adjacent either to \( u \in V(K) \) or some other vertex \( v \in V(K) \). In either case,
we get $\gamma_Y(VV(G)) = 2$, a contradiction to $\chi(G) = n - 3$.

Converse of the above is obvious.

**Theorem 5.3.6.** For a connected graph $G$, $\gamma_Y(VV(G)) + \chi(G) = 2n - 4$ if and only if $G \cong P_4, K_6, G_1, G_2, G_3, G_4$.

![Graphs G1, G2, G3, G4](image)

**Proof:** Assume $\gamma_Y(VV(G)) + \chi(G) = 2n - 4$. The possible cases are

- $\gamma_Y(VV(G)) = n - 4$ and $\chi(G) = n$, $\gamma_Y(VV(G)) = n - 3$ and $\chi(G) = n - 1$, $\gamma_Y(VV(G)) = n - 2$ and $\chi(G) = n - 2$, $\gamma_Y(VV(G)) = n - 1$ and $\chi(G) = n - 3$, $\gamma_Y(VV(G)) = n$ and $\chi(G) = n - 4$.

**Case (i):** $\gamma_Y(VV(G)) = n - 4$ and $\chi(G) = n$.

If $\chi(G) = n$, $G \cong K_n$. But $\gamma(VV(K_n)) = 2$. Therefore, $n = 6$. Hence, $G \cong K_6$.

**Case (ii):** $\gamma_Y(VV(G)) = n - 3$ and $\chi(G) = n - 1$.

$\chi(G) = n - 1$. $G$ contains a clique $K = K_{n-1}$. Let $x$ be a vertex other than
the vertices of $K$. Since $G$ is connected $x$ is adjacent to at least one vertex of $K$. But $\gamma_V(VV(G)) = 2$. Hence, $n = 5$.

**Subcase (i):** If $x$ is adjacent to all the vertices of $K$, we get $G \cong K_5$, a contradiction to $\chi(K_5) \neq n - 1$.

**Subcase (ii):** $x$ is adjacent to a vertex of $K$, we get $G_2$. If $x$ is adjacent to two vertices of $K$, we get $G_3$. If $x$ is adjacent to three vertices of $K$, we get $G_4$.

**Case (iii):** $\gamma_V(VV(G)) = n - 2$ and $\chi(G) = n - 2$.

**Case (a):** Since $\chi(G) = n - 2$, we get a clique $K = K_{n-2}$

Let $\langle S \rangle = K_2$. If $x$ and $y$ are not adjacent to any vertex of $K_{n-2}$, then we get $\gamma_{l}(G) = 4$. Therefore, $n - 2 = 4$ which gives, $n = 6$. Therefore, $G = K_4 \cup K_2$.

If $x$ is adjacent to some vertex of $K_{n-2}$ and $y$ is not adjacent to any vertex of $K_{n-2}$, then $\gamma_l(G) = 2$. Therefore, $n - 2 = 2$. Which gives $n = 4$. Therefore, $G = P_4$. If $x$ and $y$ are adjacent to same vertex of $K_{n-2}$, then $\gamma_l(G) = 2$.

Therefore, $G = K_3$ with pendant vertex. If $x$ and $y$ are adjacent to different vertices of $K_{n-2}$, then $\gamma_l(G) = 2$. Therefore, $n = 4$. Hence, $G = C_4$.

**Case (b):** $G$ contains $K_{n-3}$ and the remaining three vertices say $x, y$ and $z$
are independent and there is at least one edge between \( K_{n-3} \) and the set of three independent vertices.

Since \( G \) has no isolates, each vertex in the independent set of three elements is adjacent to a vertex in \( K_{n-3} \).

**Subcase (i):** \( x, y \) and \( z \) are adjacent to different vertices of \( K_{n-3} \). Then, \( \gamma_t(G) = 3 \). \( n - 2 = 3 \). Therefore, \( n = 5 \). This case does not arise.

**Subcase (ii):** Two of them are adjacent to the same vertex of \( K_{n-3} \) and the other is adjacent to a different vertex of \( K_{n-3} \). Then \( \gamma_t(G) = 2 \). \( n - 2 = 2 \). Therefore, we get \( n = 4 \). This case does not arise.

**Subcase (iii):** All the three vertices \( x, y \) and \( z \) are adjacent to the same vertices of \( K_{n-3} \). Therefore, \( \gamma_t(G) = 2 \). Hence, \( n = 4 \). This case does not arise.

**Case (iv):** \( \gamma Y(VV(G)) = n - 1 \) and \( \chi(G) = n - 3 \).

**Case A** \( G \) contains a clique \( K_{n-3} \) and the remaining three vertices say \( x, y \) and \( z \) are not adjacent to any of the vertices of the clique. Let \( S = \{x, y, z\} \).

**Subcase (i):** \( x, y \) and \( z \) are mutually adjacent. \( \gamma_t(G) = 4 \). Therefore, \( n - 1 = 4 \). which gives \( n = 5 \). Therefore, \( G = K_2 \cup K_3 \). Therefore, \( \chi(G) = 3 \) but \( \chi(G) = n - 3 = 5 - 3 = 2 \), a contradiction.

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Subcase (ii): $x, y, z$ form $P_3$. Therefore, $\gamma_t(G) = 4$. Therefore, $n = 5$.

$G = K_2 \cup P_3$. $\chi(G) = 2$.

$x, y$ are adjacent to same vertex and $z$ is adjacent to a different vertex.

Therefore, $\gamma_t(G) = 2$. Hence, $n = 3$, a contradiction.

$x, y$ and $z$ are adjacent to the same vertex. Then, $\gamma_t(G) = 2$. Hence, $n = 3$, a contradiction.

Subcase (iii): $x, y, z$ are adjacent to different vertices of $K_{n-3}$. If $\langle S \rangle = K_3$ then $\gamma_t(G) = 2$. Therefore, $n - 1 = 2$. Hence, $n = 3$. But $n - 3 = 0$, a contradiction.

Case BG contains a clique $K_{n-4}$ and the remaining four vertices say $x, y, u$ and $v$ form two independent sets.

Then one of $x, y, u$ and $v$ for two independent sets.

Then one of $x, y$ say $y$ is adjacent to every vertex of $K_{n-4}$ and one of $u, v$ say $u$ adjacent to every vertex of $K_{n-4}$.

Case (i): Let $x$ and $v$ be adjacent.

Suppose $x$ and $v$ are not adjacent to any vertex of $K_{n-4}$. Then $\gamma_t(G) = 4$.

Therefore, $n - 1 = 4$, $n = 5$. Therefore, $G = P_3 \cup K_2$. Since $\chi(G) = 2 = n - 3$.

Suppose $x$ is adjacent to a vertex of $K_{n-4}$ and $v$ is not adjacent to any
vertex of $K_{n-4}$. Then, $\gamma_t(G) = 2$. Therefore, $\gamma_t(G) = 2$. Therefore, $n - 1 = 2$.

Hence, $n = 3$. This case does not arise.

If $x$ and $v$ are adjacent to the same or different vertices of $K_{n-4}$, then $n = 3$, a contradiction.

Case (v): $\gamma_Y(VV(G)) = n$ and $\chi(G) = n - 4$.

$\gamma_t(G) = n$. Therefore, $G = mk_2$. Since, $\chi(G) = 2$. We get $n = 6$. Therefore, $G = 3K_2$.

From the above discussion we get characterization of graphs for which

(i) $\gamma_t(G) + \chi(G) = 2n$.
(ii) $\gamma_t(G) + \chi(G) = 2n - 1$.
(iii) $\gamma_t(G) + \chi(G) = 2n - 2$.
(iv) $\gamma_t(G) + \chi(G) = 2n - 3$.
(iv) $\gamma_t(G) + \chi(G) = 2n - 4$.

## 5.4 X-Indominate graphs

**Definition 5.4.1.** An X-dominating set which is also X-independent is called an X-independent X-dominating set. A graph is said to be X-indominate if $X(G)$ can be partitioned into X-independent X-dominating sets, otherwise $G$ is called non X-indominate.
Example 5.4.2.

\{\{x_1, x_3, x_5\}, \{x_2, x_4, x_6\}\} are X-independent sets which are also X-dominating sets. Therefore, $G = C_{12}$ given below is X-indominable.

Example 5.4.3. In the graph $G = C_{10}$ maximum X-independent set are
\{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_2, x_5\} \text{ and } \{x_3, x_5\}. \text{ These are also } X\text{-dominating set and } \gamma_X(G) = 2. \text{ Since } |X| = 5, X \text{ cannot be partitioned into } X\text{-independent } X\text{-dominating sets. Therefore, } G = C_{10} \text{ is non } X\text{-indominable.}

**Theorem 5.4.4.** If \(G\) is not \(X\)-indominable then there exists a \(X\)-indominable graph \(H\) containing \(G\) as an induced subgraph.

**Proof:** Let \(G\) be a non \(X\)-indominable graph with \(|X| = n\). Let \(X = \{u_1, u_2, \cdots, u_n\}\). Add vertices \(v_1, v_2, \cdots, v_n\) to \(X\) such that every vertex \(u_i\) is \(X\)-adjacent to vertices \(v_j, i \neq j, 1 \leq i \leq n\). \(\{u_i, v_i\} \quad i = 1 \text{ to } n\) forms \(X\)-independent \(X\)-dominating sets. Therefore, \(H\) is \(X\)-indominable and \(X(H)\) can be partitioned into \(X\)-independent \(X\)-dominating sets. \(\blacksquare\)

**Corollary 5.4.5.** The class of \(X\)-indominable graphs cannot be characterized by a family of forbidden subgraphs.

**Definition 5.4.6.** A partition \(P = \{D_1, D_2, \cdots, D_k\}\) of the vertex set \(X(G)\) of \(G\) is called an \(X\)-indomatic partition, if \(D_i\) is an \(X\)-independent \(X\)-dominating set, for each \(i = 1, 2, 3, \cdots, k\). If \(\pi_{id}(G)\) denotes the set of all \(X\)-indomatic partition of \(G\), then the number \(b_X(G) = \max_{P \in \pi_{id}(G)} |P|\) is called \(X\)-indomatic number of \(G\).

**Observation 5.4.7.** Any \(X\)-indomatic partition is an \(X\)-domatic partition. Therefore, \(b_X(G) \leq d_X(G)\).

**Theorem 5.4.8.** For any graph \(G\), \(b(G) = b_X(VE(G))\).
**Proof:** Let $b(G) = k$. There exists a partition of $V(G)$ into independent dominating sets of cardinality $k$. Let $V_1, V_2, \ldots, V_k$ be the partition of $V(G)$ into independent dominating sets. In the graph $VE(G) = (X, Y, E^I)$, $V_1, V_2, \ldots, V_k$ is a partition of $X$ into $X$-independent $X$-dominating sets. Therefore, $b_X(VE(G)) \geq b(G)$.

Conversely, let $\alpha$ be the $X$-indomatic number of $VE(G)$. $X_1, X_2, X_3, \ldots, X_\alpha$ is a partition of $X$ into $X$-independent $X$-dominating set. In $G$, $X_1, X_2, X_3, \ldots, X_\alpha$ forms a indomatic partition of $G$. Therefore, $b(G) \geq b_X(VE(G))$. Hence, $b(G) = b_X(VE(G))$.

**Theorem 5.4.9.** For any graph $G$, $b^1(G) = b_X(EV(G))$.

**Proof:** Let $b^1(G) = k$. There exists a partition of edges into independent dominating set of cardinality $k$. Let $E_1, E_2, \ldots, E_k$ be the partition of $E(G)$ into independent dominating sets. In the graph $EV(G) = (X, Y, E^I)$, $E_1, E_2, \ldots, E_k$ is a partition of $X$ into $X$-independent $X$-dominating sets. Therefore $b_X(EV(G)) \geq b(G)$.

Conversely, let $\alpha$ be the $X$-indomatic number of $EV(G)$. $X_1, X_2, X_3, \ldots, X_\alpha$ is a partition of $X$ into $X$-independent $X$-dominating set. In $G$, $X_1, X_2, X_3, \ldots, X_\alpha$ forms an edge indomatic partition of $G$. Therefore, $b(G) \geq b_X(EV(G))$. 167
Hence, $b(G) = b_X(EV(G))$. ■

**Theorem 5.4.10.** For any graph $G$, $b(G^2) = b_X(VV(G))$.

**Proof:** Let $b(G^2) = k$. There exists a partition of $V(G)$ into independent dominating sets of cardinality $k$. Let $V_1, V_2, \ldots, V_k$ be the partition of $V(G^2)$ into independent dominating sets. In the graph $VV(G) = (X, Y, E^1)$ $V_1, V_2, \ldots, V_k$ is a partition of $X$ into $X$-independent $X$-dominating sets. Therefore, $b_X(VV(G)) \geq b(G)$.

Conversely, let $\alpha$ be the $X$-indomatic number of $VV(G)$. $X_1, X_2, X_3, \ldots, X_\alpha$ is a partition of $X$ into $X$-independent $X$-dominating set. In $G^2, X_1, X_2, X_3, \ldots, X_\alpha$ forms a indomatic partition of $G^2$. Therefore, $b(G) \geq b_X(VV(G))$. Hence, $b(G) = b_X(VV(G))$. ■

**Theorem 5.4.11.** For any graph $G$, $b(G^2) = b_X(VV^+(G))$.

**Proof:** Let $b(G^2) = k$. There exists a partition of $V(G^2)$ into independent dominating sets of cardinality $k$. Let $V_1, V_2, \cdots, V_k$ be the partition of $V(G^2)$ into independent dominating sets. In the graph $VV^+(G) = (X, Y, E^1)$ $V_1, V_2, \cdots, V_k$ is a partition of $X$ into $X$-independent $X$-dominating sets. Therefore, $b_X(VV^+(G)) \geq b(G)$. 

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Conversely, let $\alpha$ be the $X$-indomatic number of $VV^+(G)$. $X_1, X_2, X_3, \ldots, X_\alpha$ is a partition of $X$ into $X$-independent $X$-dominating set. In $G^2, X_1, X_2, X_3, \ldots, X_\alpha$ forms a indomatic partition of $G^2$. Therefore, $b(G) \geq b_X(VV^+(G))$. Hence, $b(G^2) = b_X(VV^+(G))$. 

**Definition 5.4.12.** The $X$-indominable number of a non $X$-indominable graph $G$ with respect to $X$-domination, denoted by $IND_X(G)$ is defined as $X(H) - X(G)$ where $H$ is an $X$-indominable graph of least order in which $G$ can be embedded.

**Remark 5.4.13.** $1 \leq IND_X(G) \leq n$.

**Remark 5.4.14.** $IND_X(C_{2n}) = 1$ if $n$ is odd and $n \neq 3$.

Let $G = C_{2n}$, $n$ odd $n \neq 3$. Let $H$ be the graph obtained from $G$ by adding a vertex $x_{n+1}$ to $X(G)$ and $z_1, z_2, \ldots, z_{n-1}$ to $Y(G)$ and making $x_i$ and $x_{n+1}$ adjacent with $z_i, 1 \leq i \leq n - 1$. Then the partition $\pi = \{\{x_1, x_3, \ldots, x_{n-2}\}, \{x_2, x_4, \ldots, x_{n-1}\}, \{x_n, x_{n+1}\}\}$ is an $X$-indominable partition of $H$.

Therefore, $IND_X(C_{2n}) = 1$.

**Example 5.4.15.**
The graph $G = C_{10}$ is non-$X$-indominable. We add a vertex $x_6$ to $X$ and
$z_1, z_2, z_3, z_4$ to $Y$. Make $x_6$ adjacent to $z_1, z_2, z_3$ and $z_4$. $z_1$ is adjacent to $x_1, x_2$
is adjacent to $x_2, x_3$ is adjacent to $x_3, z_4$ is adjacent to $x_4$. The graph obtained
by these operation gives a $X$-indominable graph which is given below. Hence,

$IND_X(G = C_{10}) = 1$.  

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