Chapter - III

A TWO TAILED SECRETARY

PROBLEM WITH MEASURABLE

CHARACTER
CHAPTER III
A TWO TAILED SECRETARY PROBLEM WITH MEASURABLE CHARACTER

3.1 Introduction:

In original secretary problem a known number of units are to be presented one by one in random order for inspection before an observer. The total number of units is N and all N! permutations are equally likely. The unit, which is presented, must be either accepted or rejected. If the unit is accepted the process terminates resulting in the selection of that unit. If the unit is rejected, next unit is called for inspection. If the last unit is inspected it must be accepted. This problem is discussed in section 2.1.

If the observer can not afford to observe all N units, and want to terminate the process quite earlier, he will have to stop after observing at the most, say \( r_2 \) number of units (\( r_2 < N \)). We call such a secretary problem a Two-Tailed Secretary Problem. A TTSP discussed in section 2.3.3, appeared in Ubale (2000).
Looking from the realistic point of view to the problem, it is likely that the observer may have a desire to select the unit matching minimum expected standard in his mind and preferably a better than or the best one. Hence, an observer has some measurable character mind with which each unit can be compared. The analysis of origin secretary problem with measurable character is given by Kane (200).

In this chapter, we analyse TTSP with measurable character observer has to opt for TTSP due to several reasons such as available limited time, saving of cost of inspection and at the same time to the requirement of selecting a unit with minimum expected stand his mind.

3.2 Basic Definitions:

1. **Measurable Characteristic:**

   If instead of assigning ranks to every unit some score associated with them, then these scores associated with the unit known as measurable characteristics.

   The aim of the observer here is to select as far as possible an having measurable character more than 'm' say.
2. Two-Tailed Secretary Problem:

Let there be $N$ units available. First $r_1$ units constitute first which is a non-selection zone. Units from $r_1+1$, $r_1+2$ ... $r_2$ constitute selection zone while units from $r_2+1$, $r_2+2$ ... $N$ is again a non-selection zone called as second tail. Thus, these two tails constitute non-selection zones.

3.3 Stopping Rule:

Let there be $N$ units available. Observe first $r_1$ units while selecting any and note their measurable characters. Let maximum of first $r_1$ units be $M$. Observe $(r_1+1)$, $(r_1+2)$, ... till we get a unit measurable character greater than or equal to $\max(M, m)$. If all the from $r_1+1$, $r_1+2$, ... $r_2-1$ are having measurable character less than $\max(M, m)$ then, select $r_2^{th}$ unit. Here, $r_1 < r_2 < N$. The guideline of choosing $r_1$ and $r_2$ will be now optimality criterion.

Here, selected unit has measurable character greater than $\max(M, m)$ that lies in one of the $(r_1+1)$, $(r_1+2)$...$(r_2-1)$ positions. If selection is made till $(r_2-1)$ position then a unit appearing at $r_2^{th}$ position must be selected, in which case the selected unit may have $X=1, 2, \ldots, N$. 

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3.4 Notations and Terminology: Let

\[ N = \text{Total number of units to be presented.} \]
\[ r_1 = \text{Number of units in the first tail, up to which selection made.} \]
\[ r_2 = \text{the last permissible position for selecting the unit} \ (r_2 < N) \]
\[ m = \text{Minimum required expected standard of the unit, the ob needs.} \]
\[ t = \text{Number of units which have measurable character less than} \]
\[ \text{out of} \ N \text{units.} \]

Thus, we can call this \( t \) as a parameter for given \( N \) units; obviously \( t \) remains unknown during the entire selection process. But, one can estimate \( t \) by different methods. Here \( N, m \) and \( t \) are parameters with which \( r_2 \) can be chosen as per the given situation.

Here, we define two sets,

**Set A**: Set of all units, which have measurable character less than \( m \).

**Set B**: Set of all units, which have measurable character greater than or equal to \( m \).
3.5 Analysis:

Here, process of selection leads to two random variables $X$ and $Y$ given $t$, when we stop after examining $Y$ units, with $X$ as real rank $x$ of the selected unit. $x$ and $y$ are values taken by $X$ and $Y$ respectively.

In the light of above situation i.e. set $A$ and set $B$ there will be possible cases.

Case I: when $0 \leq t \leq r_1$

Case II: when $r_1 < t < N-2$

Case III: when $t = N-2$

Case IV: when $t = N-1$

Case V: when $t = N$

We know that, $t$ is parameter which is a constant for given set of units before observer. This $t$ can be estimated. Estimated value of $t$ can be used to decide which one of the above cases to be considered in analysis in that particular situation.

In this chapter we discuss explicitly the expression of the probability that we stop after examining $y$ units and the selected unit has real rank $x$, for given $r_1$, $r_2$, $N$ and $t$. We denote this probability $P_t(x, y / r_1, r_2, N)$. 

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This explicit expression of $P_t(x, y / r_1, r_2, N)$ is of vital importance because from that we derive exact marginal distribution of $X$ and $Y$, which appears in the next chapter.

Here we note that $X$ and $Y$ are two discrete random variables assuming the values $X = 1, 2, \ldots, N$ and $Y = r_1 + 1, r_1 + 2, \ldots, r_2$.

### 3.6 Joint Distribution of $X$ and $Y$:

**CASE I: ($0 \leq t \leq r_1$)**

Here, first $r_1$ units may contain all the units from set $A$, if $t = r_1$ otherwise, there will be at least one unit of set $B$ in first $r_1$ units. Now, measure of selected unit will always be greater than $M$, which is maximum of first $r_1$ units and hence, it coincides with TTSP analysed in Ubale and Kane (2000).

\[ \therefore \text{Joint probability distribution of } X \text{ and } Y \text{ for given } r_1, r_2, t \text{ and } N \text{ is given by,} \]

\[
P_t(x, y / r_1, r_2, N) = \begin{cases}
\frac{r_1 (N - y)! (x - 1)!}{(y - 1)N! (x - y)!}, & r_1 + 1 \leq y \leq r_2 - 1, y \leq x \leq N \\
\frac{r_1 (N - r_2)!}{N!} \sum_{i=0}^{N-r_2} \frac{(N - 2 - i)!}{(N - r_2 - i)!}, & y = r_2, 1 \leq x \leq N \\
0, & \text{otherwise.}
\end{cases}
\]

...(3.6.1)
The table 3.1 shows the values of $P_t(x, y/ r_1, r_2, N)$, with parameters $N=7$, $r_1 = 2$, $r_2 = 6$ and $t = 0$ which are obtained by using the formula (3.6.1)

Table 3.1

<table>
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<tr>
<th>X</th>
<th>Y</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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It can be easily verified that,

$$\sum \sum P_t(x, y/ r_1, r_2, N) = 1$$

**Remark 3.6.1:** Table for $t = 1$ and $t = 2$ are found to be similar to above table which is obvious because equation (3.6.1) is independent of $t$. 

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CASE II: \( r_1 \leq t < (N-2) \)

Here we discuss expression of probability that we stop after examining \( y \) units and the selected unit has real rank \( x \) for given \( r_1, r_2, N \) and \( t \). We denote probability by \( P_t(x, y/r_1, r_2, N) \).

There are two possibilities in this case,

(i) None of the first \( r_1 \) units belong to set \( B \).

(ii) At least one unit from set \( B \) appears in first \( r_1 \) units.

Here we note that, unit with rank \( X = 1, 2 \ldots t \). will not be selected at \( y = r_1 + 1, r_1 + 2 \ldots r_2 - 1 \) positions since unit with the rank up to \( t \) is an element of set \( A \) which will not satisfy our expected standard in mind.

It is observed that unit with rank \( X = t + 1 \) will be selected at \( y^{th} \) position, \( y = r_1 + 1, r_1 + 2 \ldots r_2 - 1 \) only in case of the first possibility stated above. Hence, we will treat it separately.

Here we stop at \( y^{th} \) position means up to \( (y-1) \) positions all the units are from set \( A \), and \( y^{th} \) unit is with rank \( x \). Number of permutations qualifying these conditions are,

\[ t^{(y-1)}(N - y)!, \ y = r_1 + 1, r_1 + 2 \ldots (t + 1) \ \text{and} \]
\[ x = (t + 1), \ \text{where,} \ a^{(b)} = a(a - 1)\ldots(a - b + 1), b \leq a. \]
For selection of unit with rank \( X = t+2 \ldots N \), we have to consider both the possibilities (i) and (ii) for \( Y = r_1+1, r_1+2, \ldots (t+1) \).

In possibility (i), we observe that, as we stop at \( y \), up to \( (y) \) positions all units are from set A and \( y^{th} \) unit is \( x \). Here, number of permutations qualifying these conditions is,

\[
t^{(y-1)}(N-y)!, \quad y = r_1 + 1, r_1 + 2 \ldots (t + 1) \quad \text{and} \quad x = (t + 2), (t + 3) \ldots N.
\]

Now, to consider the possibility when at least one unit from \( a \) appears in first \( r_1 \) units, let \( j \) be the maximum real rank of first \( r_1 \) units. Then the number of permutations qualifying these conditions is equal to

\[
\sum_{j=t+1}^{x-1} r_1(N-y)! \quad (j-1)^{(y-2)}, \quad x = t + 2, t + 3, \ldots N
\]

\[
y = r_1 + 1, r_1 + 2, \ldots t +
\]

Taking into consideration both possibilities that we stop at \( y \), \( y = r_1+1, r_1+2 \ldots t+1 \) and selected unit has real rank \( x = (t+2), (t+3) \) has probability

\[
t^{(y-1)}(N-y)! + \sum_{j=t+1}^{x-1} r_1(N-y)! \quad (j-1)^{(y-2)}
\]

\[
\frac{N!}{N!}
\]
Again we observe that unit with rank \( X = t + 2 \ldots \) \( N \) can also be selected at positions \( y = t + 2 \ldots r_2 - 1 \) only in possibility (ii) with the probability

\[
\sum_{j=t+1}^{x-1} \frac{r_1 (N - y)! (j - 1)^{(y-2)}}{N!}, \quad y = t + 2, \ldots r_2 - 1 \text{ and } y \leq x.
\]

Again unit with rank \( x = 1, 2, \ldots \) \( N \) can be selected at position \( y = r_2 \) with probability

\[
\frac{r_1 (N - r_2)!}{N!} \left( \sum_{i=0}^{N-r_2} \frac{(N - 2 - i)!}{(N - r_2 - i)!} \right)
\]

\[\therefore\text{ The joint probability of } x \text{ and } y \text{ is given by},\]
\[ P_t(x, y / r_1, r_2, N) = \begin{cases} 
0, & \text{for } x = 1, 2, \ldots, t \\
\frac{t^{(y-1)} (N-y)!}{N!} & \text{for } x = t + 1 \\
\frac{t^{(y-1)} (N-y)! + \sum_{j=t+1}^{x-1} r_1 (N-y)! (j-1)^{(y-2)}}{N!} & \text{for } x = t + 2, t + 3, \ldots, N \\
\frac{\sum_{j=t+1}^{x-1} r_1 (N-y)! (j-1)^{(y-2)}}{N!} & \text{for } x = t + 2, t + 3, \ldots, N \\
\frac{r_1 (N-r_2)!}{N!} \sum_{i=0}^{N-r_2} \frac{(N-2-i)!}{(N-r_2-i)!} & \text{for } y = r_2 \ x = 1, 2, \ldots, N \\
\end{cases} \]

For simplicity we indicate above formulae by following notations in table 3.2
$$P_t(x, y / r_1, r_2, N) = \begin{cases} 
0 & \text{for } x = 1, 2, \ldots, t \\
@ & \text{for } x = t + \\
* & \text{for } x = t + 2, t + 3, \ldots, N \\
# & \text{for } x = t + 2, t + 3, \ldots, r_2 - 1 \\
$ & \text{for } y = r_2, x = 1, 2, \ldots, N 
\end{cases}$$
Table 3.2

Joint probability distribution of X and Y for case -II

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Using formula (3.6.2) we construct table 3.3 showing values of $P_t(x, y/ r_1, r_2, N)$ with parameters $N=7$, $r_1=2$, $r_2=6$. (by running program Annexure-A2.1)

### Table 3.3:

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<td>0.0071</td>
<td>0</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>X=5</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
<td>0.2142</td>
<td>0.0047</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>X=6</td>
<td>0</td>
<td>0</td>
<td>0.0952</td>
<td>0.05</td>
<td>0.0238</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>X=7</td>
<td>0</td>
<td>0</td>
<td>0.1428</td>
<td>0.0976</td>
<td>0.0714</td>
<td>0.0571</td>
<td>0</td>
</tr>
</tbody>
</table>

### Table 3.4:

Joint probabilities of X and Y for $N=7$, $r_1=2$, $r_2=6$ and $t=4$

<table>
<thead>
<tr>
<th></th>
<th>Y=1</th>
<th>Y=2</th>
<th>Y=3</th>
<th>Y=4</th>
<th>Y=5</th>
<th>Y=6</th>
<th>Y=7</th>
</tr>
</thead>
<tbody>
<tr>
<td>X=1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
</tr>
<tr>
<td>X=2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
</tr>
<tr>
<td>X=3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
</tr>
<tr>
<td>X=4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
</tr>
<tr>
<td>X=5</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
<td>0.0285</td>
<td>0.0095</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>X=6</td>
<td>0</td>
<td>0</td>
<td>0.0952</td>
<td>0.0571</td>
<td>0.0285</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>X=7</td>
<td>0</td>
<td>0</td>
<td>0.1428</td>
<td>0.1047</td>
<td>0.0761</td>
<td>0.0571</td>
<td>0</td>
</tr>
</tbody>
</table>
It can be easily verified that,

\[ \sum \sum P_{t=3}(x, y/2, 6, 7) = 1 \]

and

\[ \sum \sum P_{t=4}(x, y/2, 6, 7) = 1 \]

**CASE III: \( t = (N-2) \)**

Here \( t = (N-2) \) means out of \( N \) units presented, \( N-2 \) units have measurable character less than 'm'. Set \( A \) consists of \((N-2)\) units which together consist of two units.

Selection is made for a unit with measurable character greater than \( \max(M, m) \). Because there are \((N-2)\) units with measurable character less than \( m \), selection is not made for unit with rank \( X = 1, 2\ldots N-2 \). i.e. unit with \( X = N \) or \( X = N-1 \) can be selected. Hence,

\[ P_t(x, y/r_1, r_2, N) = 0 \quad \text{for } 0 \leq x \leq (N-2) \text{ and } r_1 + 1 \leq y \leq r_2 - 1. \]

Let us consider selection of unit with rank \( X = (N-1) \)

**a) i)** **Selection of unit with \( X = (N-1) \) at \( r_1 + 1 \leq y \leq r_2 - 1:\)**

This is possible only when unit with rank \( X = N \) will not appear in first tail \((1 \leq y \leq r_1)\). Since it appears in the first tail then we have to stop at \( y = r_2 \).
Here we observe that as we stop at \( y \), elements up to \((y-1)\) positic
are from set A and \( y^{th} \) element is \( X = N-1 \). Therefore, number
permutations qualifying these conditions is,
\[
\binom{N-y}{y-1} (N-y)! = (N-2)\binom{N-y}{y-1} = (N-2)^{y-1} (N-y)! \]

Thus, the corresponding probability is,
\[
P_t(x, y / r_1, r_2, N) = \frac{(N-2)^{y-1}(N-y)!}{N!} \]
\[
= \frac{(N-2)! (N-y)!}{(N-2-y+1)! N!} \]
\[
= \frac{N-y}{N(N-1)} \]

ii) Selection of unit with \( X = (N-1) \) at \( y = r_2 \):

In this case a unit with rank \( X = N \) appears in the first tail or in the second tail.

i) If unit with rank \( X = N \) appears in the first tail:

Probability of this event \( = \frac{r_1(N-2)!}{N!} = \frac{r_1}{N(N-1)} \)

ii) If unit with rank \( X = N \) appears in second tail:
\[
\text{Probability of this event} = \frac{\binom{N-1}{r_1}(N-2)!}{N(N-1)} = \frac{(N-r_2)}{N(N-1)}
\]

Taking in to consideration both the possibilities we get probability that \(X=(N-1)\) selected at \(y = r_2\) as,

\[
P_1(x, y | r_1, r_2, N) = \frac{r_1}{N(N-1)} + \frac{(N-r_2)}{N(N-1)}
\]

\[
= \frac{(N-r_2 + r_1)}{N(N-1)}
\]

b) Selection of the unit with rank \(X = N\) at \(r_1 + 1 \leq y \leq r_2 - 1\).

For selection of \(X = N\) on \(y^{th}\) position we come across possibilities,

i) Second best unit i.e. unit with rank \(X = (N-1)\) belongs to the 1 tail.

Units with rank \(X = N\) will be selected on \(y^{th}\) position and \(X = (N-1)\) is among \(r_1\) units in first tail has probability,

\[
\frac{r_1(N-2)!}{N!} = \frac{r_1}{N(N-1)}
\]

ii) Second best unit \(X = (N-1)\) does not belong to first tail.

Selection of best unit \(X = N\) on \(y^{th}\) position will be done only w\(\ddot{a}\) \(X = (N-1)\) appears after \(y^{th}\) position. i.e. \(X = (N-1)\) occupy position at \(y^{th}\) position.
Here possible permutations will be,

\[ t^{(y-1)} (N-y)! \]

But, here \( t = (N-2) \)

\[ \therefore \quad t^{(y-1)} (N-y)! = (N-2)^{(y-1)} (N-y)! \]

Hence, probability that \( X = N \) is selected at \( y^{th} \) position is given by,

\[
\frac{(N-2)! (N-y)!}{(N-2-y+1)! (N)!} = \frac{(N-y)}{N(N-1)}
\]

Thus, the required probability is,

\[
P_t(x, y / r_1, r_2, N) = \frac{r_1}{N(N-1)} + \frac{(N-y)}{N(N-1)}
\]

\[
= \frac{(N-y + r_1)}{N(N-1)}, \text{ for } r_1 + 1 \leq y \leq r_2 - 1
\]

It is observed that,

\[
P_t(x, y / r_1, r_2, N) = \frac{(N-r_2 + r_1)}{N(N-1)}, \text{ for } y = r_2
\]

c) Selection of unit with rank \( X, \quad 0 \leq x \leq (N-2) \) at \( y = r_2 \)

Here there are two possibilities,

i) Unit with rank \( X = N \) lies in the first tail.

ii) Unit with rank \( X = N \) lies in the second tail.

When \( X = N \) lies in first tail, unit \( X = (N-1) \) can occupy any position in the three zones. Then also unit with rank \( X \) will be selected at \( x = r_2 \) i.e. when the best unit appears in the first tail then wherever
\( X = (N-1) \) lies, we stop at \( Y = r_2 \) and the selected unit has rank \( X \).

Best unit can occupy any of \( r_1 \) positions. Excluding best unit and selected unit at \( Y = r_2 \), all remaining \((N-2)\) units can occupy any other positions.

Probability of this event is,

\[
\frac{r_1(N-2)!}{N!} = \frac{r_1}{N(N-1)}
\]

In second possibility, unit \( X = N \) lies in second tail then second best unit \( X=(N-1) \) cannot occupy positions in between \((r_1 +1)\) to \((r_2 -1)\), otherwise it will be selected.

When best unit \( X = N \) lies in second tail, there are two possibilities for \( X = N-1 \);

i) Unit with rank \( X = N-1 \) lies in the first tail.

ii) Unit with rank \( X = N-1 \) lies in the second tail.

Consider,

i) Unit \( X = N-1 \) lies in first tail and \( X = N \) is in second tail. Unit \( X = N-1 \) can occupy \( r_1 \) positions while \( X = N \) can occupy \((N-r_2)\) positions. Except best unit, second best unit and the selected unit at \( r_2^{th} \) position, all remaining \((N-3)\) units can occupy any of positions. Number of permutations qualifying this condition is,

\[
r_1 (N-r_2) (N-3)!
\]
Hence, corresponding probability is,

\[
\frac{r_1(N-r_2)(N-3)!}{N!} = \frac{r_1(N-r_2)}{N(N-1)(N-2)}
\]

ii) Unit \(X = N\) and \(X = (N-1)\) both are in second tail.

Both the best and the second best units are in the second tail, which can occupy \((N- r_2)\) \((N-r_2 -1)\) positions. Except these two units and selected one, all units can occupy any positions giving us probability,

\[
\frac{(N-r_2)(N-r_2-1)(N-3)!}{N!} = \frac{(N-r_2)(N-r_2-1)}{N(N-1)(N-2)}
\]

Any of above three possibilities can happen which gives probability that unit will be selected at \(r_2^{th}\) position, as follows,

\[
P_t(x,y / r_1, r_2, N) = \frac{r_1}{N(N-1)} + \frac{r_1(N-r_2)}{N(N-1)(N-2)} + \frac{(N-r_2)(N-r_2-1)}{N(N-1)(N-2)}
\]

\[
= \frac{r_1}{N(N-1)} + \frac{(N-r_2)}{N(N-1)(N-2)} \left[ r_1 + (N-r_2-1) \right]
\]

\[
= \frac{1}{N(N-1)} \left[ r_1 + \frac{(N-r_2)(N+r_1-r_2-1)}{(N-2)} \right]
\]
Hence, we get joint distribution of $X$ and $Y$ i.e. $P_t(x, y/ r_1, r_2, N)$
when $t = (N-2)$ as,

$$P_t(x, y / r_1, r_2, N) = \begin{cases} \frac{(N - y)}{N(N - 1)} & \text{for } X = N - 1, r_1 + 1 \leq y \leq r_2 - 1 \\ \frac{(N - y + r_1)}{N(N - 1)} & \text{for } X = N, r_1 + 1 \leq y \leq r_2 - 1 \\ \frac{(N - r_2 + r_1)}{N(N - 1)} & \text{for } X = N - 1, N \text{ and } y = r_2 \\ \frac{1}{N(N - 1)} \left[ r_1 + \frac{(N - r_2)(N + r_1 - r_2 - 1)}{(N - 2)} \right] & \text{for } y = r_2, 1 \leq x \leq N - 2 \\ 0 & \text{Otherwise} \end{cases}$$

...(3.6.3)

Table 3.5 below shows $P_{t=(N-2)}(x, y/ r_1, r_2, N)$ for case III.
Table 3.5:

Joint probabilities of X and Y for t = (N-2)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>( r_1 )</th>
<th>( r_1^+1 )</th>
<th>( r_2-1 )</th>
<th>( r_2 )</th>
<th>( r_2^+1 )</th>
<th>...</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
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<td>0</td>
<td>0</td>
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<td>*</td>
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<td>0</td>
</tr>
<tr>
<td>( r_1^+1 )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( r_2-1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>( r_2^+1 )</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>N-2</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>*</td>
<td>0</td>
<td>...</td>
<td>0</td>
</tr>
<tr>
<td>N-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>@</td>
<td>...</td>
<td>@</td>
<td>$</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>N</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>#</td>
<td>...</td>
<td>#</td>
<td>$</td>
<td>0</td>
<td>...</td>
</tr>
</tbody>
</table>

* indicates probability \( = \frac{1}{N(N-1)}\left[r_1 + \frac{(N-r_2)(N+r_1-r_2-1)}{(N-2)}\right] \)
@ indicates probability \[ \frac{(N - y)}{N(N - 1)} \]

# indicates probability \[ \frac{(N - y + r_1)}{N(N - 1)} \]

$ indicates probability \[ \frac{(N - r_2 + r_1)}{N(N - 1)} \]

Using (3.6.3) and running program annexure A_{2.1}, we observe the values in the following table (3.6),

Table 3.6:

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.0571</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0.0952</td>
<td>0.0714</td>
<td>0.0476</td>
<td>0.0714</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0.1428</td>
<td>0.1190</td>
<td>0.0952</td>
<td>0.0714</td>
<td>0</td>
</tr>
</tbody>
</table>

It is found that,

\[ \sum \sum_{x, y} p_{t=5}(x, y / 2, 6, 7) = 0.999999999 \]
CASE IV: \( t = (N-1) \)

Here \( t = (N-1) \) means \((N-1)\) units out of \(N\) have measurable character less than 'm'. There is only one unit with measurable character greater than 'm'. Set \( A \) consists of \((N-1)\) units while set \( B \) is a single unit.

To find joint distribution of \( X \) and \( Y \) we consider two cases:

i) The best unit, unit \( X = N \), appears in any of the two tails.

ii) The best unit \( X = N \) appears in position \( y = r_1 + 1, r_1 + 2, \ldots, r_2 - 1 \).

Case I: The best unit appears in one of the two tails.

a) It appears in the first tail: In this situation the best unit \( X = N \) appear in any of the first \( r_1 \) positions. Because the best unit has already been come across, observer will not stop at \( Y = r_1 + 1, \ldots, r_2 - 1 \) positions. Hence he will stop at \( Y = r_2 \) and real rank of selected unit can be \( X = 1, 2, \ldots, (N-1) \).

Therefore,

\[
P_t(x, y / r_1, r_2, N) = \frac{r_1(N-2)!}{N!} = \frac{r_1}{N(N-1)}
\]

b) Unit \( X = N \), the best unit appears in second tail: The best unit, which
in the second tail, can occupy \((N- r_2)\) possible positions. Here, \( t = (N-1)\)

55
means all units excluding best unit belongs to set A and hence, will not be selected at \( Y = r_1+1, r_1+2 \ldots r_2-1 \), observer has to stop at position \( Y = r_2 \).

Therefore, the probability of this event is,

\[
\frac{(N-r_2)(N-2)!}{N!}
\]

\[
= \frac{N-r_2}{N(N-1)}
\]

Thus, finally the probability that a unit will be selected at \( Y = r \) when best unit appears in one of the two tails is given by,

\[
\frac{r_1}{N(N-1)} + \frac{(N-r_2)}{N(N-1)}
\]

\[
= \frac{(N-r_2+r_1)}{N(N-1)}
\]

**Case (ii):** The best unit appears at position \( Y = r_1+1, r_1+2 \ldots r_2-1 \).

Here none of the elements will have measurable character greater than 'm'. There is only one unit whose measurable character is greater than 'm'; hence unit, which will be selected, is the best unit. Further, selection can be made at any of \( r_1+1, r_1+2 \ldots r_2-1 \) positions. Hence, probability of stopping at \( Y = r_1+1, r_1+2 \ldots r_2 \) with the best unit is \((1/N)\).
\[ \therefore p_t(x,y/r_1,r_2,N) = \frac{1}{N}, \text{ for } y = r_1 + 1, r_1 + 2, \ldots r_2 - 1. \]

Hence, joint distribution of \( X \) and \( Y \) when \( t = (N-1) \) is given by,

\[
P_{t=N-1}(x,y/r_1,r_2,N) = \begin{cases} 
\frac{(N + r_1 - r_2)}{N(N - 1)} & \text{for } x = 1, 2, \ldots N - 1 \\
\frac{1}{N} & y = r_2 \text{ for } x = N \text{ and } \\
0 & y = r_1 + 1, r_1 + 2, \ldots r_2 \text{ otherwise} 
\end{cases}
\]

It can be easily verified that,

\[
\sum\sum_{x,y} p_t(x,y/r_1,r_2,N) = 1
\]

as follows;
\[ \sum_{x} \sum_{y} P_t(x, y / r_1, r_2, N) = \frac{(N + r_1 - r_2)}{N(N - 1)} (N - 1) + \frac{1}{N} (r_2 - r_1 - 1 + 1) \]

\[ = \frac{N}{N} + \left( \frac{r_1 - r_2}{N} \right) - \left( \frac{r_1 - r_2}{N} \right) \]

\[ = 1. \]
Table 3.7

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>r₁</th>
<th>r₁+1</th>
<th>r₁+2</th>
<th>...</th>
<th>r₂-1</th>
<th>r₂</th>
<th>r₂+1</th>
<th>...</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(N-r₂+r₁)</td>
<td>0</td>
<td>0</td>
<td>N</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(N-r₂+r₁)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
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<td></td>
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<td></td>
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</tr>
<tr>
<td>r₁</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(N-r₂+r₁)</td>
<td>0</td>
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</tr>
<tr>
<td>...</td>
<td></td>
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<tr>
<td>r₂</td>
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<td>0</td>
<td>0</td>
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<td>0</td>
<td>(N-r₂+r₁)</td>
<td>0</td>
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<tr>
<td>...</td>
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<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>N-1</td>
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<td>0</td>
<td>0</td>
<td>(1/N)</td>
<td>(1/N)</td>
<td></td>
<td>(1/N)</td>
<td>(N-r₂+r₁)</td>
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</tr>
<tr>
<td>N</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(1/N)</td>
<td>(1/N)</td>
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<td>(1/N)</td>
<td></td>
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</tr>
</tbody>
</table>

After running computer program in annexure A₂.₁ we get,
Table 3.8

Probabilities, $P_{t=N-1}(x, y/r_1, r_2, N)$ for $N = 7$, $r_1 = 2$, $r_2 = 6$, $t = 6 = (N-1)$

<table>
<thead>
<tr>
<th>Y</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/14</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/14</td>
</tr>
<tr>
<td>3</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/14</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>1/14</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1/14</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
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<td>1/7</td>
<td>1/7</td>
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<td>1/7</td>
<td>0</td>
</tr>
</tbody>
</table>

\[
\sum \sum_{x \ y} P_{t=6}(x, y/2, 6, 7) = \left( \frac{6}{14} \right) + \left( \frac{4}{7} \right) = \left( \frac{14}{14} \right) = 1.
\]

**CASE V: t = N**

If $t = N$, there are $N$ units in set A and no unit in set B i.e. none of the units is having measurable character greater than 'm' in the entire set of $N$ units; forcing us to stop at $y = r_2^{th}$ position only. In TTSP selecting the unit with real rank $X$, $X = 1, 2... N$, the joint probability distribution of $x$ and $y$ is;
\[ P_t(x, y/r_1, r_2, N) = (1/N). t = N \text{ at } Y = r_2, X = 1, 2, \ldots N. \quad \ldots (3.6.5) \]

Following table shows probability \( P_{t=N}(x, y/r_1, r_2, N) \) in case V.

**Table 3.9**

**Joint probability distribution of X and Y for t=N**

<table>
<thead>
<tr>
<th>Y</th>
<th>X</th>
<th>1</th>
<th>2</th>
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<th>\ldots</th>
<th>\ldots</th>
<th>\ldots</th>
<th>r_1</th>
<th>\ldots</th>
<th>r_2,1</th>
<th>r_2</th>
<th>r_2+1</th>
<th>\ldots</th>
<th>N</th>
</tr>
</thead>
<tbody>
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<td>1</td>
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<td>\ldots</td>
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<tr>
<td>r_2</td>
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<td>1/N</td>
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<td>0</td>
<td>\ldots</td>
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<td>0</td>
</tr>
</tbody>
</table>

Here, also we observe that,

\[
\sum_{x} \sum_{y} P_t(x, y/r_1, r_2, N) = 1
\]

******

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