5.1. Introduction

Statistical models have been extensively used in the study of data arising from times to failure of units/individuals such as machine components, patients. In engineering applications, while describing shock models, finite mixture of distributions has been found to be appropriate and useful. Even in studies of mortality in demography, finite mixture models have been suggested for populations whose members differ in their endowment for longevity. Hence, nowadays models with the distribution function of the random time to failure is a mixture of a family of lifetime distributions are of more interest.

Modelling lifetimes using mixture of exponential distributions is one of the major concern in reliability studies, particularly, with standby systems. Some of the distributions can also be viewed as mixture of exponentials. For example, Pareto distribution is a mixture of exponential distributions with a gamma mixing distribution. Lindley distribution can be written as a mixture of exponential and gamma distribution with shape parameter 2. Sandhya and Umamaheswari (2013a), have considered the problem of estimating reliability $P[X > Y]$ in a stress-strength model when $X$ follow finite mixture of exponential distributions and $Y$ follow exponential distribution with the assumption that $X$ and $Y$ are independent.
In this chapter, the estimation of reliability of multi-component Stress-Strength model with mixture of two exponential as distribution of stress is studied in detail. Estimation of \( R_p = P[Y < \text{Max}(X_1, X_2, \ldots, X_k)] \) and \( R_s = P[Y < \text{Min}(X_1, X_2, \ldots, X_k)] \) is studied when \( X_1, X_2, \ldots, X_k \) are independent strengths subjected to an independent stress \( Y \). Assuming that \( Y \) follow mixture of two exponential distributions, the system reliabilities are studied by assuming different distributions for strength such as exponential, two parameter exponential, power function and Lindley distribution. MLEs for the parameters and reliability functions are derived. Also, the performance of MLEs is studied by estimating the MSEs through simulations.

5.2. Estimation of Reliability when distribution of strengths follow exponential distribution

In this section, the stress-strength reliability of multi-component system with \( k \) identical components is considered with the assumption that stress and strengths are independent. The \( k \) components of the system are affected by an independent stress. Here, the strength and stress variables have respectively exponential and mixture of two exponential distributions. The reliability expression for the system considered here is derived and MLE of the reliability through the estimators of parameters characterizing the model is obtained.
5.2.1. System Reliability

Let $X_1, X_2, ..., X_k$ be independent strengths each having Exponential distribution with parameter $\lambda$, subjected to an independent stress $Y$, following a mixture of two exponential distributions with parameters $p, \theta_1, \theta_2$.

The pdf of $X_i$ is given by

$$f_i(x) = \lambda e^{-\lambda x} \quad x > 0, \lambda > 0, \quad i = 1, 2, \ldots, k$$

and the pdf of $Y$ is given by

$$g(y) = p \theta_1 e^{-\theta_1 y} + (1 - p)\theta_2 e^{-\theta_2 y} \quad \theta_1, \theta_2, y > 0, 0 \leq p \leq 1.$$  

The distribution function of $Y$ is given by

$$G(y) = 1 - p e^{-\theta_1 y} + (1 - p)e^{-\theta_2 y} \quad \theta_1, \theta_2, y > 0, 0 \leq p \leq 1.$$  

Here the system under study is having $k$ independent strengths, experiencing a single stress, which is independent of $k$ strengths.

Let $U = \text{Max}(X_1, X_2, ..., X_k)$ and $V = \text{Min}(X_1, X_2, ..., X_k)$.

Then, the distribution function of $U$ is given by

$$H_1(u) = P[U < u]$$

$$= \prod_{i=1}^{k} P(X_i < u)$$

$$= \left[1 - e^{-\lambda u}\right]^k$$

and the distribution function of $V$ is given by

$$H_2(v) = P[V < v]$$

$$= 1 - P[V \geq v]$$

$$= 1 - e^{-\lambda k v}.$$
The system reliability for parallel system is obtained as

\[ R_p = P[Y < \text{Max}(X_1, X_2, \ldots, X_k)] \]

\[ = P[Y < U] \]

\[ = \int_0^\infty \overline{H}_1(y) \, dG(y) \]

\[ = \int_0^\infty \left[ 1 - \left(1 - e^{-\lambda y}\right)^k \right] \left( p \theta_1 e^{-\theta_1 y} + (1 - p) \theta_2 e^{-\theta_2 y} \right) \, dy. \]

Since \( 0 < e^{-\lambda y} < 1 \), for \( \lambda > 0 \) and \( y > 0 \), by using the series expansion of

\[ (1 - e^{-\lambda y})^{k-1} \]

as

\[ (1 - e^{-\lambda y})^k = \sum_{l=0}^{k-1} C(k, l) (-1)^l e^{-\lambda y}, \]

the expression of \( R_p \) simplifies to

\[ R_p = 1 - \sum_{l=0}^{k} C(k, l) (-1)^l \left\{ \left. \begin{array}{c} \frac{p \theta_1 \int_0^\infty e^{-\left(\theta_1 + \lambda l\right)y} \, dy}{(1 - p) \theta_2 \int_0^\infty e^{-\left(\theta_2 + \lambda l\right)y} \, dy} \\ + \left. \right\} \right\} \]

\[ = 1 - \sum_{l=0}^{k} C(k, l) (-1)^l \left\{ \frac{p \theta_1}{\theta_1 + \lambda l} + (1 - p) \frac{\theta_2}{\theta_2 + \lambda l} \right\}, \]

Now, the system reliability for series system is obtained as

\[ R_s = P[Y < \text{Min}(X_1, X_2, \ldots, X_k)] \]

\[ = P[Y < V] \]

\[ = \int_0^\infty \overline{H}_2(y) \, dG(y) \]

\[ = \int_0^\infty e^{-(\lambda + \theta_1 \lambda) y} [p \theta_1 e^{-\theta_1 y} + (1 - p) \theta_2 e^{-\theta_2 y}] \, dy \]

\[ = p \theta_1 \int_0^\infty e^{-\left(\theta_1 + \lambda \right)y} \, dy + (1 - p) \theta_2 \int_0^\infty e^{-\left(\theta_2 + \lambda \right)y} \, dy \]

\[ = \frac{p \theta_1}{\lambda + \theta_1} + \frac{(1 - p) \theta_2}{\lambda + \theta_2}. \]
As the reliability function of both series and parallel systems involve $\lambda, p, \theta_1$ and $\theta_2$, we first consider the estimation of $\lambda, p, \theta_1$ and $\theta_2$, using method of maximum likelihood and then the system reliability estimates are obtained in the next section.

5.2.2. Maximum Likelihood Estimators for parameters $\lambda, p, \theta_1, \theta_2$ and Reliability

Let $X_{i1}, X_{i2}, ..., X_{ik}$ ($i = 1, 2, ..., m$) be a random sample on strengths of $m$ systems that are exponentially distributed with parameter $\lambda$. Let there be only two sub populations with mixing proportions $p$ and $1 - p$ and $g_1(y), g_2(y)$ be exponential densities with parameters $\theta_1$ and $\theta_2$ respectively where $0 < p < 1$. Assuming that we have a complete sample, the likelihood of the sample is given by

$$L(p, \theta_1, \theta_2, \lambda | y_{11}, y_{12}, ..., y_{1n_1}; y_{21}, y_{22}, ..., y_{2n_2}; x_{i1}, x_{i2}, ..., x_{ik})$$

$$= \prod_{i=1}^{n} \left\{ \sum_{j=1}^{2} p_j f(y_i, \theta_j) \right\} \prod_{i=1}^{k} \prod_{j=1}^{n} (\lambda e^{-\lambda x_{ij}})$$

$$= \frac{n!}{n_1! n_2!} \left(1 - p\right)^{n_2} \theta_1^{n_1} \theta_2^{n_2} \exp\left(-\sum_{j=1}^{n_1} \theta_1 y_{1j} - \sum_{j=1}^{n_2} \theta_2 y_{2j}\right)$$

$$\times \lambda^{nm} \exp\left(-\lambda \sum_{i=1}^{k} \sum_{j=1}^{m} x_{ij}\right)$$

where $\mathbf{y} = (y_{11}, y_{12}, ..., y_{1n_1}; y_{21}, y_{22}, ..., y_{2n_2})$ and $n_1 + n_2 = n$.

Then log-likelihood function is given by

$$\ln(L) = \ln \left( \frac{n!}{n_1! n_2!} \right) + n_1 \ln(p) + n_1 \ln(\theta_1) + n_2 \ln(1 - p) + n_2 \ln(\theta_2)$$

$$- \theta_1 \sum_{j=1}^{n_1} y_{1j} - \theta_2 \sum_{j=1}^{n_2} y_{2j} + km \ln(\lambda) - \lambda \sum_{i=1}^{k} \sum_{j=1}^{m} x_{ij}.$$
Likelihood equations \( \frac{\partial}{\partial p} \ln L = 0, \frac{\partial}{\partial \theta_1} \ln L = 0, \frac{\partial}{\partial \theta_2} \ln L = 0 \) and \( \frac{\partial}{\partial \lambda} \ln L = 0 \) gives MLEs of \( p, \theta_1, \theta_2 \) and \( \lambda \) respectively as

\[
\hat{p} = \frac{n_1}{n}, \quad \hat{\theta}_1 = \left( \frac{\sum_{j=1}^{n_1} y_{1j}}{n_1} \right)^{-1}, \quad \hat{\theta}_2 = \left( \frac{\sum_{j=1}^{n_2} y_{2j}}{n_2} \right)^{-1} \text{ and } \lambda = \left( \frac{\sum_{j=1}^{k} \sum_{i=1}^{m} x_{ij}}{mk} \right)^{-1}.
\]

Hence, MLEs of \( R_p \) and \( R_s \) are respectively given by

\[
\hat{R}_p = 1 - \sum_{l=0}^{k} C(k, l) (-1)^l \left\{ \frac{\hat{p}}{\hat{\theta}_1 + \lambda l} + (1 - \hat{p}) \frac{\hat{\theta}_2}{\hat{\lambda} + \hat{\theta}_2} \right\},
\]

\[
\hat{R}_s = \frac{\hat{p} \hat{\theta}_2}{(\hat{\lambda} + \hat{\theta}_2)} + (1 - \hat{p}) \frac{\hat{\theta}_1}{(\hat{\lambda} + \hat{\theta}_1)}.
\]

### 5.2.3. Simulation Study

A simulation study is conducted to evaluate MSEs of reliabilities of series and parallel systems by generating 100000 samples of size \( m = 10, n = 15 \) for different values of \( k \) and the parameters \( (p, \theta_1, \theta_2, \lambda) \) as specified in tables. Note that the values written in parentheses represent the corresponding values for \( k = 3 \) in the tables 5.1 and 5.2. MATLAB code written for this simulation is given in Appendix XIV.

**Table 5.1: MLEs and MSEs for estimates of \( R_p \) and \( R_s \)**

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( R_s )</th>
<th>( \hat{R}_s )</th>
<th>( R_p )</th>
<th>( \hat{R}_p )</th>
<th>( (R_s - \hat{R}_s)^2 )</th>
<th>( (R_p - \hat{R}_p)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.6229 (0.5250)</td>
<td>0.6263 (0.5350)</td>
<td>0.9105 (0.9564)</td>
<td>0.9036 (0.9506)</td>
<td>0.005930 (0.005929)</td>
<td>0.001985 (0.000851)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3576 (0.2714)</td>
<td>0.3732 (0.2899)</td>
<td>0.6924 (0.7737)</td>
<td>0.6936 (0.7724)</td>
<td>0.006283 (0.004619)</td>
<td>0.007535 (0.005935)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.2513 (0.1833)</td>
<td>0.2692 (0.2014)</td>
<td>0.5487 (0.6295)</td>
<td>0.5584 (0.6369)</td>
<td>0.004839 (0.003150)</td>
<td>0.009013 (0.008220)</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1443 (0.1013)</td>
<td>0.1612 (0.1161)</td>
<td>0.3582 (0.4221)</td>
<td>0.3773 (0.4416)</td>
<td>0.002686 (0.001636)</td>
<td>0.007683 (0.008203)</td>
</tr>
<tr>
<td>1.5</td>
<td>0.1013 (0.070)</td>
<td>0.1159 (0.0822)</td>
<td>0.2654 (0.3161)</td>
<td>0.2865 (0.3390)</td>
<td>0.001776 (0.001057)</td>
<td>0.006014 (0.006785)</td>
</tr>
</tbody>
</table>
Observation: It is observed from the table 5.1 that for fixed values of the stress parameters there is decrease in the values of $R_s$ and $R_p$ with increase in the values of strength parameters.

Table 5.2: MLEs and MSEs for estimates of $R_p$ and $R_s$

\[ p = 0.2, \lambda = 0.1, k = 2(k = 3), m = 10, n = 15 \]

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$R_s$</th>
<th>$\bar{R}_s$</th>
<th>$R_p$</th>
<th>$\bar{R}_p$</th>
<th>$(R_s - \bar{R}_s)^2$</th>
<th>$(R_p - \bar{R}_p)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.3</td>
<td>0.584</td>
<td>0.5884</td>
<td>0.8869</td>
<td>0.8814</td>
<td>0.006550</td>
<td>0.002580</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6714</td>
<td>0.6778</td>
<td>0.6778</td>
<td>0.9286</td>
<td>0.9247</td>
<td>0.00523</td>
<td>0.001329</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7222</td>
<td>0.7272</td>
<td>0.7272</td>
<td>0.9444</td>
<td>0.9418</td>
<td>0.004144</td>
<td>0.0009078</td>
</tr>
<tr>
<td>0.9</td>
<td>0.7545</td>
<td>0.7592</td>
<td>0.7592</td>
<td>0.9521</td>
<td>0.9499</td>
<td>0.003483</td>
<td>0.0007481</td>
</tr>
<tr>
<td>1.0</td>
<td>0.7666</td>
<td>0.7716</td>
<td>0.7716</td>
<td>0.9545</td>
<td>0.9527</td>
<td>0.003176</td>
<td>0.0006967</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6229</td>
<td>0.6262</td>
<td>0.6262</td>
<td>0.9105</td>
<td>0.9035</td>
<td>0.005971</td>
<td>0.0020</td>
</tr>
<tr>
<td>0.5</td>
<td>0.7143</td>
<td>0.7151</td>
<td>0.7151</td>
<td>0.9524</td>
<td>0.9469</td>
<td>0.004613</td>
<td>0.00080</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7651</td>
<td>0.7651</td>
<td>0.7651</td>
<td>0.9682</td>
<td>0.9637</td>
<td>0.000036</td>
<td>0.000437</td>
</tr>
<tr>
<td>0.9</td>
<td>0.7974</td>
<td>0.7974</td>
<td>0.7974</td>
<td>0.9759</td>
<td>0.9722</td>
<td>0.000029</td>
<td>0.0002871</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8095</td>
<td>0.8095</td>
<td>0.8095</td>
<td>0.9784</td>
<td>0.9749</td>
<td>0.000027</td>
<td>0.0002482</td>
</tr>
</tbody>
</table>

Observation: If strength parameter is fixed and stress parameters $\theta_i$'s are increasing, then values of $R_s$ and $R_p$ are increasing.
5.3. Estimation of Reliability when distribution of strengths follow two parameter exponential distribution

In this section, the reliability of \( k \) component system with identical components and an independent stress acting on system is considered. Here, the strength and stress variables have respectively two-parameter exponential and mixture of two exponential distributions. The reliability expressions for the systems considered here are derived and MLEs of the reliability functions through the estimators of parameters characterizing the model are obtained.

5.3.1. Reliability Function

Let \( X_1, X_2, \ldots, X_k \) be strengths having two parameter exponential distribution with location parameter \( \mu \) and scale parameter \( \theta \), subjected to the stress \( Y \), following mixture of two exponential distributions with parameters \( p, \theta_1, \theta_2 \).

The pdf of \( X_i \) is given by
\[
f_i(x, \mu, \theta) = \frac{1}{\theta} e^{-(x-\mu)/\theta} \quad x > \mu \geq 0, \theta > 0, \quad i = 1, 2, \ldots, k
\]
and the pdf of \( Y \) is given by
\[
g(y) = p \theta_1 e^{-\theta_1 y} + (1 - p) \theta_2 e^{-\theta_2 y} \quad \theta_1, \theta_2, y > 0, 0 \leq p \leq 1.
\]

The distribution function of \( Y \) is given by
\[
G(y) = 1 - p e^{-\theta_1 y} + (1 - p) e^{-\theta_2 y} \quad \theta_1, \theta_2, y > 0, 0 \leq p \leq 1.
\]

Let \( U = \text{Max}(X_1, X_2, \ldots, X_k) \) and \( V = \text{Min}(X_1, X_2, \ldots, X_k) \).

Then the distribution function of \( U \) is given by
\[
H_3(u) = P[U < u] = \prod_{i=1}^{k} P(X_i < u)
\]
and the distribution function of $V$ is given by

$$H_V(v) = P[V < v]$$

$$= 1 - e^{-k(v-\mu)/\theta} \quad v > \mu \geq 0, \theta > 0.$$ 

The system reliability for parallel system is obtained as

$$R_p = P[Y < \text{Max}(X_1, X_2, ..., X_k)]$$

$$= \int_\mu^\infty G(u)dH_3(u)$$

$$= \int_\mu^\infty \left[1 - p e^{-\theta_1 u} - (1 - p)e^{-\theta_2 u}\right]_0^k$$

$$\times \left(1 - e^{-(u-\mu)/\theta}\right)^{k-1} e^{-(u-\mu)/\theta} du.$$ 

Using \(1 - e^{-(u-\mu)/\theta}\)^{k-1} = \sum_{l=0}^{k-1} C(k - 1, l) (-1)^l e^{-l(u-\mu)/\theta}\), the expression for \(R_p\) simplifies to

$$R_p = \frac{k}{\theta} \sum_{l=0}^{k-1} C(k - 1, l) (-1)^l$$

$$\times \int_\mu^\infty \left[1 - p e^{-\theta_1 u} - (1 - p)e^{-\theta_2 u}\right] \exp \left(-(l + 1) \left(\frac{u-\mu}{\theta}\right)\right) du$$

$$= \frac{k}{\theta} \sum_{l=0}^{k-1} C(k - 1, l) (-1)^l [I_1 - pI_2 - (1 - p)I_3]$$

where

$$I_1 = \int_\mu^\infty \exp \left(-(l + 1) \left(\frac{u-\mu}{\theta}\right)\right) du$$

$$= \frac{\theta}{l+1},$$

$$I_2 = \int_\mu^\infty \exp \left(- \left(\theta_1 u + (l + 1) \left(\frac{u-\mu}{\theta}\right)\right)\right) du$$

$$= \exp \left[\frac{(l+1)\mu}{\theta}\right] \int_\mu^\infty \exp \left[- \left(\theta_1 + \frac{(l+1)}{\theta}\right) u\right] du$$
\[ I_3 = \int_{\mu}^{\infty} \exp \left[ - \left( \theta_2 u + (l + 1) \left( \frac{\mu - \lambda}{\theta} \right) \right) \right] du \]

\[ = \frac{e^{-\theta_2 \mu}}{\theta_2 + \frac{(l + 1)}{\theta}}. \]

Hence,

\[ R_p = k \sum_{l=0}^{k-1} C(k - 1, l) (-1)^l \left[ \frac{1}{l+1} - \frac{p e^{-\theta_1 \mu}}{\theta (\theta_1 + \frac{(l+1)}{\theta})} - \frac{(1-p)e^{-\theta_2 \mu}}{\theta (\theta_2 + \frac{(l+1)}{\theta})} \right]. \]

Now, the system reliability for series system is obtained as

\[ R_s = P[Y < \min(X_1, X_2, ..., X_k)] \]

\[ = P[Y < \mathcal{V}] \]

\[ = \int_{\mu}^{\infty} G(v) \, dH_4(v) \]

\[ = \int_{\mu}^{\infty} \left[ 1 - p e^{-\theta_1 v} + (1 - p)e^{-\theta_2 v} \right] dH_4(v) \]

\[ = 1 - \frac{p k}{\theta} e^{\mu k/\theta} \int_{\mu}^{\infty} \exp[-(\theta_1 + (k/\theta))v]dv \]

\[ - \frac{(1-p)k}{\theta} e^{\mu k/\theta} \int_{\mu}^{\infty} \exp[-(\theta_2 + (k/\theta))v]dv \]

\[ = 1 - \frac{k}{\theta} e^{\mu k/\theta} \left[ p \frac{\exp[-\left( \frac{\theta_1 + (k/\theta)}{\theta_1 + (k/\theta)} \right)]}{\theta_1 + (k/\theta)} + (1 - p) \frac{\exp[-\left( \frac{\theta_2 + (k/\theta)}{\theta_2 + (k/\theta)} \right)]}{\theta_2 + (k/\theta)} \right]. \]
5.3.2. Maximum Likelihood Estimators for parameters $\mu, \theta, p, \theta_1, \theta_2$ and Reliability

Let $X_{i1}, X_{i2}, ..., X_{ik}$ $(i = 1, 2, ..., m)$ be a random sample on strengths of $m$ systems following two parameter exponential distribution with parameters $\mu$ and $\theta$. Let there be only two sub populations with mixing proportions $p$ and $1 - p$ and $g_1(y)$, $g_2(y)$ be exponential densities with parameters $\theta_1$ and $\theta_2$ respectively where $0 < p < 1$.

The likelihood function of the sample is given by

$$L(p, \theta_1, \theta_2, \mu, \theta \mid y_{11}, y_{12}, ..., y_{1n_1}; y_{21}, y_{22}, ..., y_{2n_2}; x_{i1}, x_{i2}, ..., x_{ik})$$

$$= \prod_{i=1}^{n} \left( \sum_{j=1}^{k} p_j f(y_i, \theta_j) \right) \prod_{j=1}^{k} \prod_{i=1}^{n} \left( \frac{1}{\theta} e^{-\frac{(x_{ij}-\mu)}{\theta}} \right)$$

$$= \frac{n!}{n_1! n_2!} p^{n_1} (1 - p)^{n_2} \theta_1^{n_1} \theta_2^{n_2} \exp \left( -\sum_{j=1}^{k} \theta_1 y_{1j} - \sum_{j=1}^{k} \theta_2 y_{2j} \right)$$

$$\times \left( \frac{1}{\theta} \right)^{km} \exp \left( -\frac{1}{\theta} \sum_{i=1}^{k} \sum_{j=1}^{m} (x_{ij} - \mu) \right)$$

where $y = (y_{11}, y_{12}, ..., y_{1n_1}; y_{21}, y_{22}, ..., y_{2n_2})$ and $n_1 + n_2 = n$.

Then log-likelihood function is given by

$$\ln(L) = n \ln \left( \frac{n!}{n_1! n_2!} \right) + n_1 \ln(p) + n_2 \ln(1 - p) + n_2 \ln(\theta_2)$$

$$- \theta_1 \sum_{j=1}^{n} y_{1j} - \theta_2 \sum_{j=1}^{n} y_{2j} - k \ln(\theta) - \frac{1}{\theta} \sum_{i=1}^{k} \sum_{j=1}^{m} (x_{ij} - \mu).$$

Likelihood equations $\frac{\partial}{\partial \theta} \ln L = 0$, $\frac{\partial}{\partial \theta_1} \ln L = 0$, $\frac{\partial}{\partial \theta_2} \ln L = 0$, and $\frac{\partial}{\partial \mu} \ln L = 0$ gives MLEs of $p, \theta_1, \theta_2$, and $\theta$ as

$$\hat{p} = \frac{n_1}{n}, \quad \hat{\theta}_1 = \left( \frac{\sum_{j=1}^{n} y_{1j}}{n_1} \right)^{-1}, \quad \hat{\theta}_2 = \left( \frac{\sum_{j=1}^{n} y_{2j}}{n_2} \right)^{-1} \quad \text{and} \quad \hat{\theta} = \frac{\sum_{i=1}^{k} \sum_{j=1}^{m} (x_{ij} - \bar{\mu})}{km}$$

where $\bar{\mu} = \min_{1 \leq i \leq k, 1 \leq j \leq m} x_{ij}$.
Hence, MLEs of $R_p$ and $R_s$ are respectively given by

$$
\hat{R}_p = k \sum_{l=0}^{k-1} C(k-1, l) (-1)^l \left[ \frac{1}{\theta} \frac{\beta e^{-\theta_1 \beta}}{\theta_1 + (l+1) \beta} - \frac{(1-\beta) e^{-\theta_2 \beta}}{\theta_2 + (l+1) \beta} \right],
$$

$$
\hat{R}_s = 1 - \frac{k}{\theta} e^{R_p / \theta} \left[ \hat{\beta} \exp \left[ -\left( \theta_1 + \frac{k}{\theta} \right) \beta \right] + (1 - \hat{\beta}) \exp \left[ -\left( \theta_2 + \frac{k}{\theta} \right) \beta \right] \right].
$$

### 5.3.3. Simulation Study

A simulation study is conducted to evaluate MSEs of reliabilities of series and parallel systems by generating 100000 samples of size $m = 10, n = 15$ for different values of $k$ and the parameters $(p, \theta_1, \theta_2, \mu, \theta)$ as specified in tables. Note that the values written in parentheses represent the corresponding values for $k = 3$ in the tables 5.3, 5.4 and 5.5. MATLAB code written for this simulation is given in Appendix XV.

#### Table 5.3: MLEs and MSEs for estimates of $R_p$ and $R_s$

$p = 0.2, \theta_1 = 0.5, \theta_2 = 0.3, k = 2(k = 3), m = 10, n = 15$

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\theta$</th>
<th>$R_s$</th>
<th>$\hat{R}_s$</th>
<th>$R_p$</th>
<th>$\hat{R}_p$</th>
<th>$(R_s - \hat{R}_s)^2$</th>
<th>$(R_p - \hat{R}_p)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>0.2</td>
<td>0.5036</td>
<td>0.5279</td>
<td>0.5341</td>
<td>0.5554</td>
<td>0.008349</td>
<td>0.008406</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.4984)</td>
<td>(0.5227)</td>
<td>(0.5441)</td>
<td>(0.5647)</td>
<td>(0.008298)</td>
<td>(0.008393)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.5262</td>
<td>0.5495</td>
<td>0.5952</td>
<td>0.6085</td>
<td>0.008436</td>
<td>0.008373</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.5139)</td>
<td>(0.5386)</td>
<td>(0.6134)</td>
<td>(0.6290)</td>
<td>(0.008456)</td>
<td>(0.008303)</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.5531</td>
<td>0.5766</td>
<td>0.6540</td>
<td>0.6655</td>
<td>0.008513</td>
<td>0.008021</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.5332)</td>
<td>(0.5581)</td>
<td>(0.6845)</td>
<td>(0.6943)</td>
<td>(0.008528)</td>
<td>(0.007709)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>0.5881</td>
<td>0.6112</td>
<td>0.7224</td>
<td>0.7285</td>
<td>0.008557</td>
<td>0.007202</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.5594)</td>
<td>(0.5840)</td>
<td>(0.7606)</td>
<td>(0.7638)</td>
<td>(0.008574)</td>
<td>(0.006471)</td>
</tr>
<tr>
<td>5.0</td>
<td>0.2</td>
<td>0.8111</td>
<td>0.8141</td>
<td>0.8223</td>
<td>0.8240</td>
<td>0.005471</td>
<td>0.005210</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8091)</td>
<td>(0.8119)</td>
<td>(0.8260)</td>
<td>(0.8269)</td>
<td>(0.005543)</td>
<td>(0.005146)</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>0.8194</td>
<td>0.8217</td>
<td>0.8439</td>
<td>0.8432</td>
<td>0.005263</td>
<td>0.004666</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8148)</td>
<td>(0.8180)</td>
<td>(0.8517)</td>
<td>(0.8508)</td>
<td>(0.005352)</td>
<td>(0.004433)</td>
</tr>
<tr>
<td></td>
<td>0.9</td>
<td>0.8293</td>
<td>0.8314</td>
<td>0.8669</td>
<td>0.8643</td>
<td>0.00494</td>
<td>0.003994</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8220)</td>
<td>(0.8249)</td>
<td>(0.8784)</td>
<td>(0.8752)</td>
<td>(0.005175)</td>
<td>(0.003682)</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>0.8423</td>
<td>0.8442</td>
<td>0.8927</td>
<td>0.8884</td>
<td>0.004579</td>
<td>0.003226</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8316)</td>
<td>(0.8342)</td>
<td>(0.9072)</td>
<td>(0.9020)</td>
<td>(0.004888)</td>
<td>(0.002788)</td>
</tr>
</tbody>
</table>
Observations: It is observed from the table 5.3 that for fixed the values of stress parameters and location parameter of strength, there is an increase in the values of $R_s$ and $R_p$ for increase in the values of shape parameter.

Table 5.4: MLEs and MSEs for estimates of $R_p$ and $R_s$

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$R_s$</th>
<th>$\bar{R}_s$</th>
<th>$R_p$</th>
<th>$\bar{R}_p$</th>
<th>$(R_s - \bar{R}_s)^2$</th>
<th>$(R_p - \bar{R}_p)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.6496</td>
<td>0.6684</td>
<td>0.6815</td>
<td>0.6963</td>
<td>0.008092</td>
<td>0.007692</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.6440)</td>
<td>(0.6627)</td>
<td>(0.6918)</td>
<td>(0.7052)</td>
<td>(0.008149)</td>
<td>(0.007547)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>0.7695</td>
<td>0.7785</td>
<td>0.7978</td>
<td>0.8027</td>
<td>0.006205</td>
<td>0.005584</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.7644)</td>
<td>(0.7734)</td>
<td>(0.8069)</td>
<td>(0.8105)</td>
<td>(0.006360)</td>
<td>(0.005414)</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
<td>0.8483</td>
<td>0.8497</td>
<td>0.8715</td>
<td>0.8696</td>
<td>0.004311</td>
<td>0.003710</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8441)</td>
<td>(0.8458)</td>
<td>(0.8788)</td>
<td>(0.8761)</td>
<td>(0.004439)</td>
<td>(0.003516)</td>
</tr>
<tr>
<td>1.1</td>
<td>1.1</td>
<td>0.9002</td>
<td>0.8966</td>
<td>0.9182</td>
<td>0.9123</td>
<td>0.002849</td>
<td>0.002361</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8968)</td>
<td>(0.8934)</td>
<td>(0.9238)</td>
<td>(0.9175)</td>
<td>(0.002974)</td>
<td>(0.002217)</td>
</tr>
</tbody>
</table>

Table 5.5: MLEs and MSEs for estimates of $R_p$ and $R_s$

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$R_s$</th>
<th>$\bar{R}_s$</th>
<th>$R_p$</th>
<th>$\bar{R}_p$</th>
<th>$(R_s - \bar{R}_s)^2$</th>
<th>$(R_p - \bar{R}_p)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.6815</td>
<td>0.6710</td>
<td>0.6815</td>
<td>0.6986</td>
<td>0.008415</td>
<td>0.007958</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.6440)</td>
<td>(0.6647)</td>
<td>(0.6918)</td>
<td>(0.7069)</td>
<td>(0.008418)</td>
<td>(0.007744)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>0.7695</td>
<td>0.7794</td>
<td>0.7978</td>
<td>0.8033</td>
<td>0.006309</td>
<td>0.005664</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.7644)</td>
<td>(0.7747)</td>
<td>(0.8069)</td>
<td>(0.8113)</td>
<td>(0.006396)</td>
<td>(0.005424)</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
<td>0.8483</td>
<td>0.8497</td>
<td>0.8715</td>
<td>0.8693</td>
<td>0.004381</td>
<td>0.003777</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8441)</td>
<td>(0.8456)</td>
<td>(0.8788)</td>
<td>(0.8756)</td>
<td>(0.004488)</td>
<td>(0.003566)</td>
</tr>
<tr>
<td>1.1</td>
<td>1.1</td>
<td>0.9002</td>
<td>0.8963</td>
<td>0.9182</td>
<td>0.9118</td>
<td>0.002896</td>
<td>0.002411</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8968)</td>
<td>(0.8929)</td>
<td>(0.9238)</td>
<td>(0.9167)</td>
<td>(0.003032)</td>
<td>(0.002275)</td>
</tr>
</tbody>
</table>

Observations: From the tables 5.4 and 5.5 it is observed that values of $R_s$ and $R_p$ are increasing for fixed values of strength parameters and proportion parameter of stress with an increase in the value of scale parameter of stress.
5.4. Estimation of Reliability when distribution of strengths follow Lindley distribution

In this section, the stress-strength reliability of multi-component system with \( k \) identical components is considered with the assumption that stress and strengths are independent. The \( k \) components of the system are affected by an independent stress. Here, the strength and stress variables follow respectively Lindley and mixture of two exponential distributions. The reliability expressions for the system are derived and MLEs of the reliability functions through the estimators of parameters characterizing the model are obtained.

5.4.1. System Reliability

Let \( X_1, X_2, \ldots, X_k \) be strengths having Lindley distribution with scale parameter \( \lambda \) exposed to an independent stress \( Y \), following mixture of two exponential distributions with parameters \( p, \theta_1, \theta_2 \).

The pdf of \( X_i \) is given by

\[
f_i(x, \lambda) = \frac{\lambda^2}{(1+\lambda)}(1+x) \ e^{-\lambda x} \quad x, \lambda > 0, \ i = 1,2, \ldots, k
\]

and the pdf of \( Y \) is given by

\[
g(y) = p \ \theta_1 e^{-\theta_1 y} + (1-p) \theta_2 e^{-\theta_2 y} \quad \theta_1, \theta_2, y > 0, \ 0 \leq p \leq 1.
\]

The distribution function of \( Y \) is given by

\[
G(y) = 1 - p \ e^{-\theta_1 y} + (1-p) e^{-\theta_2 y} \quad \theta_1, \theta_2, y > 0, \ 0 \leq p \leq 1.
\]
Then the distribution function of $U = \text{Max}(X_1, X_2, ..., X_k)$ is given by

$$H_5(u) = P[U < u]$$

$$= \prod_{i=1}^{k} P(X_i < u)$$

$$= \left(1 - e^{-\lambda u} \left(1 + \frac{\lambda u}{1+\lambda}\right)\right)^k, \quad u, \lambda > 0.$$  

The distribution function of $V = \text{Min}(X_1, X_2, ..., X_k)$ is given by

$$H_6(v) = P[V < v]$$

$$= 1 - e^{-\lambda v k} \left(1 + \frac{\lambda v}{1+\lambda}\right)^k.$$  

Now, the system reliability for parallel system is obtained as

$$R_p = P[Y < \text{Max}(X_1, X_2, ..., X_k)]$$

$$= P[Y < U]$$

$$= \int_0^\infty H_5(y) dG(y)$$

$$= \int_0^\infty \left\{1 - \left[1 - e^{-\lambda y} \left(1 + \frac{\lambda y}{1+\lambda}\right)\right]^k\right\} \left[p \theta_1 e^{-\theta_1 y} + (1-p)\theta_2 e^{-\theta_2 y}\right] dy.$$  

Using \(1 - e^{-\lambda y} \left(1 + \frac{\lambda y}{1+\lambda}\right)\) \(= \sum_{l=0}^{k} C(k, l) (-1)^l e^{-\lambda y} \left(1 + \frac{\lambda y}{1+\lambda}\right)^l\),

the expression for $R_p$ simplifies to

$$R_p = 1 - \sum_{l=0}^{k} C(k, l) (-1)^l \left[p \theta_1 \int_0^\infty e^{-(\lambda l + \theta_1) y} \left(1 + \frac{\lambda y}{1+\lambda}\right)^l dy + (1-p)\theta_2 \int_0^\infty e^{-(\lambda l + \theta_2) y} \left(1 + \frac{\lambda y}{1+\lambda}\right)^l dy\right]$$

153
Now, the system reliability for series system is obtained as

\[
R_s = P[Y < \text{Min}(X_1, X_2, \ldots, X_k)]
\]

\[
= P[Y < V]
\]

\[
= \int_0^\infty \bar{H}_6(y) g(y) \, dy \quad \text{where} \quad \bar{H}_6(y) = 1 - H_6(y)
\]

\[
= \int_0^\infty e^{-\lambda ky} \left(1 + \frac{\lambda y}{1+\lambda}\right)^k \left[p \theta_1 e^{-\theta_1 y} + (1 - p) \theta_2 e^{-\theta_2 y}\right] \, dy
\]

\[
= \sum_{l=0}^k C(k,l) \left(\frac{\lambda}{1+\lambda}\right)^l
\]

\[
\times \left[p \theta_1 \int_0^\infty e^{-(\lambda k + \theta_1 y)} y^l \, dy + (1 - p) \theta_2 \int_0^\infty e^{-(\lambda k + \theta_2 y)} y^l \, dy\right]
\]

\[
= \sum_{l=0}^k \frac{k!}{(k-l)!} \left(\frac{\lambda}{1+\lambda}\right)^l \left[\frac{p \theta_1}{(\lambda k + \theta_1)^{l+1}} + \frac{(1-p) \theta_2}{(\lambda k + \theta_2)^{l+1}}\right].
\]

5.4.2. Maximum Likelihood Estimators for parameters \(\lambda, p, \theta_1, \theta_2\) and Reliability

Let \(X_{i1}, X_{i2}, \ldots, X_{ik} (i = 1, 2, \ldots, m)\) be a random sample on strengths of \(m\) systems that are following Lindley distribution with parameter \(\lambda\). Let there be only two sub populations with mixing proportions \(p\) and \(1 - p\) and \(g_1(y)\), \(g_2(y)\) be exponential densities with parameters \(\theta_1\) and \(\theta_2\) respectively where \(0 < p < 1\).
The likelihood of the sample is given by

$$L(p, \theta_1, \theta_2, \lambda | y_{11}, y_{12}, \ldots, y_{n_1}; y_{21}, y_{22}, \ldots, y_{2n_2}; x_{i1}, x_{i2}, \ldots, x_{ik})$$

$$= \prod_{i=1}^{n_1} \{ \sum_{j=1}^{p} f(y_i, \theta_1) \} \prod_{i=1}^{n_2} \{ \sum_{j=1}^{p} f(y_i, \theta_2) \} \left( \frac{\lambda^2}{(1+\lambda)} \right) (1 + x_{ij}) e^{-\lambda x_{ij}}$$

$$= \frac{n_1}{n_1!n_2!} p^{n_1} (1-p)^{n_2} \theta_1^{n_1} \theta_2^{n_2} \exp(-\sum_{j=1}^{n_1} \theta_1 y_{1j} - \sum_{j=1}^{n_2} \theta_2 y_{2j})$$

$$\times \frac{\lambda^{2km}}{(1+\lambda)^{km}} \left( \prod_{i=1}^{k} \prod_{j=1}^{m} (1 + x_{ij}) \right) e^{-\lambda \sum_{i=1}^{k} \sum_{j=1}^{m} x_{ij}}$$

where $y = (y_{11}, y_{12}, \ldots, y_{n_1}; y_{21}, y_{22}, \ldots, y_{2n_2})$ and $n_1 + n_2 = n$.

Then log-likelihood function is given by

$$\ln(L) = \ln \left( \frac{n_1}{n_1!n_2!} \right) + n_1 \ln(p) + n_1 \ln(\theta_1) + n_2 \ln(1-p) + n_2 \ln(\theta_2)$$

$$- \theta_1 \sum_{j=1}^{n_1} y_{1j} - \theta_2 \sum_{j=1}^{n_2} y_{2j} + 2k \ln(\lambda) - k \ln(1+\lambda)$$

$$+ \sum_{i=1}^{k} \sum_{j=1}^{m} \ln(1+x_{ij}) - \lambda \sum_{i=1}^{k} \sum_{j=1}^{m} x_{ij}.$$ 

Likelihood equations $\frac{\partial}{\partial p} \ln L = 0$, $\frac{\partial}{\partial \theta_1} \ln L = 0$, $\frac{\partial}{\partial \theta_2} \ln L = 0$ and $\frac{\partial}{\partial \lambda} \ln L = 0$ results in MLEs of $p, \theta_1, \theta_2$ and $\lambda$ as

$$\hat{p} = \frac{n_1}{n}, \quad \hat{\theta}_1 = \left( \frac{\sum_{j=1}^{n_1} y_{1j}}{n_1} \right)^{-1}, \quad \hat{\theta}_2 = \left( \frac{\sum_{j=1}^{n_2} y_{2j}}{n_2} \right)^{-1} \quad \text{and} \quad \hat{\lambda} = \frac{(1-\bar{x}) + \sqrt{\bar{x}^2 + 4\bar{x} + 1}}{2\bar{x}}$$

where $\bar{x} = \frac{\sum_{i=1}^{k} \sum_{j=1}^{m} x_{ij}}{mk}$.

Hence, MLEs of $R_p$ and $R_s$ are respectively given by

$$\hat{R}_p = k \hat{\lambda} \sum_{r=0}^{k-1} \sum_{r=0}^{l+1} C(k-1, l) (-1)^l \frac{(l+1)!}{(l+1-r)!} \left( \frac{\hat{\lambda}}{\hat{\lambda} + \hat{\theta}_1} \right)^r \left( \frac{1}{\hat{\lambda} + \hat{\theta}_2} \right)^{r+1} \left( \frac{1-\hat{p}}{(l+1)\hat{\lambda} + \hat{\theta}_1} \right)^{r+1} \left( \frac{1-\hat{p}}{(l+1)\hat{\lambda} + \hat{\theta}_2} \right)^{r+1}$$
A simulation study is conducted to evaluate MSEs of reliabilities of series and parallel systems by generating 100,000 samples of size \( m = 10, n = 15 \) for different values of \( k \) and the parameters \( (p, \theta_1, \theta_2, \lambda) \) as specified in tables. Note that the values written in parentheses represent the corresponding values for \( k = 3 \) in the tables 5.6, 5.7, 5.8 and 5.9. MATLAB codes written for this simulation is given in Appendix XVI.

**Table 5.6: MLEs and MSEs for estimates of \( R_p \) and \( R_s \)**  
\( \lambda = 0.1, p = 0.2, k = 2(k = 3), m = 10, n = 15 \)

<table>
<thead>
<tr>
<th>( \theta_1 )</th>
<th>( \theta_2 )</th>
<th>( R_s )</th>
<th>( \hat{R}_s )</th>
<th>( R_p )</th>
<th>( \hat{R}_p )</th>
<th>( (R_s - \hat{R}_s)^2 )</th>
<th>( (R_p - \hat{R}_p)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.6085 (0.5155)</td>
<td>0.8579 (0.9076)</td>
<td>0.8541 (0.9018)</td>
<td>0.007616 (0.007440)</td>
<td>0.003858 (0.002454)</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>0.8548 (0.8072)</td>
<td>0.9831 (0.9944)</td>
<td>0.9786 (0.9915)</td>
<td>0.002642 (0.003470)</td>
<td>0.0002546 (0.0002796)</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.9199 (0.8892)</td>
<td>0.9953 (0.9991)</td>
<td>0.9933 (0.9982)</td>
<td>0.001058 (0.001564)</td>
<td>0.000382 (0.000067)</td>
<td></td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>0.9981 (0.9997)</td>
<td>0.9971 (0.9994)</td>
<td>0.000520 (0.00080)</td>
<td>0.000050 (0.000009)</td>
<td>0.000009 (0.000001)</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.7: MLEs and MSEs for estimates of $R_p$ and $R_s$

$\lambda = 0.1, p = 0.6, k = 2(k = 3), m = 10, n = 15$

<table>
<thead>
<tr>
<th>$\theta_2$</th>
<th>$\theta_2$</th>
<th>$R_s$</th>
<th>$\bar{R}_s$</th>
<th>$R_p$</th>
<th>$\bar{R}_p$</th>
<th>$(R_s - \bar{R}_s)^2$</th>
<th>$(R_p - \bar{R}_p)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.5966</td>
<td>0.6085</td>
<td>0.8580</td>
<td>0.8534</td>
<td>0.007975</td>
<td>0.004015</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.5155)</td>
<td>(0.5337)</td>
<td>(0.9076)</td>
<td>(0.9013)</td>
<td>(0.007877)</td>
<td>(0.002562)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>0.8579</td>
<td>0.8547</td>
<td>0.9831</td>
<td>0.9784</td>
<td>0.002756</td>
<td>0.0002677</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8066)</td>
<td>(0.8068)</td>
<td>(0.9944)</td>
<td>(0.9913)</td>
<td>(0.003678)</td>
<td>(0.000077)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.9237</td>
<td>0.9196</td>
<td>0.9953</td>
<td>0.9932</td>
<td>0.001131</td>
<td>0.0000409</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.8920)</td>
<td>(0.8890)</td>
<td>(0.9991)</td>
<td>(0.9982)</td>
<td>(0.001628)</td>
<td>(0.000006)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>0.9508</td>
<td>0.9471</td>
<td>0.9981</td>
<td>0.9971</td>
<td>0.000544</td>
<td>0.000009</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.9288)</td>
<td>(0.9254)</td>
<td>(0.9997)</td>
<td>(0.9994)</td>
<td>(0.00086)</td>
<td>(0.000001)</td>
</tr>
</tbody>
</table>

Observations: From the tables 5.6 and 5.7 it is observed that for fixed values of strength parameters and mixing parameter of stress, values of $R_s$ and $R_p$ are increasing with the increasing values of scale parameters of stress.

Table 5.8: MLEs and MSEs for estimates of $R_p$ and $R_s$

$\theta_1 = \theta_2 = 0.1, p = 0.2, k = 2(k = 3), m = 10, n = 15$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$R_s$</th>
<th>$\bar{R}_s$</th>
<th>$R_p$</th>
<th>$\bar{R}_p$</th>
<th>$(R_s - \bar{R}_s)^2$</th>
<th>$(R_p - \bar{R}_p)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5966</td>
<td>0.6085</td>
<td>0.8579</td>
<td>0.8541</td>
<td>0.007616</td>
<td>0.003858</td>
</tr>
<tr>
<td></td>
<td>(0.5155)</td>
<td>(0.5339)</td>
<td>(0.9076)</td>
<td>(0.9023)</td>
<td>(0.0075)</td>
<td>(0.002428)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3778</td>
<td>0.3981</td>
<td>0.6593</td>
<td>0.6684</td>
<td>0.007266</td>
<td>0.008269</td>
</tr>
<tr>
<td></td>
<td>(0.3051)</td>
<td>(0.3277)</td>
<td>(0.7273)</td>
<td>(0.7343)</td>
<td>(0.005815)</td>
<td>(0.006976)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2681</td>
<td>0.2901</td>
<td>0.5204</td>
<td>0.5390</td>
<td>0.005614</td>
<td>0.009335</td>
</tr>
<tr>
<td></td>
<td>(0.2086)</td>
<td>(0.2297)</td>
<td>(0.5869)</td>
<td>(0.6039)</td>
<td>(0.004039)</td>
<td>(0.008963)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2041</td>
<td>0.2246</td>
<td>0.4245</td>
<td>0.4469</td>
<td>0.004208</td>
<td>0.008837</td>
</tr>
<tr>
<td></td>
<td>(0.1548)</td>
<td>(0.1735)</td>
<td>(0.4855)</td>
<td>(0.5079)</td>
<td>(0.002832)</td>
<td>(0.009053)</td>
</tr>
</tbody>
</table>
Table 5.9: MLEs and MSEs for estimates of $R_p$ and $R_s$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$R_s$</th>
<th>$\bar{R}_s$</th>
<th>$R_p$</th>
<th>$\bar{R}_p$</th>
<th>$(R_s - \bar{R}_s)^2$</th>
<th>$(R_p - \bar{R}_p)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5966</td>
<td>0.6085</td>
<td>0.8580</td>
<td>0.8534</td>
<td>0.0007975</td>
<td>0.0004015</td>
</tr>
<tr>
<td></td>
<td>(0.5155)</td>
<td>(0.5338)</td>
<td>(0.9076)</td>
<td>(0.9014)</td>
<td>(0.007837)</td>
<td>(0.002539)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3778</td>
<td>0.3993</td>
<td>0.6593</td>
<td>0.6698</td>
<td>0.0057801</td>
<td>0.0008735</td>
</tr>
<tr>
<td></td>
<td>(0.3051)</td>
<td>(0.3282)</td>
<td>(0.7273)</td>
<td>(0.7352)</td>
<td>(0.006155)</td>
<td>(0.007323)</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2681</td>
<td>0.2901</td>
<td>0.5204</td>
<td>0.5398</td>
<td>0.0055894</td>
<td>0.0009875</td>
</tr>
<tr>
<td></td>
<td>(0.2086)</td>
<td>(0.2295)</td>
<td>(0.5869)</td>
<td>(0.6057)</td>
<td>(0.004143)</td>
<td>(0.009403)</td>
</tr>
<tr>
<td>0.4</td>
<td>0.2041</td>
<td>0.2241</td>
<td>0.4245</td>
<td>0.4477</td>
<td>0.0004345</td>
<td>0.0009375</td>
</tr>
<tr>
<td></td>
<td>(0.1548)</td>
<td>(0.1727)</td>
<td>(0.4855)</td>
<td>(0.5092)</td>
<td>(0.002814)</td>
<td>(0.009561)</td>
</tr>
</tbody>
</table>

Observations: It is observed from the tables 5.8 and 5.9 that if the values of stress parameters are fixed then values of $R_s$ and $R_p$ are decreasing when the values of scale parameters of strength are increasing.

5.5. Estimation of Reliability when distribution of strengths follow Power function distribution

In this section, the stress-strength reliability of $k$ component system with identical components and an independent stress acting on system is considered. Here, the strength and stress variables have Power function distribution and mixture of two exponential distributions respectively. The reliability expressions for the system considered here are derived and MLEs of the reliability functions through the estimators of parameters characterizing the model are obtained.
5.5.1. System Reliability

Let $X_1, X_2, \ldots, X_k$ be strengths having power function distribution with scale parameter $\theta$ and shape parameter $\alpha$, subjected to an independent stress $Y$, following mixture of two exponential distributions with parameters $p, \theta_1$ and $\theta_2$.

The pdf of $X_i$ is given by

$$f_i(x, \theta, \alpha) = \frac{\alpha}{\theta} \left(\frac{x}{\theta}\right)^{\alpha-1} \quad 0 < x < \theta, \quad \alpha > 0, \quad i = 1, 2, \ldots, k$$

and the pdf of $Y$ is given by

$$g(y) = p \theta_1 e^{-\theta_1 y} + (1 - p) \theta_2 e^{-\theta_2 y} \quad \theta_1, \theta_2, y > 0, \quad 0 \leq p \leq 1.$$ 

The distribution function of $Y$ is given by

$$G(y) = 1 - p e^{-\theta_1 y} + (1 - p) e^{-\theta_2 y} \quad \theta_1, \theta_2, y > 0, \quad 0 \leq p \leq 1.$$ 

Then the distribution function of $U = \text{Max}(X_1, X_2, \ldots, X_k)$ is given by

$$H_7(u) = P[U < u]$$

$$= \prod_{i=1}^{k} P(X_i < u)$$

$$= \left(\frac{u}{\theta}\right)^{k\alpha} \quad 0 < u < \theta$$

and the distribution function of $V = \text{Min}(X_1, X_2, \ldots, X_k)$ is given by

$$H_8(v) = P[V < v]$$

$$= 1 - P[V \geq v]$$

$$= 1 - \prod_{i=1}^{k} \left[1 - \left(\frac{v}{\theta}\right)^{\alpha}\right]$$

$$= \left[1 - \left(\frac{v}{\theta}\right)^{\alpha}\right]^k.$$
Thus, the system reliability for parallel system is given by

\[ R_p = P[Y < \text{Max}(X_1, X_2, ..., X_k)] \]

\[ = \int_0^\theta G(u) \, dH_\gamma(u) \]

\[ = \int_0^\theta \left( 1 - pe^{-\theta_1 u} - (1 - p)e^{-\theta_2 u} \right) \frac{ka}{\theta_1} \frac{ka}{\theta_2} e^{-tu} u^{ka-1} \, du \]

\[ = \frac{ka}{\theta_1 \theta_2} \left[ \frac{\theta_2}{\theta_1} \int_0^\theta e^{-t u} u^{ka-1} \, dt - \frac{(1-p)\theta_2}{\theta_2} \int_0^\theta e^{-t u} u^{ka-1} \, dt \right] \]

\[ = 1 - \frac{pka}{(\theta_1 \theta_2)ka} \gamma(ka, \theta_1) - \frac{(1-p)ka}{(\theta_2)ka} \gamma(ka, \theta_2) \]

where \( \gamma(s,x) \) is the lower incomplete gamma function defined as

\[ \gamma(s,x) = \int_0^x e^{-t} t^{s-1} \, dt. \]

The system reliability for series system is obtained as

\[ R_s = P[Y < \text{Min}(X_1, X_2, ..., X_k)] \]

\[ = \int_0^\infty H_\lambda(y) \, g(y) \, dy \quad \text{where} \quad H_\lambda(y) = 1 - H_\lambda(y) \]

\[ = \int_0^\infty \left[ 1 - \left( \frac{1}{\lambda} \right)^\lambda y \right]^k \left[ p \theta_1 e^{-\theta_1 y} + (1 - p)\theta_2 e^{-\theta_2 y} \right] dy \]

\[ = \sum_{i=0}^k \lambda^i \left( \frac{1}{\lambda} \right)^{ai} \left[ p \theta_1 e^{-\theta_1 y} + (1 - p)\theta_2 e^{-\theta_2 y} \right] dy \]

\[ = \sum_{i=0}^k \lambda^i \left( \frac{1}{\lambda} \right)^{ai} \left[ p \theta_1 e^{-\theta_1 y} y^{ai} \right. \left. + (1 - p)\theta_2 e^{-\theta_2 y} y^{ai} \right] dy \]

\[ = \sum_{i=0}^k \lambda^i \left( \frac{1}{\lambda} \right)^{ai} \left[ p \theta_1 \frac{\Gamma(ai+1)}{\theta_1^{ai+1}} + (1 - p)\theta_2 \frac{\Gamma(ai+1)}{\theta_2^{ai+1}} \right] \]

\[ = \sum_{i=0}^k \lambda^i \left( \frac{1}{\lambda} \right)^{ai} \left[ p \frac{(ai+1)}{(\theta_1)^{ai+1}} + (1 - p)\frac{(ai+1)}{(\theta_2)^{ai+1}} \right] \]

\[ = \sum_{i=0}^k \lambda^i \left( \frac{1}{\lambda} \right)^{ai} \Gamma(ai+1) \left[ p \frac{(1-p)}{(\theta_1)^{ai}} + \frac{(1-p)}{(\theta_2)^{ai}} \right]. \]
5.5.2. Maximum Likelihood Estimators for parameters $a, \theta, p, \theta_1, \theta_2$ and Reliability

Let $X_{i1}, X_{i2}, \ldots, X_{ik}$ ($i = 1, 2, \ldots, m$) be a random sample on strengths of $m$ systems that are following power function distribution with scale parameter $\theta$ and shape parameter $a$. Let there be only two sub populations with mixing proportions $p$ and $1 - p$ and $g_1(y)$, $g_2(y)$ be exponential densities with parameters $\theta_1$ and $\theta_2$ respectively where $0 < p < 1$.

The likelihood of the sample is given by

\[
L(p, \theta_1, \theta_2, \lambda | y_{11}, y_{12}, \ldots, y_{1n_1}; y_{21}, y_{22}, \ldots, y_{2n_2}; x_{i1}, x_{i2}, \ldots, x_{ik})
\]

\[
= \prod_{i=1}^{n_1} \left( \sum_{j=1}^{2} p_j f(y_i, \theta_j) \right) \prod_{i=1}^{k} \prod_{j=1}^{m} \left( \frac{a}{\theta} \right)^{a-1} \frac{\theta_1^{n_1} \theta_2^{n_2} \exp\left( -\sum_{j=1}^{n_1} \theta_1 y_{1j} - \sum_{j=1}^{n_2} \theta_2 y_{2j} \right)}{\prod_{i=1}^{k} \prod_{j=1}^{m} x_{ij}^{a-1}}
\]

where $y = (y_{11}, y_{12}, \ldots, y_{1n_1}; y_{21}, y_{22}, \ldots, y_{2n_2})$ and $n_1 + n_2 = n$.

Then log-likelihood function is given by

\[
ln(L) = ln \left( \frac{n_1}{n_{11} n_{12}^{n_2}} \right) + n_1 ln(p) + n_1 ln(\theta_1) + n_2 ln(1 - p) + n_2 ln(\theta_2)
- \theta_1 \sum_{j=1}^{n_1} y_{1j} - \theta_2 \sum_{j=1}^{n_2} y_{2j} + kmln(a) - akmln(\theta)
+ (a - 1) \sum_{i=1}^{k} \sum_{j=1}^{m} ln(x_{ij}).
\]

Likelihood function can be maximized by taking

\[
\hat{\theta} = Max(x_{i1}, x_{i2}, \ldots, x_{im}) \quad \forall \; i = 1, 2, \ldots, k.
\]
Also, Likelihood equations \[ \frac{\partial}{\partial p} \ln L = 0, \quad \frac{\partial}{\partial \theta_1} \ln L = 0, \quad \frac{\partial}{\partial \theta_2} \ln L = 0, \]
and \[ \frac{\partial}{\partial a} \ln L = 0 \]
results in MLEs of \( p, \theta_1, \theta_2, \) and \( a \) respectively as

\[ \hat{p} = \frac{n_1}{n}, \quad \hat{\theta}_1 = \left( \frac{\sum_{j=1}^{n_1} y_{1j}}{n_1} \right)^{-1}, \quad \hat{\theta}_2 = \left( \frac{\sum_{j=1}^{n_2} y_{2j}}{n_2} \right)^{-1} \quad \text{and} \quad \hat{a} = \frac{km}{k \ln(\hat{\theta}) - \sum_{i=1}^{n} \sum_{j=1}^{m} \ln(x_{ij})} \]

Hence, MLE of \( R_p \) is given by

\[ \hat{R}_p = 1 - \frac{\hat{p} \hat{\alpha} \hat{\beta}}{(\hat{\theta}_1) \hat{\beta} \hat{\alpha}} \gamma(k \hat{\alpha}, \hat{\theta}_1) - \frac{(1-\hat{p}) \hat{\alpha} \hat{\beta}}{(\hat{\theta}_2) \hat{\beta} \hat{\alpha}} \gamma(k \hat{\alpha}, \hat{\theta}_2) \]

and that of \( R_s \) is given by

\[ \hat{R}_s = \sum_{i=0}^{k} C(k, i)(-1)^i \Gamma(\hat{\alpha} + 1) \left[ \frac{\hat{p}}{(\hat{\theta}_1) \hat{\alpha}} + \frac{(1-\hat{p})}{(\hat{\theta}_2) \hat{\alpha}} \right]. \]

5.5.3. Simulation Study

A simulation study is conducted to evaluate MSEs of reliabilities of series and parallel systems by generating 100,000 samples of size \( m = 10, n = 15 \) for different values of \( k \) and the parameters \( (p, \theta_1, \theta_2, a, \theta) \) as specified in tables. Note that the values written in parentheses represent the corresponding values for \( k = 3 \) in the tables 5.10 and 5.11. MATLAB codes written for this simulation is given in Appendix XVII.
Table 5.10: MLEs and MSEs for estimates of $R_p$ and $R_s$

$\theta_1 = 0.5, \theta_2 = 0.3, p = 0.2, a = 6, \theta = 8, k = 2, m = 10, n = 15$

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$R_s$</th>
<th>$\bar{R}_s$</th>
<th>$R_p$</th>
<th>$\bar{R}_p$</th>
<th>$(R_s - R_p)^2$</th>
<th>$(R_p - \bar{R}_p)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.6439</td>
<td>0.6624</td>
<td>0.7028</td>
<td>0.7146</td>
<td>0.0082</td>
<td>0.007387</td>
</tr>
<tr>
<td></td>
<td>(0.6659)</td>
<td>(0.6800)</td>
<td>(0.7102)</td>
<td>(0.7193)</td>
<td>(0.0078)</td>
<td>(0.007212)</td>
</tr>
<tr>
<td>6</td>
<td>0.7814</td>
<td>0.7875</td>
<td>0.8339</td>
<td>0.8336</td>
<td>0.006148</td>
<td>0.004963</td>
</tr>
<tr>
<td></td>
<td>(0.7814)</td>
<td>(0.7875)</td>
<td>(0.8339)</td>
<td>(0.8335)</td>
<td>(0.006105)</td>
<td>(0.004905)</td>
</tr>
<tr>
<td>8</td>
<td>0.8638</td>
<td>0.8636</td>
<td>0.9060</td>
<td>0.9010</td>
<td>0.004010</td>
<td>0.0028667</td>
</tr>
<tr>
<td></td>
<td>(0.8808)</td>
<td>(0.8801)</td>
<td>(0.9108)</td>
<td>(0.9066)</td>
<td>(0.003484)</td>
<td>(0.002667)</td>
</tr>
<tr>
<td>10</td>
<td>0.9139</td>
<td>0.9393</td>
<td>0.9464</td>
<td>0.9394</td>
<td>0.002556</td>
<td>0.001628</td>
</tr>
<tr>
<td></td>
<td>(0.9275)</td>
<td>(0.9223)</td>
<td>(0.9498)</td>
<td>(0.9428)</td>
<td>(0.00211)</td>
<td>(0.001484)</td>
</tr>
</tbody>
</table>

Observation: It is observed from the table 5.10 that for fixed value of the stress parameters and shape parameter of strength, values of $R_s$ and $R_p$ are increasing with the increasing values of scale parameter of strength.

Table 5.11: MLEs and MSEs for estimates of $R_p$ and $R_s$

$\theta_1 = 0.5, p = 0.2 (p = 0.5), a = 6, \theta = 8, k = 2, m = 10, n = 15$

<table>
<thead>
<tr>
<th>$\theta_2$</th>
<th>$R_s$</th>
<th>$\bar{R}_s$</th>
<th>$R_p$</th>
<th>$\bar{R}_p$</th>
<th>$(R_s - R_p)^2$</th>
<th>$(R_p - \bar{R}_p)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.8637</td>
<td>0.8636</td>
<td>0.9060</td>
<td>0.9010</td>
<td>0.004010</td>
<td>0.002867</td>
</tr>
<tr>
<td></td>
<td>(0.8963)</td>
<td>(0.8955)</td>
<td>(0.9315)</td>
<td>(0.9262)</td>
<td>(0.002994)</td>
<td>(0.002041)</td>
</tr>
<tr>
<td>0.5</td>
<td>0.9505</td>
<td>0.9442</td>
<td>0.9738</td>
<td>0.9659</td>
<td>0.001334</td>
<td>0.00073</td>
</tr>
<tr>
<td></td>
<td>(0.9505)</td>
<td>(0.9435)</td>
<td>(0.9738)</td>
<td>(0.9651)</td>
<td>(0.001378)</td>
<td>(0.00078)</td>
</tr>
<tr>
<td>0.7</td>
<td>0.9769</td>
<td>0.9704</td>
<td>0.9897</td>
<td>0.9834</td>
<td>0.0005766</td>
<td>0.0002896</td>
</tr>
<tr>
<td></td>
<td>(0.9670)</td>
<td>(0.9602)</td>
<td>(0.9838)</td>
<td>(0.9766)</td>
<td>(0.0009086)</td>
<td>(0.0004796)</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9853</td>
<td>0.9802</td>
<td>0.9935</td>
<td>0.9887</td>
<td>0.0003634</td>
<td>0.0001965</td>
</tr>
<tr>
<td></td>
<td>(0.9723)</td>
<td>(0.9667)</td>
<td>(0.9861)</td>
<td>(0.9805)</td>
<td>(0.0007528)</td>
<td>(0.0003955)</td>
</tr>
</tbody>
</table>

Observation: The values of $R_s$ and $R_p$ are increasing for the increasing values of $\theta_2$ with the fixed values of strength parameters, values of stress parameters $p$ and $\theta_1$. 

163
clear all;
clc

p=input('Enter p value:');
theta1=input('Enter theta1 value:');
theta2 = input('Enter theta2 value:');
lambda= input('Enter lambda value:');
m=input('Enter m value:');
n=input('Enter n value:');
k=input('Enter k value:');
iterationno=input('Enter iterationno:');
T1=0;
T2=0;
T3=0;
for l=0:k-1
    T1=T1+nchoosek(k-1,l)*((-1)^l)/((l+1)*lambda);
    T2=T2+nchoosek(k-1,l)*((-1)^l)/((l+1)*lambda+theta1);
    T3=T3+nchoosek(k-1,l)*((-1)^l)/((l+1)*lambda+theta2);
End
R= zeros(iterationno,4);
for t=1:iterationno
    % generation of exponential rvs
    X= exprnd(1/lambda,k,m);
    xij= sum(sum(X));
    lambdacap=m*k/xij;
    Rp=k*lambda*(T1-p*T2-(1-p)*T3);
    Rs=((p*theta1)/(lambda*k+theta1))+(((1-p)*theta2)/(lambda*k+theta2));
end

Appendix XIV
MATLAB program for the computation of reliability estimation and MSE: Strength-Exponential and Stress-Mixture of two Exponential.
% generation of mixture of expo rvs
r1=rand(1,n);
U1=exprnd(1/theta1,1,n);
U2=exprnd(1/theta2,1,n);
y1j=zeros(1,n);
y2j=zeros(1,n);
count=0;
for j=1:n
    if r1(1,j)<=p
        y1j(1,j)=U1(1,j);
count=count+1;
    else
        y2j(1,j)=U2(1,j);
    end
end
Y1j=sum(y1j);
Y2j=sum(y2j);
p_cap=count/n;
theta1cap=count/Y1j;
theta2cap=(n-count)/Y2j;

% estimation of R_cap and Rcap
T11=0;
T12=0;
T13=0;
for l=0:k-1
    T11=T11+nchoosek(k-1,l)*((-1)^l)/((l+1)*lambdacap);
    T12=T12+nchoosek(k-1,l)*((-1)^l)/((l+1)*lambdacap+theta1cap);
    T13=T13+nchoosek(k-1,l)*((-1)^l)/((l+1)*lambdacap+theta2cap);
end
Rcap=k*lambdacap*(T11-p_cap*T12-(1-p_cap)*T13);
\[ R_{\text{scap}} = \frac{(pcap \cdot \theta_1 \text{cap})}{(\lambda \text{cap} \cdot k + \theta_1 \text{cap})} + \frac{(1 - pcap \cdot \theta_2 \text{cap})}{(\lambda \text{cap} \cdot k + \theta_2 \text{cap})}; \]

\[ R(t,:) = [R_{\text{pcap}}, R_{\text{scap}}, (R_{\text{pcap}} - Rp)^2, (R_{\text{scap}} - Rs)^2 ]; \]

\textbf{end}

format long

Rs

Rp
disp('R_{\text{pcap}}, R_{\text{scap}}, (R_{\text{pcap}} - Rp)^2, (R_{\text{scap}} - Rs)^2');

mean(R)
clear all;
clc;
p=input('Enter p value:');
theta1=input('Enter theta1 value:');
theta2=input('Enter theta2 value:');
theta=input('Enter theta value:');
muu=input('Enter muu value:');
m=input('Enter m value:');
n=input('Enter n value:');
k=input('Enter k value:');
iterationno=input('Enter iterationno:');
T1=0;
for l=0:k-1
    Term=1/(l+1);
    Term1=p*exp(-muu*theta1)/(theta*(theta1+((l+1)/theta)));
    Term2=(1-p)*exp(-muu*theta2)/(theta*(theta2+((l+1)/theta)));
    T1=T1+nchoosek(k-1,l)*((-1)^l)*(Term-Term1-Term2);
end
Rp=k*T1;
Term1=p*exp(-muu*(theta1+(k/theta)))/((k/theta)+theta1);
Term2=(1-p)*exp(-muu*(theta2+(k/theta)))/((k/theta)+theta2);
Rs=1-(k*exp(k*muu/theta)/theta)*(Term1+Term2);
R=zeros(iterationno,4);
for t=1:iterationno
    % generation of two parameter exponential rvs
    r1=rand(k,m);
    x=zeros(k,m);
for i=1:k
    for j=1:m
        x(i,j)=muu-theta*log(1-r1(i,j));
    end
end
muucap=min(min(x));
theta=mean(mean(x))-muucap;
% generation of mixture of expo rvs
r2=rand(1,n);
U1=exprnd(1/theta1,1,n);
U2=exprnd(1/theta2,1,n);
y1j=zeros(1,n);
y2j=zeros(1,n);
count=0;
for j=1:n
    if r2(1,j)<=p
        y1j(1,j)=U1(1,j);
        count=count+1;
    else
        y2j(1,j)=U2(1,j);
    end
end
Y1j=sum(y1j);
Y2j=sum(y2j);
pcap=count/n;
theta1cap=count/Y1j;
theta2cap=(n-count)/Y2j;
% estimation of R1cap and R2cap
T1=0;
for l=0:k-1
    Term=1/(l+1);
Term1=pcap*exp(-muucap*theta1cap)/(thetacap*(theta1cap+((l+1)/thetacap)));
Term2=(1-pcap)*exp(-muucap*theta2cap)/(thetacap*(theta2cap+((l+1)/thetacap)));
T1=T1+nchoosek(k-1,l)*((-1)^l)*(Term-Term1-Term2);
end
Rpcap=k*T1;
Term1=pcap*exp(-muucap*(theta1cap+(k/thetacap)))/((k/thetacap)+theta1cap);
Term2=(1-pcap)*exp(-muucap*(theta2cap+(k/thetacap)))/((k/thetacap)+theta2cap);
Rscap=1-(k*exp(k*muucap/thetacap)/thetacap)*(Term1+Term2);
R(t,:)=[Rpcap, Rscap,(Rpcap-Rp)^2,(Rscap-Rs)^2 ];
end
format long
Rs
Rp
disp('Rpcap, Rscap,(Rpcap-Rp)^2,(Rscap-Rs)^2');
mean(R)
clear all;
clc;
p=input('Enter p value:');
theta1=input('Enter theta1 value:');
theta2 = input('Enter theta2 value:');
lambda=input('Enter lambda value:');
m=input('Enter m value:');
n=input('Enter n value:');
k=input('Enter k value:');
itervationno=input('Enter iterationno:');
R=zeros(iterationno,4);
T1=0;
T2=0;
for l=0:k-1
    for r=0:l+1
        Term11=nchoosek(k-1,l)*((-1)^l)*factorial(l+1)
        *((lambda/(lambda+1))^r)/factorial(l+1-r);
        Term12=(1/(((l+1)*lambda)^(r+1)))-(p/(((l+1)*lambda+theta1)^(r+1)))
            -((1-p)/(((l+1)*lambda+theta2)^(r+1)));
        T1=T1+Term11*Term12;
    end
end
for l=0:k-1
    for s=0:l
        Term13=nchoosek(k-1,l)*((-1)^l)*factorial(l)
        *((lambda/(lambda+1))^(s+1))/factorial(l-s);
Term14 = \frac{1}{((l+1)\lambda)^{(s+1)}} - \frac{p}{((l+1)\lambda+\theta_1)^{(s+1)}} - \frac{1-p}{((l+1)\lambda+\theta_2)^{(s+1)}}

T2 = T2 + Term13 * Term14;

end

end

Rp = k*\lambda*T1 - k*T2;

T3 = 0;

for l = 0:k

  Term21 = factorial(k)/factorial(k-l);
  Term22 = ((\lambda/(\lambda+1))^l);
  Term23 = (p*\theta_1/((\lambda*k+\theta_1)^{(l+1)}));
  Term24 = ((1-p)*\theta_2/((\lambda*k+\theta_2)^{(l+1)}));
  T3 = T3 + (Term21 * Term22 * (Term23 + Term24));

end

Rs = T3;

for t = 1:iterationno

  % generation of lindley rvs
  r1 = rand(k,m);
  G1 = gamrnd(1,1/\lambda, [k,m]);
  G2 = gamrnd(2,1/\lambda, [k,m]);
  xij = zeros(k,m);
  for i = 1:k

    for j = 1:m

      if r1(i,j) <= \lambda/(1+\lambda)
        xij(i,j) = G1(i,j);
      else
        xij(i,j) = G2(i,j);
      end

    end

  end

  average = mean(mean(xij,1));
avg\lambda_{cap} = ((1-\text{average}) + \sqrt{\text{average}^2 + 6\times\text{average} + 1})/(2\times\text{average});

\% generation of mixture of expo rvs
r2 = \text{rand}(1,n);
U1 = \text{exprnd}(1/\theta_1,1,n);
U2 = \text{exprnd}(1/\theta_2,1,n);
y1j = \text{zeros}(1,n);
y2j = \text{zeros}(1,n);
count = 0;
\text{for } j = 1:n
\quad \text{if } r2(1,j) \leq p
\quad \quad y1j(1,j) = U1(1,j);
\quad \quad \text{count} = \text{count} + 1;
\quad \text{else}
\quad \quad y2j(1,j) = U2(1,j);
\quad \text{end}
\text{end}
Y1j = \text{sum}(y1j);
Y2j = \text{sum}(y2j);
\text{pcap} = \text{count}/n;
\theta_1\text{cap} = \text{count}/Y1j;
\theta_2\text{cap} = (n-\text{count})/Y2j;
\% estimation of R\cap a \text{nd } R\text{scap}
T1 = 0;
T2 = 0;
\text{for } l = 0:k-1
\quad \text{for } r = 0:l+1
\quad \quad \text{Term11} = n\text{choose}(k-1,l)\times((-1)^l)\times\text{factorial}(l+1)
\quad \quad \times((\text{avg}\lambda_{cap}/(\text{avg}\lambda_{cap}+1))^{r})/\text{factorial}(l+1-r);
\quad \text{Term12} = (1/(((l+1)\times\text{avg}\lambda_{cap})^{(r+1)}))
\quad \quad -((p/(((l+1)\times\text{avg}\lambda_{cap}+\theta_1\text{cap})^{(r+1)}))
\quad \quad -((1-p)/(((l+1)\times\text{avg}\lambda_{cap}+\theta_2\text{cap})^{(r+1)}));
T1 = T1 + Term11 * Term12;
end
end
for l = 0:k-1
    for s = 0:l
        Term13 = nchoosek(k-1,l) * ((-1)^l) * factorial(l)
        * ((avglambdacap/(avglambdacap+1))^l((s+1))/factorial(l-s));
        Term14 = (1/((l+1)*avglambdacap)^(s+1)))
        - (p/(((l+1)*avglambdacap+theta1cap)^(s+1)))
        - ((1-p)/(((l+1)*avglambdacap+theta2cap)^(s+1)));
        T2 = T2 + Term13 * Term14;
    end
end
Rpcap = k * avglambdacap * T1 - k * T2;
T3 = 0;
for l = 0:k
    Term21 = factorial(k) / factorial(k-l);
    Term22 = ((avglambdacap/(avglambdacap+1))^l);
    Term23 = (p * theta1cap / ((avglambdacap * k + theta1cap)^(l+1)));
    Term24 = ((1-p) * theta2cap / ((avglambdacap * k + theta2cap)^(l+1)));
    T3 = T3 + (Term21 * Term22 * (Term23 + Term24));
end
Rscap = T3;
R(t,:) = [Rpcap, Rscap, (Rpcap-Rp)^2, (Rscap-Rs)^2];
end
Rp
Rs
format long
disp('Rpcap, Rscap, (Rpcap-Rp)^2, (Rscap-Rs)^2');
mean(R,1)
Appendix XVII
MATLAB program for the computation of reliability estimation and MSE:
Strength-Power function and Stress-Mixture of two Exponential.

clear all;
clc;
p=input('Enter p value:');
theta1=input('Enter theta1 value:');
theta2 = input('Enter theta2 value:');
a=input('Enter a value:');
theta = input('Enter theta value:');
m=input('Enter m value:');
n=input('Enter n value:');
k=input('Enter k value:');
iterationno=input('Enter iterationno:');
f1 = @(x)exp(-x).x.^(k*a-1);
lowergamma1 = integral(f1,0,theta*theta1);
f1 = @(x)exp(-x).x.^(k*a-1);
lowergamma2 = integral(f1,0,theta*theta2);
Rp=1-(p*k*a/((theta*theta1)^(k*a)))*(lowergamma1) -((1-p)*k*a/((theta*theta2)^(k*a)))*(lowergamma2);

Term11=0;
Term12=0;
for l=0:k-1
    f1 = @(x)exp(-x).x.^(a*(l+1)-1);
    lowergamma11 = integral(f1,0,theta*theta1);
    f1 = @(x)exp(-x).x.^(a*(l+1)-1);
    lowergamma12 = integral(f1,0,theta*theta2);
    Term1=(lowergamma11/((theta*theta1)^(a*(l+1))));
    Term2=(lowergamma12/((theta*theta2)^(a*(l+1))));
    Term11=Term11+nchoosek(k-1,l)*((-1)^l)*Term1;
    Term12=Term12+nchoosek(k-1,l)*((-1)^l)*Term2;
end
Rs=1-(p*k*a*Term11)-((1-p)*k*a*Term12);
R=zeros(iterationno,4);
for t=1:iterationno
    % generation of power function rvs
    r1=rand(k,m);
    x=zeros(k,m);
    for i=1:k
        for j=1:m
            x(i,j)=theta*(r1(i,j)^(1/a));
        end
    end

    thetacap=max(max(x));
    product=prod(prod(x));
    acap=(k*m)/(k*m*log(thetacap)-log(product));
    % generation of mixture of expo rvs
    r2=rand(1,n);
    U1=exprnd(1/theta1,1,n);
    U2=exprnd(1/theta2,1,n);
    y1j=zeros(1,n);
    y2j=zeros(1,n);
    count=0;
    for j=1:n
        if r2(1,j)<=p
            y1j(1,j)=U1(1,j);
            count=count+1;
        else
            y2j(1,j)=U2(1,j);
        end
    end

    Y1j=sum(y1j);
Y2j=sum(y2j);
pcap=count/n;
theta1cap=count/Y1j;
theta2cap=(n-count)/Y2j;

%estimation of Rpcap and Rscap
f1 = @(x)exp(-x).*x.^(k*acap-1);
lowergamma1cap= integral(f1,0,thetacap*theta1cap);
f1 = @(x)exp(-x).*x.^(k*acap-1);
lowergamma2cap= integral(f1,0,thetacap*theta2cap);
Rpcap=1-(pcap*k*acap/((thetacap*theta1cap)^(k*acap)))*(lowergamma1cap)
-((1-pcap)*k*acap/((thetacap*theta2cap)^(k*acap)))*(lowergamma2cap);
Term11=0;
Term12=0;
for l=0:k-1
  f1 = @(x)exp(-x).*x.^(acap*(l+1)-1);
  lowergamma11cap= integral(f1,0,thetacap*theta1cap);
  f1 = @(x)exp(-x).*x.^(acap*(l+1)-1);
  lowergamma12cap= integral(f1,0,thetacap*theta2cap);
  Term1=(lowergamma11cap/((thetacap*theta1cap)^(acap*(l+1))));
  Term2=(lowergamma12cap/((thetacap*theta2cap)^(acap*(l+1))));
  Term11=Term11+nchoosek(k-1,l)*((-1)^l)*Term1;
  Term12=Term12+nchoosek(k-1,l)*((-1)^l)*Term2;
end
Rscap=1-(pcap*k*acap*Term11)-((1-pcap)*k*acap*Term12);
R(t,:)=[Rpcap, Rscap,(Rpcap-Rp)^2,(Rscap-Rs)^2 ];
end
Rp
Rs
format long
disp('Rpcap, Rscap,(Rpcap-Rp)^2,(Rscap-Rs)^2');
mean(R,1)