CHAPTER IV
T₁-CLASS OF LINEAR ESTIMATORS

4.0 Summary.

When elements of a finite universe are sampled without replacement with varying probabilities of selection at each draw Horvitz and Thompson have pointed out the existence of three distinct classes of linear estimators. Following the logic of Horvitz and Thompson, Koop (1963) formulated the seven classes of linear estimators. Prabhu- Ajgaonkar (1962) studied the T₁-class in detail and introduced the criteria of empty and non-empty classes. In this chapter we study a sampling procedure consisting of selecting first \( m \) elements with varying probabilities with replacement, the \( m+1 \) th element with varying probability without replacement and the subsequent \( n-m-1 \) elements with equal probability without replacement. The estimators engendered by the T₁-class of linear estimators are considered. It is demonstrated that for this sampling procedure the T₁-class is non-empty and interestingly enough that there exists the minimum-variance-linear unbiased estimator.
4.1. **Introduction.**

Sample surveys often encountered in practice, employ equal probabilities for selection of sampling elements and linear estimators for estimating the values of the characteristics under consideration. With the emergence of novel technique introduced for the first time by Hansen and Hurwitz (1943) of incorporating the available ancillary information into a sampling procedure and eventually drawing the population elements with varying probabilities, the problem of what to use for estimator of the population characteristics under consideration, came into prominence. Hansen and Hurwitz (1943) presented an unbiased estimator when only one population element was drawn. Midzuno (1950) developed an unbiased estimator when a combination of population elements was sampled with varying probabilities without replacement.

Horvitz and Thompson (1952) considered various aspects of the problem and discovered the existence of three distinct classes of linear estimators for providing a sample appraisal of the population total. Subsequently these classes of linear estimators have been scrutinized by Horvitz and Thompson (1952), Koop (1963) and Prabhu Ajgaonkar (1962, 1965, 1967 a,b,c, (1970 a,b and 1971 a). For Horvitz and Thompson's \( T_1 \)-class of linear estimators the following results have been established:
i) For some sampling procedures the $T_1$-class does not possess an unbiased estimator. Subsequently the class is termed an empty class for the given sampling procedure.

ii) The $T_1$-class is non-empty for Ikeda-Sen's sampling procedure. Further the criteria of minimum variance furnishes a unique and serviceable estimator.

iii) For Midzuno's sampling procedure the $T_1$-class is non-empty. However the criteria of minimum variance does not render a unique and serviceable estimator. Moreover the criterion of necessary best estimator proposed by Prabhu-Ajgaonkar (1965) does not render a unique and serviceable estimator.

4.2. General discussion of $T_1$-class Estimators.

Let $Y_1$ denote the value of the characteristic $Y$ under consideration for the $i$th unit of the population for $i=1,2,\ldots,N$. It is required to estimate the population total

$$Y = \sum_{i=1}^{N} Y_i, \quad i=1,2,\ldots,N.$$  \hspace{1cm} (4.2.1)

For this purpose $n$ draws are made randomly with varying probabilities. Let $p_{ir}$ denote the probability of drawing the $i$th element of the population at the $r$th draw. Let $P$
denote the probability matrix where \( r \)th column represents the probabilities of including \( N \) units at the \( r \)th draw for \( r=1,2,\ldots,n \). Thus

\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1r} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2r} & \cdots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p_{i1} & p_{i2} & \cdots & p_{ir} & \cdots & p_{in} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p_{N1} & p_{N2} & \cdots & p_{Nr} & \cdots & p_{Nn}
\end{bmatrix}
\]

Let \( Y_r \) denote the outcome at the \( r \)th draw. Following Horvitz and Thompson (1952) we can represent \( T_1 \)-class estimator, symbolically as

\[
T_1 = \sum_{r=1}^{n} \alpha_r Y_r
\]

where \( \alpha_r \) for \( r=1,2,\ldots,n \) is the weight associated with the element turned up at the \( r \)th draw in the sample. Thus all weights \((\alpha_1,\alpha_2,\ldots,\alpha_n)\) occurring in the estimator are fixed in advance. Now
\[ E(T_1) = \sum_{r=1}^{n} \alpha_r E(y_r) \]

\[ = \sum_{r=1}^{n} \alpha_r \sum_{i=1}^{N} Y_i p_{ir} \]

\[ = \sum_{i=1}^{N} Y_i \sum_{r=1}^{n} \alpha_r p_{ir} \]

(4.2.3)

In order that \( T_1 \) to be unbiased estimator of the population total \( Y \) whatever may be \( Y_i \) for \( i=1,2,\ldots,N \); we must have

\[ \sum_{r=1}^{n} \alpha_r p_{ir} = 1, \quad i=1,2,\ldots,N \]

or in matrix notation,

\[
\begin{bmatrix}
  p_{11} & p_{12} & \cdots & p_{1n} \\
  p_{21} & p_{22} & \cdots & p_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  p_{N1} & p_{N2} & \cdots & p_{Nn}
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \vdots \\
  \alpha_n
\end{bmatrix}
= 
\begin{bmatrix}
  1 \\
  \alpha_2 \\
  \vdots \\
  1
\end{bmatrix}
\]

(4.2.4)

This system of linear equations will have a consistent solution in \( \alpha_1, \alpha_2, \ldots, \alpha_n \) if the rank of the coefficient matrix
\[
P = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1n} \\
p_{21} & p_{22} & \cdots & p_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn}
\end{bmatrix}
\]

is the same as the rank of the augmented matrix

\[
P^* = \begin{bmatrix}
p_{11} & p_{12} & \cdots & p_{1n} & 1 \\
p_{21} & p_{22} & \cdots & p_{2n} & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
p_{n1} & p_{n2} & \cdots & p_{nn} & 1
\end{bmatrix}
\]

It is easy to see that the \( T_1 \)-class estimators do not exist for all probability systems \( P \). For example if there is sampling with replacement and if probability of selecting ith unit at the rth draw is \( p_i \) for all \( r \) then the rank of the augmented matrix is not the same as that of the coefficient matrix. Thus the equations for the condition of unbiasedness are inconsistent and there is therefore no \( T_1 \)-class estimators for this probability system. However if we have \( p_{ir} = p_r \) then \( T_1 \)-class estimators exist. Since \( \sum_{i=1}^{N} p_r = Np_r = 1 \), therefore \( p_{ir} = 1/N \) for all \( i, r \). Thus this is the case of simple random sampling with or without replacement.
4.3. **Sampling Procedure.**

First, $m$ draws are carried out with varying probabilities with replacement, the $m+1$ th draw with varying probabilities without replacement and the remaining $n-m-1$ draws with simple random sampling without replacement.

Let $p_{ij}$ $(i=1,2,...,N; j=1,2,...,n)$ be the probability of selecting the $i$th population element at the $j$th draw and

$$P = (p_{ij})$$

be the probability matrix associated with this sampling procedure.

Let $p_{ij} = p_i$ for $i=1,2,...,N$ and $j=1,2,...,m$ for the next $n-m$ draws ($n > m$) the Ikeda-Sen's sampling procedure has been followed,

$$p_{ij} = \begin{cases} p_i & \text{for } j=m+1, \text{ and } i=1,2,...,N \\ \frac{1-p_i}{N-1} & \text{for } j=m+2,m+3,...,n \text{ and } i=1,2,...,N. \end{cases}$$

The probability matrix for the above scheme is given by

$$P = \begin{bmatrix} p_1 & \ldots & p_1 & c_1 & \ldots & c_1 \\ p_2 & \ldots & p_2 & c_2 & \ldots & c_2 \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ \vdots & & \vdots & \ddots & \vdots & \vdots \\ p_N & \ldots & p_N & c_N & \ldots & c_N \end{bmatrix}$$

where $c_i = \frac{1-p_i}{N-1}$, $i=1,2,...,N$. 
Since column operations do not change the rank of matrix, subtracting the first column from the immediate remaining \( m \) columns and \( m+2 \) th column from the remaining subsequent columns, we get

\[
\begin{bmatrix}
p_1 & 0 & \cdots & 0 & c_1 & 0 & \cdots & 0 \\
p_2 & 0 & \cdots & 0 & c_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
p_N & 0 & \cdots & 0 & c_N & 0 & \cdots & 0 \\
\end{bmatrix}
\]

Here it is noted that since \( p_1 \neq p_2 \) the rank of the above matrix is two.

Now consider the rank of the augmented matrix of order three

\[
\begin{bmatrix}
p_1 & c_1 & 1 \\
p_2 & c_2 & 1 \\
p_3 & c_3 & 1 \\
\end{bmatrix}
= p_1 (c_2 - c_3) - c_1 (p_2 - p_3) + 1 (p_2 c_3 - p_3 c_2)
\]

\[
= p_1 \left( \frac{1-p_2}{N-1} - \frac{1-p_3}{N-1} \right) - \frac{1-p_1}{N-1} (p_2 - p_3) + 1 \left( p_2 \frac{1-p_3}{N-1} - p_3 \frac{1-p_2}{N-1} \right)
\]

\[
= \frac{1}{N-1} \left( p_1 p_2 - p_1 + p_1 p_3 - p_2 + p_1 p_2 - p_1 p_3 + p_2 p_3 - p_3 + p_2 p_3 \right)
\]

\[
= 0
\]
Thus the rank of the augmented matrix is also two. Since the rank of the augmented matrix is the same as that of the probability matrix, we have for this sampling procedure the $T_1$-class is nonempty. Thus the system of equations given in (4.2.4) becomes

\[
\begin{bmatrix}
  p_1 & c_1 \\
  p_2 & c_2 \\
  \vdots & \vdots \\
  p_N & c_N
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \vdots \\
  \alpha_n
\end{bmatrix}
= 
\begin{bmatrix}
  1 \\
  1 \\
  \vdots \\
  1
\end{bmatrix}
\]

4.4 Unbiased Estimators.

Since the rank of the coefficient matrix is two, there are two effective equations of unbiasedness.

Subtracting from the $i$th row $i > 1$, $p_i/p_1$ times first row, we get,

\[
\begin{bmatrix}
  p_1 & c_1 \\
  0 & \delta_2 \\
  \vdots & \vdots \\
  0 & \delta_N
\end{bmatrix}
\begin{bmatrix}
  \alpha_1 \\
  \alpha_2 \\
  \vdots \\
  \alpha_n
\end{bmatrix}
= 
\begin{bmatrix}
  1 \\
  1 - \frac{p_2}{p_1} \\
  \vdots \\
  1 - \frac{p_N}{p_1}
\end{bmatrix}
\]

where $\delta_i = c_i - \frac{p_i c_1}{p_1}$, $i = 1, 2, \ldots, N$
Since \( p_1 \neq p_2 \), \( \delta_2 \) is not equal to zero. Now subtracting from the \( i \)th row \( i > 2 \), \( \delta_1/\delta_2 \) times second row, we obtain

\[
\begin{bmatrix}
p_1 & \cdots & p_1 & c_1 & \cdots & c_1 \\
0 & \cdots & 0 & \delta_2 & \cdots & \delta_2 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\vdots \\
\vdots \\
\alpha_n \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
1 - \frac{p_2}{p_1} \\
0 \\
\vdots \\
\vdots \\
0 \\
\end{bmatrix}
\]

Thus the effective conditions on \( \alpha_i \)'s are

\[
p_1(\alpha_1+\alpha_2^{m+1}+\cdots+\alpha_{m+1}) + c_1(\alpha_{m+2}^{m+2}+\cdots+\alpha_n) = 1 \quad (4.4.1)
\]

and

\[
\delta_2(\alpha_{m+2}^{m+2} + \alpha_{m+3}^{m+3} + \cdots + \alpha_n) = 1 - \frac{p_2}{p_1} \quad (4.4.2)
\]

where

\[
\delta_2 = c_2 - \frac{p_2}{p_1} c_1 = \frac{1-p_2}{N-1} - \frac{p_2(1-p_1)}{p_1(N-1)} = \frac{p_1-p_2}{(N-1)p_1}
\]

Therefore from equation (4.4.2) we have

\[
\frac{p_1-p_2}{(N-1)p_1} (\alpha_{m+2} + \alpha_{m+3} + \cdots + \alpha_n) = \frac{p_1-p_2}{p_1}
\]

i.e.

\[
\alpha_{m+2} + \alpha_{m+3} + \cdots + \alpha_n = N-1 \quad (4.4.3)
\]
Substituting this value in equation (4.4.1), we have

\[ p_1(\alpha_1 + \alpha_2 + \ldots + \alpha_{m+1}) + c_1(N-1) = 1 \]
\[ p_1(\alpha_1 + \alpha_2 + \ldots + \alpha_{m+1}) = 1 - \frac{(1-p_1)(N-1)}{N-1} \]

i.e.,
\[ \alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} = 1 \]

Thus the effective conditions on \( \alpha_r \)'s are reduced to

\[
\begin{align*}
\alpha_1 + \alpha_2 + \ldots + \alpha_{m+1} &= 1 \\
\alpha_{m+2} + \alpha_{m+3} + \ldots + \alpha_n &= N-1
\end{align*}
\]

(4.4.4)

4.5 Determination of Unique Estimator.

Now consider,

\[ \text{Var}(t_1) = \text{Var} \left( \sum_{r=1}^{n} \alpha_r y_r \right) \]
\[ = \sum_{r=1}^{n} \alpha_r^2 \text{Var}(y_r) + \sum_{r \neq s=1}^{n} \alpha_r \alpha_s \text{Cov}(y_r y_s) \]
\[ = \sum_{r=1}^{m+1} \alpha_r^2 \text{Var}(y_r) + \sum_{r=m+2}^{n} \alpha_r^2 \text{Var}(y_r) \]
\[ + 2 \alpha_1 \sum_{r=m+2}^{n} \alpha_r \text{Cov}(y_1, y_r) + 2 \alpha_2 \sum_{r=m+2}^{n} \alpha_r \text{Cov}(y_2, y_r) \]
\[ + \ldots \ldots \ldots + 2 \alpha_{m+1} \sum_{r=m+2}^{n} \alpha_r \text{Cov}(y_{m+1}, y_r) \]
\[ + \sum_{r \neq s=1}^{m+1} \alpha_r \alpha_s \text{Cov}(y_r y_s) + \sum_{r+s=m+2}^{n} \alpha_r \alpha_s (y_r y_s) \]
Now let

\[
\text{Var}(y_r) = 6_1^2 \quad \text{for } r = 1, 2, \ldots, m+1
\]

\[
\text{Var}(y_r) = 6_2^2 \quad \text{for } r = m+2, \ldots, n
\]

\[
\text{Cov}(y_r, y_s) = 6_{12} \quad \text{for } r \neq s = 1, 2, \ldots, m+1
\]

\[
\text{Cov}(y_r, y_s) = 6_{13} \quad \text{for } r \neq s = m+2, \ldots, n
\]

\[
\text{Cov}(y_r, y_s) = 6_{14} \quad \text{for } r = m+2, \ldots, n \quad s = 1, 2, \ldots, m+1
\]

Therefore

\[
\text{Var}(t_1) = 6_1^n \sum_{r=1}^{m+1} \alpha_r^2 + 6_2^n \sum_{r=m+2}^{n} \alpha_r^2 + 2(\alpha_1 + \ldots + \alpha_{m+1})\sum_{r=m+2}^{n} \alpha_r + \sum_{r=s=1}^{m+1} \alpha_r \alpha_s + 26_{14}(N-1)
\]

\[
= 6_1^n \sum_{r=1}^{m+1} \alpha_r^2 + 6_2^n \sum_{r=m+2}^{n} \alpha_r^2 + \sum_{r=s=m+2}^{n} \alpha_r \alpha_s + 26_{14}(N-1)
\]

\[
+ 6_{12} \left\{ \left( \sum_{r=1}^{m+1} \alpha_r^2 - \sum_{r=1}^{m+1} \alpha_r^2 \right) \right\} + 6_{13} \left\{ \left( \sum_{r=m+2}^{n} \alpha_r^2 - \sum_{r=m+2}^{n} \alpha_r^2 \right) \right\}
\]

\[
= (6_1^2 - 6_{12}) \sum_{r=1}^{m+1} \alpha_r^2 + (6_2^2 - 6_{13}) \sum_{r=m+2}^{n} \alpha_r^2 + 26_{14}(N-1) + 6_{12} + 6_{13}(N-1)^2
\]
Now we select that set of \( \alpha_r \)'s which minimizes \( \text{Var}(t_1) \).

For this purpose let

\[
Q = \text{Var}(t_1) - 2 \lambda \sum_{r=1}^{m+1} \alpha_r - 2 \mu \sum_{\alpha=m+2}^{n} \alpha_r
\]  

\( (4.5.1) \)

where \( \lambda \) and \( \mu \) are Lagrange's undetermined multipliers.

Now differentiating equation (4.5.1) with respect to \( \alpha_r \)'s for \( r=1,2,\ldots,m+1 \) and \( r = m+2,m+3,\ldots,n \); we have

\[
\frac{\delta Q}{\delta \alpha_r} = 2 (\theta_1^2 - \theta_{12}) \alpha_r - 2 \lambda \text{ for } r=1,2,\ldots,m+1
\]  

\( (4.5.2) \)

and

\[
\frac{\delta Q}{\delta \alpha_r} = 2 (\theta_2^2 - \theta_{13}) \alpha_r - 2 \mu \text{ for } r = m+2,\ldots,n
\]  

\( (4.5.3) \)

Now equating equation (4.5.2) to zero, we have

\[
\alpha_r = \frac{\lambda}{\theta_1^2 - \theta_{12}}
\]

Summing over \( r=1,2,\ldots,m+1 \)

\[
\sum_{r=1}^{m+1} \alpha_r = \frac{(m+1)\lambda}{\theta_1^2 - \theta_{12}}
\]

\[
\lambda = \frac{\theta_1^2 - \theta_{12}}{m+1}
\]

\[
\alpha_r = \frac{\theta_1^2 - \theta_{12}}{m+1} = \frac{1}{(\theta_1^2 - \theta_{12})(m+1)}
\]

\[
\alpha_r = \frac{1}{m+1} \text{ for } r=1,2,\ldots,m+1.
\]  

\( (4.5.4) \)
Now equating equation (4.5.3) to zero, we have

\[ \alpha_r = \frac{\mu}{\sigma_r^2} \]

Summing over \( r = m+2, m+3, \ldots, n \)

\[ \sum_{r=m+2}^{n} \alpha_r = \frac{(n-m-1) \mu}{\sigma_r^2 - \sigma_{13}^2} \]

i.e.,

\[ \mu = \frac{(N-1)(\sigma_r^2 - \sigma_{13}^2)}{n-m-1} \]

\[ \alpha_r = \frac{(N-1)(\sigma_r^2 - \sigma_{13}^2)}{(n-m-1)(\sigma_r^2 - \sigma_{13}^2)} \]

\[ \alpha_r = \frac{N-1}{n-m-1} \text{ for } r = m+2, \ldots, n \quad (4.5.5) \]

Substituting these values of \( \alpha_r \)'s in

\[ t_1 = \sum_{r=1}^{n} \alpha_r y_r \]

we have the best minimum variance unbiased estimator for the \( T_1 \)-class for the above sampling procedure and is given by

\[ t_1 = \frac{1}{m+1} (y_1 + y_2 + \ldots + y_{m+1}) + \frac{N-1}{n-m-1} \sum_{r=m+2}^{n} y_r \quad (4.5.6) \]

and the variance of this estimator is
\[ \text{Var}(t_1) = \frac{1}{m+1} (6_1^2 - 6_{12}) + \frac{(N-1)^2}{n-m-1} (6_2^2 - 6_{13}) \]

\[ + 2(N-1) 6_{14} + 6_{12} 6_1 + (N-1)^2 6_{13} \quad (4.5.7) \]

It may be noted that if \( m = 0 \) the equation (4.5.6) reduces to

\[ t_1 = y_1 + \frac{N-1}{n-N} \sum_{r=2}^{n} y_r \]

which is the best estimator in Horvitz and Thompson's \( T_1 \)-class of linear estimators when Ikeda-Sen's sampling procedure is adopted. This estimator has been derived by Prabhu-Ajgaonkar (1965).

Now

\[ \text{Var}(y_r) = \sum_{i=1}^{N} p_i (y_i - \bar{y}_{N_1})^2, \quad r = 1, 2, \ldots, m+1 \]

\[ = 6_1^2. \]

\[ = \sum_{i=1}^{N} p_i (1-p_i) y_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{N} y_i y_j p_i p_j \quad (4.5.8) \]

where \( \bar{y}_{N_1} = \sum_{i=1}^{N} y_i p_i \).
\[ \text{Var}(y_r) = \sum_{i=1}^{N} c_i (Y_i - \bar{Y}_N)^2 \] for \( r = m+2, \ldots, n \)

\[ = \frac{2}{62} \]

\[ = \frac{N}{\sum_{i=1}^{N} (1-p_i)(N-2+p_i)} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{(1-p_i)(1-p_j)}{(N-1)^2} Y_i Y_j \right) \]

where \( \bar{Y}_N = \sum_{i=1}^{N} c_i Y_i \cdot \)

Now we will calculate \( \text{Cov}(y_r, y_s) \) for \( r = 1, 2, \ldots, m+1 \) and \( s = m+2, \ldots, n \)

\[ \text{E}(y_1, y_s) = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{Y_i Y_j}{N-1} \{ \text{probability that } Y_1 \text{ and } Y_j \text{ are selected at first and } s \text{ th draw where } s > m+1 \} \]

\[ = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{Y_i Y_j}{N-1} \frac{p_i}{p_i} \]

\[ = \frac{1}{N-1} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{Y_i Y_j}{N-1} p_i \]

\[ = \frac{1}{N-1} \left\{ \left( \sum_{i=1}^{N} Y_i p_i \right) \left( \sum_{i=1}^{N} Y_i - \sum_{i=1}^{N} p_i Y_i \right) \right\} \]

\[ = \frac{1}{N-1} \left\{ \sum_{i=1}^{N} \bar{Y}_N \bar{Y}_N - \sum_{i=1}^{N} p_i Y_i^2 \right\} \]

(4.5.9)

where \( \bar{Y}_N = \frac{1}{N} \sum_{i=1}^{N} Y_i \cdot \)
We also require

\[
E(Y_r)E(Y_s) = \sum_{j=1}^{N} p_j Y_j \sum_{j=1}^{N} c_j Y_j \quad \text{for } s > m+1
\]

\[
= \bar{Y}_{N_1} \left( \sum_{j=1}^{N} \frac{1-p_j}{N-1} Y_j \right)
\]

\[
= \bar{Y}_{N_1} \left( \sum_{j=1}^{N} \frac{Y_j}{N-1} - \sum_{j=1}^{N} p_j Y_j \right)
\]

\[
= \frac{1}{N-1} \left\{ N \bar{Y}_{N_1} \bar{Y}_{N_1} - \frac{\bar{Y}_{N_1}^2}{N-1} \right\}
\]

From equations (4.5.9) and (4.5.10) we have

\[
\text{Cov}(y_r, y_s) = \frac{N \bar{Y}_{N_1}^2 - \sum_{i=1}^{N} \frac{2}{N} \sum_{i=1}^{N} \frac{2}{N} \bar{Y}_{N_1} + \bar{Y}_{N_1}^2}{N-1}
\]

\[
= - \frac{1}{N-1} \left\{ \sum_{i=1}^{N} p_i Y_i - \left( \sum_{i=1}^{N} p_i Y_i \right)^2 \right\}
\]

\[
= - \frac{1}{N-1} \sum_{i=1}^{N} p_i (Y_i - \bar{Y}_{N_1})^2
\]

\[
= - \frac{6_1^2}{(N-1)} = 6_{14}
\]

which is true for all \(r=1,2,\ldots,m+1\) and \(s=m+2,\ldots,n\). Now we will calculate \(\text{Cov}(y_r, y_s)\) where \(r \neq s = 1,2,\ldots,m+1\). For this we require
\[ E(y_{r} y_{s}) = \sum_{i=1}^{N} \sum_{j=1}^{N} Y_i Y_j \{ \begin{array}{l}
\text{Probability that } Y_i \text{ and } Y_j \text{ are} \\
\text{selected at } r \text{th and } s \text{th draws} \\
r, s = 1, 2, \ldots, m+1 
\end{array} \]

\[ = \sum_{i=1}^{N} \sum_{j=1}^{N} Y_i Y_j p_i p_j \]

Since sampling is with replacement, the two draws are independent. Therefore

\[ \text{Cov}(y_{r}, y_{s}) = 0 = \delta_{12}, \text{ for } r, s = 1, 2, \ldots, m+1. \quad (4.5.12) \]

Now we will calculate \( \text{Cov}(y_{r} y_{s}) \), \( r \neq s, r, s > m+1 \) for this we need

\[ E(y_{r} y_{s}) = \sum_{i=1}^{N} \sum_{j=1}^{N} Y_i Y_j \{ \begin{array}{l}
\text{Probability that } Y_i \text{ and } Y_j \text{ are} \\
\text{selected at draws } > m+1 
\end{array} \}

\[ = \sum_{i=1}^{N} \sum_{j=1}^{N} Y_i Y_j \frac{1-p_i p_j}{(N-1)(N-2)} \]

We also require
\[ E(y_x)E(y_s) = \sum_{i=1}^{N} \frac{(1-p_i)^2}{(N-1)^2} y_i^2 + \sum_{i \neq j=1}^{NN} \frac{(1-p_i)(1-p_j)}{(N-1)^2} y_i y_j \]

Therefore we have

\[ b_{13} = E(y_x y_s) - E(y_x)E(y_s) \]

\[ = \sum_{i+j=1}^{NN} \frac{1-p_i-p_j}{(N-1)(N-2)} y_i y_j - \sum_{i=1}^{N} \frac{(1-p_i)^2}{(N-1)^2} y_i \]

\[ - \sum_{i+j=1}^{NN} \frac{(1-p_i)(1-p_j)}{(N-1)^2} y_i y_j \]

\[ = \sum_{i+j=1}^{NN} \frac{1-p_i-p_j-(N-2)p_i p_j}{(N-1)^2(N-2)} y_i y_j - \sum_{i=1}^{N} \frac{(1-p_i)^2}{(N-1)^2} y_i \]

\[ (4.5.13) \]

Now we will substitute these values in equation (4.5.7).

Hence we have
\[
\text{Var}(t_1) = \frac{1}{m+1} \left\{ \sum_{i=1}^{N} p_i(1-p_i) \frac{2}{Y_i} \sum_{i+j=1}^{N} p_i p_j Y_i Y_j - \frac{0}{(N-1)^2} \right\} \\
+ \frac{(N-1)^2}{(n-m-1)} \left\{ \sum_{i=1}^{N} \frac{(1-p_i)(N-2+p_i)}{(N-1)^2} \right\} \frac{2}{Y_i Y_j} \sum_{i+j=1}^{N} \frac{p_i p_j}{(n-2)} \left\{ \frac{1-\frac{p_i}{(1-p_i)^2}}{2} \right\} \frac{2}{Y_i} \\
- \sum_{i=1}^{N} \sum_{i+j=1}^{N} \frac{1-p_i p_j - (N-2) p_i p_j}{(n-2)(N-1)^2} \frac{2}{Y_i Y_j} \frac{1-\frac{p_i}{(1-p_i)^2}}{2} \frac{2}{Y_i} \right\} \\
+ \frac{2(N-1)}{(n-1)} \left\{ \frac{-\sum_{i=1}^{N} p_i(1-p_i) \frac{2}{Y_i} \sum_{i+j=1}^{N} p_i p_j Y_i Y_j}{+0} \right\} \\
+ \frac{(N-1)^2}{(n-m-1)} \left\{ \frac{1-p_i p_j - (N-2) p_i p_j}{(n-2)(N-1)^2} \frac{2}{Y_i Y_j} \right\} \sum_{i=1}^{N} \frac{(1-p_i)^2}{2} \frac{2}{Y_i} \right\} \\
= \sum_{i=1}^{N} \left\{ \frac{p_i(1-p_i)}{m+1} \right\} \frac{2}{Y_i} + \frac{\sum_{i+j=1}^{N} \frac{p_i p_j}{n-m-1} \left\{ \frac{(1-p_i)(1-p_j)}{n-m-1} \right\} \frac{2}{Y_i Y_j}}{1-p_i p_j - (N-2) p_i p_j - \frac{(N-2) p_i p_j}{(n-2)(n-m-1)}} + 2p_i p_j \frac{1-p_i p_j - (N-2) p_i p_j}{(n-2)} \frac{2}{Y_i Y_j} \right\} \\
(4.5.14)
\]

It may be noted that if \( m = 0 \), the equation (4.5.14) reduces to

\[
\text{Var}(t_1) = \sum_{i=1}^{N} \frac{(N-n)(1-p_i)}{n-1} \frac{2}{Y_i} \sum_{i+j=1}^{N} \frac{(1-p_i)(1-p_j)}{(n-1)(N-2)} \frac{2}{Y_i Y_j} \\
(4.5.15)
\]

which is the variance of the best estimator in Horvitz and Thompson's \( T_1 \)-class of linear estimators when Ikeda-Sen's sampling procedure is adopted.