Chapter 3

Mixed Type Duality For Control Problems With Generalized Invexity
3.1. Introductory Remarks

Recently Husain et al [33] constructed the following dual in the spirit of Mond and Weir [66] for relaxing the invexity requirements further.

Problem (CD) (Dual): Maximize \( \int_a^b f(t,x,u)dt \)

subject to

\[ x(a) = \alpha, x(b) = \beta, \quad (3.1) \]
\[ f_s(t,x,u) + \lambda(t) g_s(t,x,u) + \mu(t) h_s(t,x,u) + \dot{\mu}(t) = 0, \quad t \in I, \quad (3.2) \]
\[ f_u(t,x,u) + \lambda(t) g_u(t,x,u) + \mu(t) h_u(t,x,u) = 0, \quad t \in I, \quad (3.3) \]
\[ \int_a^b \left( \lambda(t) g(t,x,u) + \mu(t) (h(t,x,u) - \dot{x}) \right) dt \geq 0, \quad (3.4) \]
\[ \lambda(t) \geq 0, t \in I. \quad (3.5) \]

In this chapter mixed type duality is presented in order to combine the Wolfe type duality and Mond-weir type duality [33].

3.2. Mixed Type Duality

We propose the following mixed type dual (Mix CD) to the control problem (CP) and establish usual duality results:

(Mix CD): Maximize \( \int_a^b \left[ f(t,x,u) + \sum_{i \in I_0} \mu^i(t) \left( h^i(t,x,u) - \dot{x}^i \right) + \sum_{j \in J_0} \lambda^j(t) g^j(t,x,u) \right] dt \)

subject to
\[ x(a) = \alpha, x(b) = \beta \quad \text{(3.6)} \]

\[ f_x(t, x, u) + \mu(t) \begin{bmatrix} h_x(t, x, u) + \lambda(t) g_x(t, x, u) + \mu(t) \end{bmatrix} = 0, t \in I \quad \text{(3.7)} \]

\[ f_x(t, x, u) + \mu(t) \begin{bmatrix} h_x(t, x, u) + \lambda(t) g_x(t, x, u) \end{bmatrix} = 0, t \in I \quad \text{(3.8)} \]

\[ \int_{a}^{b} \left( \sum_{i \in I_\alpha} \mu_i(t)(h_i(t, x, u) - \hat{x}_i(t)) + \sum_{j \in J_\alpha} \lambda_j(t)g_j(t, x, u) \right) dt \geq 0, \alpha = 1, 2, \ldots r \quad \text{(3.9)} \]

\[ \lambda(t) \geq 0, t \in I \quad \text{(3.10)} \]

where for \( N = \{1, 2, \ldots, n\} \) and \( K = \{1, 2, \ldots, k\} \),

(i) \( I_\alpha \subseteq M, \alpha = 0, 1, 2, \ldots r \)

and \( I_\alpha \cap I_\beta = \emptyset, \alpha \neq \beta \) and \( \bigcup_{\alpha=0}^{r} I_\alpha = N \).

(ii) \( J_\alpha \subseteq K, \alpha = 0, 1, 2, \ldots r \) with \( J_\alpha \cap J_\beta = \emptyset, \alpha \neq \beta \) and \( \bigcup_{\alpha=0}^{r} J_\alpha = K \), and

(iii) \( r = \max(r_1, r_2) \), where \( r_1 \) is the number of disjoint subsets of \( M \) and \( r_2 \) is the number of disjoint subsets of \( K \). Then \( I_\alpha \) or \( J_\alpha \) is empty for \( \alpha > \min(r_1, r_2) \).

**Theorem 3.1 (Weak duality):** Let \((\bar{x}, \bar{u})\) be feasible for (CP) and \((x, u, \lambda, \mu)\) be feasible for (Mix CD). If for all feasible \((\bar{x}, \bar{u}, x, u, \lambda, \mu)\),

\[ \int_{a}^{b} \left( f + \sum_{i \in I_\alpha} \mu_i(t)(h_i(t, x, u) - \hat{x}_i(t)) + \sum_{j \in J_\alpha} \lambda_j(t)g_j(t, x, u) \right) dt \]

is pseudoinvex and

\[ \int_{a}^{b} \left( \sum_{i \in I_\alpha} \mu_i(t)(h_i(t, x, u) - \hat{x}_i(t)) + \sum_{j \in J_\alpha} \lambda_j(t)g_j(t, x, u) \right) dt \]

is quasi-invex with respect to the same \( \eta \) and \( \xi \), then

\[ \inf (\text{CP}) \geq \sup (\text{Mix CD}) \]

**Proof:** Since \((\bar{x}, \bar{u})\) be feasible for (CP) and \((x, u, \lambda, \mu)\) be feasible for...
(Mix CD), we have
\[
\int_{a}^{b} \left( \sum_{i=I_a}^{i} \mu_i(t)(h_i'(t,x,u) - \dot{x}'_i) + \sum_{j=J_a}^{j} \lambda_j(t)g_j(t,x,u) \right) \, dt
\]
\[
\leq \int_{a}^{b} \left( \sum_{i=I_a}^{i} \mu_i(t)(h_i(t,x,u) - \dot{x}_i) + \sum_{j=J_a}^{j} \lambda_j(t)g_j(t,x,u) \right) \, dt, \quad \alpha = 1, 2, \ldots, r
\]

By quasi-invexity of \[ \int_{a}^{b} \left( \sum_{i=I_a}^{i} \mu_i(t)h'_i(t,x,u) + \sum_{j=J_a}^{j} \lambda_j(t)g'_j(t,x,u) \right) \, dt, \quad \alpha = 1, 2, \ldots, r \] this inequality yields,
\[
0 \geq \int_{a}^{b} \left[ \eta^T \left( \sum_{i=I_a}^{i} \mu_i(t)h'_i(t,x,u) + \sum_{j=J_a}^{j} \lambda_j(t)g'_j(t,x,u) \right) - \left( \frac{d\eta}{dt} \right) \sum_{i=I_a}^{i} \mu_i(t)h'_i(t,x,u) + \sum_{j=J_a}^{j} \lambda_j(t)g'_j(t,x,u) \right] \, dt
\]
\[
= \int_{a}^{b} \left[ \eta^T \left( \sum_{i=I_a}^{i} \mu_i(t)h'_i(t,x,u) + \sum_{j=J_a}^{j} \lambda_j(t)g'_j(t,x,u) \right) \right] \, dt - \eta \sum_{i=I_a}^{i} \mu_i \left|_{t=a}^{t=b} \right.
\]

(By integration by parts)
\[
= \int_{a}^{b} \left[ \eta^T \left( \sum_{i=I_a}^{i} \mu_i(t)h'_i(t,x,u) + \sum_{j=J_a}^{j} \lambda_j(t)g'_j(t,x,u) \right) \right] \, dt
\]
\[
(\text{using } \eta = 0, \text{ at } t = a \text{ and } t = b)
\]
\[
= \int_{a}^{b} \left[ \eta^T \left( \sum_{i=I_a}^{i} \mu_i(t)h'_i(t,x,u) + \sum_{j=J_a}^{j} \lambda_j(t)g'_j(t,x,u) \right) + \sum_{i=J_a}^{j} \lambda_j(t)g'_j(t,x,u) \right] \, dt
\]

Using (2.5) and (2.6), this implies
\[
\int_a^b \eta^T \left( \sum_{i \in I_i} (\mu_i'(t)h_i'(t,x,u) + \mu_i'(t)) + \sum_{j \in J_j} \lambda_j'(t)g_j'(t,x,u) \right) \\
+ \xi^T \left( \sum_{i \in I_i} \mu_i'(t)h_i'(t,x,u) + \sum_{j \in J_j} \lambda_j'(t)g_j'(t,x,u) \right) 
\]
\[\text{dt} \geq 0\]

This, because of pseudo-invexity of \[\int_a^b \left( f + \sum_{i \in I_i} \mu_i'(t)\left(h_i'(t,x,u) - \hat{x}'\right) + \sum_{j \in J_j} \lambda_j'(t)g_j'(t,x,u) \right) \text{dt} \] yields,

\[
\int_a^b \left( f(t,x,u) + \sum_{i \in I_i} \mu_i'(t)\left(h_i'(t,x,u) - \hat{x}'\right) + \sum_{j \in J_j} \lambda_j'(t)g_j'(t,x,u) \right) \text{dt} \geq \int_a^b \left( f(t,x,u) + \sum_{i \in I_i} \mu_i'(t)(h_i'(t,x,u) - \hat{x}') + \sum_{j \in J_j} \lambda_j'(t)g_j'(t,x,u) \right) \text{dt} \hspace{1cm} (3.11)
\]

Since \(\mu(t)^T(h(t,x,u) - \hat{x}) = 0\), and \(\lambda(t)^T g(t,x,u) \leq 0\), these respectively imply

\[
\sum_{i \in I_i} \mu_i'(t)(h_i'(t,x,u) - \hat{x}') = 0 \quad \text{and} \quad \sum_{j \in J_j} \lambda_j'(t)g_j'(t,x,u) \leq 0, \quad t \in I
\]

Consequently (3.11) gives

\[
\int_a^b f(t,x,u) \text{dt} \geq \int_a^b \left( f(t,x,u) + \sum_{i \in I_i} \mu_i'(t)(h_i'(t,x,u) - \hat{x}') + \sum_{j \in J_j} \lambda_j'(t)g_j'(t,x,u) \right) \text{dt}
\]

That is,

\[\text{infimum (CP)} \geq \text{Supremum (Mix CD)}\]

**Theorem 3.2 (Strong Duality):** If \((\bar{x}, \bar{u})\) is an optimal solution of (CP) and is normal, then there exist piecewise smooth \(\bar{\mu} : I \to \mathbb{R}^n\) and \(\bar{\lambda} : I \to \mathbb{R}^p\) such that \((\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})\) be feasible and the corresponding values of (CP) and (Mix CD) are equal.

If, also \(\int_a^b \left( f + \sum_{i \in I_i} \mu_i'(h_i' - \hat{x}') + \sum_{j \in J_j} \lambda_j'g_j' \right) \text{dt} \) is pseudoinvex and \(\int_a^b \left( f + \sum_{i \in I_i} \mu_i'(h_i' - \hat{x}') + \sum_{j \in J_j} \lambda_j'g_j' \right) \text{dt} \) is quasi-invex with respect to the same \(\eta\) and \(\xi\), then \((\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})\) is an optimal solution of (Mix CD).
**Proof:** Since \((\bar{x}, \bar{u})\) is an optimal solution to (CP) and is normal then from Proposition 2.1, there exist piecewise smooth \(\bar{\mu} : I \to \mathbb{R}^n\) and \(\bar{\lambda} : I \to \mathbb{R}^p\) such that

\[
f_x(t, \bar{x}, \bar{u}) + \bar{n}(t)^T h_x(t, \bar{x}, \bar{u}) + \mu(t) + \bar{\lambda}(t)^T g(t, \bar{x}, \bar{u}) = 0, \quad t \in I \tag{3.12}
\]

\[
f_x(t, \bar{x}, \bar{u}) + \bar{n}(t)^T h_x(t, \bar{x}, \bar{u}) + \bar{\lambda}(t)^T g(t, \bar{x}, \bar{u}) = 0, \quad t \in I \tag{3.13}
\]

\[
\bar{\lambda}(t)^T g(t, \bar{x}, \bar{u}) = 0, \quad t \in I \tag{3.14}
\]

\[
\bar{\lambda}(t) \geq 0, \quad t \in I \tag{3.15}
\]

The relation (3.13) implies \(\sum_{j \in J^0} \lambda^j(t)g^j(t, \bar{x}, \bar{u}) = 0\) and \(\sum_{j \in J^0} \bar{\lambda}^j(t)g^j(t, \bar{x}, \bar{u}) = 0\), \(\alpha = 1, 2, \ldots, r\).

Also \(\bar{\mu}(t)^T (h(t, \bar{x}, \bar{u}) - \bar{x}) = 0\), implies \(\sum_{j \in J^0} \mu^j(t)(h^j(t, \bar{x}, \bar{u}) - \bar{x}^j) = 0, \quad t \in I\) and \(\sum_{i \in I^0} \mu^i(t)(h^i(t, \bar{x}, \bar{u}) - \bar{x}^i) = 0, \quad t \in I\).

Consequently,

\[
\sum_{i \in I^0} \bar{\mu}^i(t)(h^i(t, \bar{x}, \bar{u}) - \bar{x}^i) = 0, \quad t \in I \quad \text{and} \quad \sum_{j \in J^0} \bar{\lambda}^j(t)g^j(t, \bar{x}, \bar{u}) = 0, \quad t \in I
\]

imply

\[
\int_a^b \left[ \sum_{i \in I^0} \bar{\mu}^i(t)(h^i(t, \bar{x}, \bar{u}) - \bar{x}^i) + \sum_{j \in J^0} \lambda^j(t)g^j(t, \bar{x}, \bar{u}) \right] dt = 0 \tag{3.15}
\]

From the relations (3.11), (3.12), (3.14) and (3.15), it implies that \((\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})\) is feasible for (Mix CD) and the corresponding objective values of (CP) and (Mix CD) are equal in view of \(\sum_{i \in I^0} \bar{\mu}^i(t)(h^i(t, \bar{x}, \bar{u}) - \bar{x}^i) = 0\) and \(\sum_{j \in J^0} \bar{\lambda}^j(t)g^j(t, \bar{x}, \bar{u}) = 0, \quad t \in I\).
If \( \int_a^b \left[ f + \sum_{i \in I_a} \mu_i(t)(h'_i - \dot{x}_i) + \sum_{j \in J_a} \lambda^j(t)g^j \right] dt \) is pseudoinvex and \( \int_a^b \left[ \sum_{i \in I_a} \mu_i(t)(h'_i - \dot{x}_i) + \sum_{j \in J_a} \lambda^j(t)g^j \right] dt \) \( \alpha = 1, 2, \ldots, r \) is quasi-invex with respect to the same \( \eta \) and \( \xi \), then from Theorem 3.1, \((\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})\) must be an optimal solution of \((\text{Mix CD})\).

**Theorem 3.3 (Strict Converse duality):** Let \((\bar{x}, \bar{u})\) be an optimal solution of \((\text{CP})\) and normality condition be satisfied at \((\bar{x}, \bar{u})\). Let \((\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})\) be an optimal solution of \((\text{Mix CD})\). If for all feasible \((\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})\),

\[
\int_a^b \left[ \sum_{i \in I_a} \hat{\mu}_i'(t)(h'_i - \dot{x}_i) + \sum_{j \in J_a} \hat{\lambda}^j(t)g^j \right] dt \quad \alpha = 1, 2, \ldots, r
\]

is quasi-invex and

\[
\int_a^b \left[ f + \sum_{i \in I_a} \hat{\mu}_i(t)(h'_i - \dot{x}_i) + \sum_{j \in J_a} \hat{\lambda}^j(t)g^j \right] dt
\]

is strictly pseudoinvex with respect to the same \( \eta \) and \( \xi \), then \((\hat{x}, \hat{u}) = (\bar{x}, \bar{u})\), i.e., \((\hat{x}, \hat{u})\) is an optimal solution of \((\text{CP})\).

**Proof:** We assume that \((\hat{x}, \hat{u}) \neq (\bar{x}, \bar{u})\) and show that this assumption leads to a contradiction. Since \((\bar{x}, \bar{u})\) is an optimal solution of \((\text{CP})\) and is normal, it follows by strong duality (Theorem 3.2) that there exist piecewise smooth \( \mu : I \rightarrow \mathbb{R}^n \) and \( \lambda : I \rightarrow \mathbb{R}^c \) such that \((\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})\) is an optimal solution of \((\text{Mix CD})\) and

\[
\int_a^b f(t, \bar{x}, \bar{u}) dt = \int_a^b \left[ f(t, \bar{x}, \bar{u}) + \sum_{i \in I_a} \hat{\mu}_i'(t)(h'_i(t, \hat{x}, \hat{u}) - \dot{x}_i) + \sum_{j \in J_a} \hat{\lambda}^j(t)g^j(t, \hat{x}, \hat{u}) \right] dt \quad (3.17)
\]

Also since \((\bar{x}, \bar{u})\) and \((\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})\) are feasible for \((\text{CP})\) and \((\text{Mix CD})\), therefore, for \( \alpha = 1, 2, \ldots, r \)

\[
\int_a^b \left[ \sum_{i \in I_a} \hat{\mu}_i'(t)(h'_i(t, \bar{x}, \bar{u}) - \dot{x}_i) + \sum_{j \in J_a} \hat{\lambda}^j(t)g^j(t, \bar{x}, \bar{u}) \right] dt
\]

\[
\leq \int_a^b \left[ \sum_{i \in I_a} \hat{\mu}_i'(t)(h'_i(t, \hat{x}, \hat{u}) - \dot{x}_i) + \sum_{j \in J_a} \hat{\lambda}^j(t)g^j(t, \hat{x}, \hat{u}) \right] dt \quad (3.18)
\]
This, because of quasi-invexity of \( \int_{a}^{b} \left( \sum_{i \in I_{a}} \hat{\mu}^{i}(t) \left( h^{i} - \hat{\lambda}^{i} \right) \right) dt, \alpha = 1, 2, \ldots, r \) is quasi-invex for all feasible \( (\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}) \) with respect to \( \eta \) and \( \xi \), therefore, (3.18) implies that for \( \alpha = 1, 2, \ldots, r \),

\[
\int_{a}^{b} \left[ \eta^{T} \left( \sum_{i \in I_{a}} \hat{\mu}^{i}(t) h^{i}(t, \hat{x}, \hat{u}) + \sum_{j \in J_{a}} \hat{\lambda}^{j}(t) g^{j}(t, \hat{x}, \hat{u}) \right) - \sum_{i \in I_{a}} \left( \frac{d \eta}{dt} \right)^{T} \hat{\mu}^{i} + \xi^{T} \left( \sum_{i \in I_{a}} \hat{\mu}^{i}(t) h^{i}(t, \hat{x}, \hat{u}) - \hat{\lambda}^{i}(t) \right) \right] dt \leq 0
\]

(By integration by parts)

\[
\int_{a}^{b} \left[ \eta^{T} \left( \sum_{i \in I_{a}} \hat{\mu}^{i}(t) h^{i}(t, \hat{x}, \hat{u}) + \hat{\lambda}^{i}(t) \right) + \sum_{j \in J_{a}} \hat{\lambda}^{j}(t) g^{j}(t, \hat{x}, \hat{u}) \right] dt - \eta \sum_{i \in I_{a}} \hat{\mu}^{i}(t) \bigg|_{t=a}^{b} \leq 0
\]

(Using \( \eta = 0 \) at \( t = a, t = b \))

\[
or \int_{a}^{b} \left[ \eta^{T} \left( \sum_{i \in I_{a}} \hat{\mu}^{i}(t) h^{i}(t, \hat{x}, \hat{u}) + \hat{\lambda}^{i}(t) \right) + \sum_{j \in J_{a}} \hat{\lambda}^{j}(t) g^{j}(t, \hat{x}, \hat{u}) \right] dt \leq 0
\]

Since \((\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})\) is feasible for \( \text{(Mix CD)} \), therefore, by using (3.7) and (3.8) in the above inequality, we have
\[
\int_a^b \left[ \eta^T \left( f_x(t, \hat{x}, \hat{u}) + \sum_{i \in I_0} \left( \hat{\mu}^i(t) h^i_x(t, \hat{x}, \hat{u}) + \hat{\mu}^i(t) \right) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j_x(t, \hat{x}, \hat{u}) \right) \right. \\
\left. + \xi^T \left( f_u(t, \hat{x}, \hat{u}) + \sum_{i \in I_0} \hat{\mu}^i(t) h^i_u(t, \hat{x}, \hat{u}) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j_u(t, \hat{x}, \hat{u}) \right) \right] dt \geq 0
\]

This, because of strict pseudo-invexity of \( f \) with respect to \( \eta \) and \( \xi \), yields

\[
\int_a^b \left[ f(t, \bar{x}, \bar{u}) + \sum_{i \in I_0} \left( \hat{\mu}^i(t) \left( h^i_x(t, \bar{x}, \bar{u}) - \bar{x} \right) \right) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j_x(t, \bar{x}, \bar{u}) \right] dt \\
> \int_a^b \left[ f(t, \hat{x}, \hat{u}) + \sum_{i \in I_0} \left( \hat{\mu}^i(t) \left( h^i_x(t, \hat{x}, \hat{u}) - \hat{x} \right) \right) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j_x(t, \hat{x}, \hat{u}) \right] dt
\]

Since \( \sum_{i \in I_0} \hat{\mu}^i \left( h^i_x(t, \bar{x}, \bar{u}) - \bar{x} \right) = 0 \) and \( \sum_{j \in J_0} \hat{\lambda}^j(t) g^j_x(t, \hat{x}, \hat{u}) \), which are consequence of feasibility of \((\bar{x}, \bar{u})\) for \((CP)\) and \((\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})\) for \((Mix\ CD)\), we have

\[
\int_a^b f(t, \bar{x}, \bar{u}) dt > \int_a^b f(t, \hat{x}, \hat{u}) + \sum_{i \in I_0} \left( \hat{\mu}^i(t) \left( h^i_x(t, \hat{x}, \hat{u}) - \hat{x} \right) \right) + \sum_{j \in J_0} \hat{\lambda}^j(t) g^j_x(t, \hat{x}, \hat{u}) \right] dt
\]

This is a contradiction to (3.10). Hence \((\hat{x}, \hat{u}) = (\bar{x}, \bar{u})\), i.e., \((\hat{x}, \hat{u})\) must be an optimal solution of \((CP)\). Thus the theorem follows.

We now write for simplicity of the notation

\[
\psi_1 \equiv \psi_1(t, x, u, \lambda, \mu, \hat{\mu}) = f_x + \mu(t)^T (h_x(t, x, u) + \hat{\mu}) + \lambda^T (t) g(t, x, u),
\]

\[
\psi_2 \equiv \psi_2(t, x, u, \lambda, \mu) = f_x + \mu(t)^T h_x(t, x, u) + \hat{\mu} + \lambda^T (t) g(t, x, u)
\]

where \(f_x = f_x(t, x, u), f_u = f_u(t, x, u), g_x = g_x(t, x, u)\) and \(h_x = h_x(t, x, u)\).

Consider \(\psi_1(t, x(t), u(t), \lambda(\cdot), \mu(\cdot), \hat{\mu}(\cdot))\) as defining a mapping \(Q_1: X \times U \times Y \times Z \rightarrow B\), where \(Y\) is the space of piecewise smooth functions \(\lambda: I \rightarrow \mathbb{R}^k\), \(Z\) is the space of differentiable function \(\mu: I \rightarrow \mathbb{R}^n\) and \(B\) is a banach
space; X and U are already defined. Also consider \( \psi_2(t, x(\cdot), u(\cdot), \lambda(\cdot), \mu(\cdot)) \) as defining a mapping \( Q_2: X \times U \times Y \times Z \to C \) where C is another banach space. In order to apply Proposition 2.1 to the problem (CD), some assumptions on \( \psi_1(\cdot) = 0 \) and \( \psi_2(\cdot) = 0 \) are in order. For this it suffices to assume that Frechet derivatives.

\[
Q_1' = (Q_{1x}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}), Q_{1u}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}), Q_{1\lambda}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}), Q_{1\mu}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}))
\]

\[
Q_2' = (Q_{2x}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}), Q_{2u}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}), Q_{2\lambda}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}), Q_{2\mu}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}))
\]

have weak *closed range. For notational convenience, we shall write in the sequel \( Q_1' = (Q_{1x}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}), Q_{1u}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}), Q_{1\lambda}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}), Q_{1\mu}(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})) \), \( \bar{g}_s = g_s(t, \bar{x}, \bar{u}) \)

\( \bar{h}_s = h_s(t, \bar{x}, \bar{u}) \), etc.

**Theorem 3.4 (Converse duality):** Let \( f, g, \) and \( h \) be twice continuously differentiable and \( (\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu}) \) be an optimal solution of (Mix CD). Let the Frechet derivatives \( Q_1' \) and \( Q_2' \) have weak closed range. Assume that

\[
(H_1): \int_a^b \sigma(t)^T (M(t)) \sigma(t) dt = 0 \Rightarrow \sigma(t) = 0, t \in I
\]

where \( \sigma(t) \in \mathbb{R}^{n_m} \) and

\[
M(t) = \left[ \begin{array}{c} \tilde{f}_{xx} + \bar{\mu}(t)^T h_{xx} + \bar{\lambda}(t)^T g_{xx}, \tilde{f}_{ux} + \bar{\mu}(t)^T h_{ux} + \bar{\lambda}(t)^T g_{ux} \\ \tilde{f}_{ux} + \bar{\mu}(t)^T h_{ux} + \bar{\lambda}(t)^T g_{ux}, \tilde{f}_{uu} + \bar{\mu}(t)^T h_{uu} + \bar{\lambda}(t)^T g_{uu} \end{array} \right]
\]

\[
(H_2): \left\{ \sum_{i \in I_u} (\mu'(t) h_i'(t, \bar{x}, \bar{u}) + \dot{\mu}(t)) + \sum_{j \in J_u} \lambda'(t) g_j'(t, \bar{x}, \bar{u}), \alpha = 1, 2, \ldots, r \right\} \text{ and }
\]

\[
\left\{ \sum_{i \in I_u} \mu'(t) h_i'(t, \bar{x}, \bar{u}) + \dot{\mu}(t) + \sum_{j \in J_u} \lambda'(t) g_j(t, \bar{x}, \bar{u}), \alpha = 1, 2, \ldots, r \right\} \text{ are linearly independent and }
\]

\[
(H_3) \mu(a) = 0 = \mu(b)
\]
If, for all feasible \((\bar{x}, \bar{u}, x, u, \lambda, \mu)\), \(\int_{a}^{b} \left( f + \sum_{i \in I_{a}} \mu'(t)(h' - \dot{x}') + \sum_{j \in J_{a}} \lambda'(t)g'_{j} \right) dt \) is pseudoinvex and \(\int_{a}^{b} \left( \sum_{i \in I_{a}} \mu'(t)(h' - \dot{x}') + \sum_{j \in J_{a}} \lambda'(t)g'_{j} \right) dt \) is quasi-invex with respect to the same \(\eta\) and \(\xi\), then \((\bar{x}, \bar{u})\) is an optimal solution of (CP).

**Proof:** Since \((\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})\) is an optimal solution to (CP), therefore, by Proposition 2.1, there exist \(\tau \in \mathbb{R}\), \(\gamma_{\alpha} \in \mathbb{R}, \alpha = 1, 2, \ldots, r\), and piecewise smooth \(\beta: I \rightarrow \mathbb{R}^{n}\) and \(\theta: I \rightarrow \mathbb{R}^{m}\) such that

\[
\tau \left( f_{i} + \sum_{i \in I_{a}} \left( \mu'(t)h'_{i} + \dot{\mu}'(t) \right) + \sum_{j \in J_{a}} \lambda'(t)g'_{j} \right) + \beta(t)^{T} \left( f_{i} + \lambda(t)^{T}g_{i} + \mu(t)^{T}h_{i} \right) \\
+ \theta(t)^{T} \left( f_{i} + \lambda(t)^{T}g_{i} + \mu(t)^{T}h_{i} \right)
\]

\[
+ \sum_{\alpha=1}^{r} \gamma_{\alpha} \left( \sum_{i \in I_{a}} \left( \mu'(t)h'_{i} + \dot{\mu}'(t) \right) + \sum_{j \in J_{a}} \lambda'(t)g'_{j} \right) = 0, \quad t \in I
\]  
(3.19)

\[
\tau \left( f_{i} + \sum_{i \in I_{a}} \mu'(t)h'_{i} + \sum_{j \in J_{a}} \lambda'(t)g'_{j} \right) + \beta(t)^{T} \left( f_{i} + \lambda(t)^{T}g_{i} + \mu(t)^{T}h_{i} \right) \\
+ \theta(t)^{T} \left( f_{i} + \lambda(t)^{T}g_{i} + \mu(t)^{T}h_{i} \right)
\]

\[
+ \sum_{\alpha=1}^{r} \gamma_{\alpha} \left( \sum_{i \in I_{a}} \mu'(t)h'_{i} + \sum_{j \in J_{a}} \lambda'(t)g'_{j} \right) = 0, \quad t \in I
\]  
(3.20)

\[
\tau(h' - \dot{x}') + \beta(t)^{T}h'_{i} - \beta'(t) + \theta(t)^{T}h'_{i} = 0, \quad i \in I_{0}
\]  
(3.21)

\[
\beta(t)^{T}h'_{i} - \beta'(t) + \theta(t)^{T}h'_{i} + \gamma_{\alpha}(h' - \dot{x}') = 0, \quad i \in I_{a}, \quad \alpha = 1, 2, \ldots, r
\]  
(3.22)

\[
\gamma_{a} \left( g_{i} + \beta(t)^{T}g_{i} + \theta(t)^{T}g_{i} + \eta'(t) \right) = 0, i \in I_{a}
\]  
(3.23)

\[
\beta(t)^{T}g_{i} + \theta(t)^{T}g_{i} + \gamma_{\alpha}g_{i} + \eta'(t) = 0, i \in J_{a}, \alpha = 1, 2, \ldots, r
\]  
(3.24)

\[
\gamma_{a} \int_{a}^{b} \left( \sum_{i \in I_{a}} \mu'(t)(h' - \dot{x}') + \sum_{j \in J_{a}} \lambda'(t)g'_{j} \right) dt = 0, \quad \alpha = 1, 2, \ldots, r
\]  
(3.25)

\[
\eta(t)^{T}\lambda(t) = 0, \quad t \in I,
\]  
(3.26)
\[ (\tau, \gamma, \eta(t)) \geq 0, \ t \in I \] (3.27)
\[ (\tau, \beta(t), \theta(t), \gamma, \eta(t)) \neq 0, \ t \in I \] (3.28)

Multiplying (3.21) by \( \mu(t), i \in I_0 \) and \( t \in I \), and summing over \( i \in I_0 \) and then integrating, we have
\[
\tau \int_a^b \sum_{i \in I_0} \mu(t) (h - \hat{x}) dt + \int_a^b \beta(t) \sum_{i \in I_0} \left( \mu(t) h_i^t + \hat{\mu}(t) \right) dt + \theta(t) \left( \int_a^b \sum_{i \in I_0} \mu(t) h_i^t \right) dt - \int_a^b \mu(t) \beta(t) dt = 0
\]

Using \((H_3)\), we have
\[
\tau \int_a^b \sum_{i \in I_0} \mu'(t) (h - \hat{x}) dt + \int_a^b \beta(t) \sum_{i \in I_0} \left( \mu'(t) h_i^t + \hat{\mu}'(t) \right) dt + \theta(t) \left( \int_a^b \sum_{i \in I_0} \mu'(t) h_i^t \right) dt = 0 \quad (3.29)
\]

Multiplying (3.22) by \( \mu(t), i \in I_0 \) and \( t \in I \), and summing over \( I \in I_\alpha \) and then integrating, we have
\[
\int_a^b \beta(t) \left( \sum_{i \in I_0} \mu'(t) h_i^t + \hat{\mu}'(t) \right) dt + \theta(t) \left( \int_a^b \sum_{i \in I_0} \mu'(t) h_i^t \right) dt + \gamma a \int_a^b \left( \sum_{i \in I_0} \mu'(t) (h - \hat{x}) \right) dt = 0, \quad \alpha = 1, 2, ..., r \quad (3.30)
\]

Similarly from (3.23) and (3.24) together with (3.26), it implies respectively
\[
\tau \int_a^b \sum_{i \in I_0} \lambda_i(t) g_i^j dt + \int_a^b \beta(t) \left( \sum_{i \in I_0} \lambda_i(t) g_i^j \right) dt + \theta(t) \left( \int_a^b \sum_{i \in I_0} \lambda_i(t) g_i^j \right) dt = 0, \quad (3.31)
\]
and
\[
\int_a^b \beta(t) \left( \sum_{i \in I_0} \lambda_i(t) g_i^j \right) dt + \theta(t) \left( \int_a^b \sum_{i \in I_0} \lambda_i(t) g_i^j \right) dt + \gamma a \int_a^b \left( \sum_{i \in I_0} \lambda_i(t) g_i^j \right) dt = 0,
\]
\[
\alpha = 1, 2, ..., r \quad (3.32)
\]

Adding (3.29) to (3.31) and (3.30) to (3.32), we have
\[
\tau \int_a^b \left( \sum_{i \in I_0} \mu^i(t)(h^i + \dot{x}^i) + \sum_{j \in J_0} \lambda^j(t) g^j \right) dt + \int_a^b \left( \beta(t)^T \left( \sum_{i \in I_0} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_0} \lambda^j(t) g^j \right) + \theta(t)^T \left( \sum_{i \in I_a} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_a} \lambda^j(t) g^j \right) \right) dt = 0 \quad (3.33)
\]

and
\[
\gamma_a \int_a^b \left( \sum_{i \in I_a} \mu^i(t)(h^i - \dot{x}^i) + \sum_{j \in J_a} \lambda^j(t) g^j \right) dt + \int_a^b \left( \beta(t)^T \left( \sum_{i \in I_a} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_a} \lambda^j(t) g^j \right) + \theta(t)^T \left( \sum_{i \in I_a} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_a} \lambda^j(t) g^j \right) \right) dt = 0, \quad \alpha = 1,2,\ldots,r \quad (3.34)
\]

Using (3.25) in (3.34), we have
\[
\int_a^b \left( \beta(t)^T \left( \sum_{i \in I_a} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_a} \lambda^j(t) g^j \right) + \theta(t)^T \left( \sum_{i \in I_a} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_a} \lambda^j(t) g^j \right) \right) dt = 0, \quad \alpha = 1,2,\ldots,r
\]

This can be written as
\[
\int_a^b \left( \beta(t), \theta(t) \right)^T \left( \sum_{i \in I_a} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_a} \lambda^j(t) g^j \right) \left( \sum_{i \in I_a} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_a} \lambda^j(t) g^j \right) dt = 0, \quad \alpha = 1,2,\ldots,r \quad (3.35)
\]

Using (3.7) and (3.8) in (3.19) and (3.20) respectively
\[
\sum_{\alpha=1}^r (\gamma_{\alpha} - \tau) \left( \sum_{i \in I_0} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_0} \lambda^j(t) g^j \right) + \beta(t)^T \left( f_{xx} + \lambda(t)^T g_{xx} + \mu(t)^T h_{xx} \right) + \theta(t)^T \left( f_{ux} + \lambda(t)^T g_{ux} + \mu(t)^T h_{ux} \right) = 0, \quad t \in I
\]

and
\[
\sum_{\alpha=1}^r (\gamma_{\alpha} - \tau) \left( \sum_{i \in I_a} \mu^i(t) h^i + \dot{\mu}^i(t) + \sum_{j \in J_a} \lambda^j(t) g^j \right) + \beta(t)^T \left( f_{xx} + \lambda(t)^T g_{xx} + \mu(t)^T h_{xx} \right) + \theta(t)^T \left( f_{ux} + \lambda(t)^T g_{ux} + \mu(t)^T h_{ux} \right) = 0, \quad t \in I
\]
Combining these relations, we have

\[ \sum_{\alpha=1}^{r}(\gamma_{\alpha} - \tau) \left( \sum_{i \in I_{\alpha}} (\mu'(t)h'_i + \mu'(t)) + \sum_{j \in J_{\alpha}} \lambda'(t)g'_j \right) \]

\[ + \left( \sum_{i \in I_{\alpha}} \mu'(t)h'_i + \sum_{j \in J_{\alpha}} \lambda'(t)g'_j \right) = 0, \quad t \in I \tag{3.36} \]

Pre-multiplying (3.36) by \((b(t), \theta(t))^T\) and then using (3.35), we have

\[ \int_{a}^{b} \left( (b(t), \theta(t))^T M(t) \left( \begin{array}{c} \beta(t) \\ \theta(t) \end{array} \right) \right) dt = 0, \]

where \( M(t) = \begin{pmatrix} f_{xx} + \lambda(t)^T g_{xx} + \mu(t)^T h_{xx}, f_{ux} + \lambda(t)^T g_{ux} + \mu(t)^T h_{ux} \\ f_{ux} + \lambda(t)^T g_{ux} + \mu(t)^T h_{ux}, f_{uu} + \lambda(t)^T g_{uu} + \mu(t)^T h_{uu} \end{pmatrix} \)

This, in view of \((H_1)\), yields

\[ \sigma(t) = \left( \begin{array}{c} \beta(t) \\ \theta(t) \end{array} \right) = 0, \quad t \in I. \]

That is,

\[ \beta(t) = 0 = \theta(t), \tag{3.37} \]

Using (3.21) in (3.22), we have

\[ \sum_{\alpha=1}^{r}(\gamma_{\alpha} - \tau) \left( \sum_{i \in I_{\alpha}} (\mu'(t)h'_i + \mu'(t)) + \sum_{j \in J_{\alpha}} \lambda'(t)g'_j \right) = 0. \]

This, because of the hypothesis \((H_2)\), gives

\[ \gamma_{\alpha} = \tau, \quad \alpha = 1, 2, \ldots, r \tag{3.38} \]

If \(\tau = 0\), then \(\gamma_{\alpha} = 0, \quad \alpha = 1, 2, \ldots, r\) from (3.38), \(\eta = 0\) from (3.23) and (3.24), consequently, \((\tau, \gamma, \ldots, \gamma_r, \beta(t), \theta(t), \eta(t)) = 0, \quad t \in I\) but this contradicts (3.28).
Hence $\tau = \gamma_\alpha > 0$, $\alpha = 1, 2, \ldots, r$.

Using (3.37) in (3.21) and (3.22) along with $\tau > 0$, $\gamma_\alpha (\alpha, 1, 2 \ldots r)$, we have

$$h'_i - \dot{x}_i = 0, i \in I_0 \quad \text{and} \quad h'_i - \dot{x}_i = 0, i \in I_u, \quad \alpha = 1, 2, \ldots, r$$

This implies

$$h(t, \bar{x}, \bar{u}) - \dot{x}(t) = 0, \quad t \in I$$

(3.39)

Using (3.37) in (3.23) and (3.24) together with $\tau > 0$, $\gamma_\alpha > 0$, $\alpha = 1, 2, \ldots, r$, we have

$$g(t, \bar{x}, \bar{u}) \leq 0, \quad t \in I$$

(3.40)

The relation (3.39) and (3.40) implies that, $(\bar{x}, \bar{u})$ is feasible for (CP).

Using (3.37) with $\tau > 0$ in (3.33), we have

$$\int_a^b \left( \sum_{i \in I_u} \mu'_i(t)(h'_i - \dot{x}_i) + \sum_{j \in J_u} \lambda^j(t) g^j(t, \bar{x}, \bar{u}) \right) dt = 0$$

This accomplishes the equality of objective values of (CP) and (Mix CD), i.e.,

$$\int_a^b f(t, \bar{x}, \bar{u}) dt = \int_a^b \left( f(t, \bar{x}, \bar{u}) + \sum_{i \in I_u} \mu'_i(t)(h'_i - \dot{x}_i) + \sum_{j \in J_u} \lambda^j(t) g^j(t, \bar{x}, \bar{u}) \right) dt$$

If, all feasible $(x, u, \lambda, \mu)$, $\int_a^b \left( f + \sum_{i \in I_u} \mu'_i(h'_i - \dot{x}_i) + \sum_{j \in J_u} \lambda^j g^j \right) dt$ is pseudoinvex and

$\int_a^b \left( \sum_{i \in I_u} \mu'_i h'_i + \sum_{j \in J_u} \lambda^j g^j \right) dt$ is quasi-invex with respect to the same $\eta$ and $\xi$, then from Theorem 3.1 , $(\bar{x}, \bar{u})$ is an optimal solution of (CP).
3.3. Control Problem with Free Boundary Conditions

The duality results established in the preceding section can be applied to the control problem with free boundary conditions. If the “targets” x(a) and x(b) are not restricted, we have

Problem PF (Primal): Maximize \( \int_a^b f(t,x,u)dt \)

subject to

\[ h(t,x,u) = \dot{x}, \ t \in I \]
\[ g(t,x,u) \leq 0, \ t \in I \]

This duality now includes the transversality \( \mu(t) = 0, \ t = a \) and \( t = b \) as new constraints. This implies

Problem DF (Dual):

Maximize \( \int_a^b \left[ f(t,x,u) + \sum_{i \in I_a} \mu_i(t)(h_i(t,x,u) - \dot{x}) + \sum_{j \in J_a} \lambda_j(t)g_j(t,x,u) \right] dt \)

subject to

\[ \mu(a) = 0 = \mu(b) \]
\[ f_\alpha(t,x,u) + \mu(t)^T h_\alpha(t,x,u) + \lambda(t)^T g_\alpha(t,x,u) + \bar{\mu}(t) = 0, \ t \in I \]
\[ f_\nu(t,x,u) + \mu(t)^T h_\nu(t,x,u) + \lambda(t)^T g_\nu(t,x,u) = 0, \ t \in I \]
\[ \int_a^b \left( \sum_{i \in I_a} \mu_i(t)(h_i(t,x,u) - \dot{x}) + \sum_{j \in J_a} \lambda_j(t)g_j(t,x,u) \right) dt \geq 0, t \in I \]

\[ \lambda(t) \geq 0, \ t \in I \]
3.4 Related Control Problems and Mathematical Programming

We now consider some special cases of (Mix CD). If \( I_0 = N \) and \( J_0 = K \), then (Mix CD) becomes the following Wolfe type dual, considered by Mond and Smart [61] under invexity of

\[
\int_a^b f(t,x,u)dt + \int_a^b \mu^T (h - \dot{x})dt \quad \text{and} \quad \int_a^b \lambda^T g dt
\]

subject to

\[
x(a) = \alpha, x(b) = \beta
\]

\[
f_x(t,x,u) + \mu(t)^T h_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \dot{\mu}(t) = 0, \quad t \in I
\]

\[
f_u(t,x,u) + \mu(t)^T h_u(t,x,u) + \lambda(t)^T g_u(t,x,u) = 0, \quad t \in I
\]

\[
\lambda(t) \geq 0, \quad t \in I
\]

If \( I_0 = \emptyset \) and \( J_0 = \emptyset \), then (Mix CD) becomes following Mond – Weir type dual recently considered by Husain et al [33] in order to relax invexity requirement on suitable forms of functionals involved in the formulation of the dual:

\[
\text{(M-WCD): Maximize} \quad \int_a^b f(t,x,u)dt
\]

subject to

\[
x(a) = \alpha, x(b) = \beta
\]

\[
f_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \mu(t)^T h_x + \dot{\mu}(t) = 0, \quad t \in I
\]

\[
f_u(t,x,u) + \lambda(t)^T g_u(t,x,u) + \mu(t)^T h_u = 0, \quad t \in I
\]

\[
\int_a^b \left( \sum_{i \in I_u} \mu(t)^T (h - \dot{x}) + \sum_{j \in J_u} \lambda(t)^T g(t,x,u) \right) dt \geq 0, \quad t \in I
\]
If $f$, $g$ and $h$ are independent of $t$ (without any loss of generality, assume $b - a = 1$), then the control problems (CP) and Mix (CD) reduce to a pair of static primal and dual of mathematical programming, considered by Mond and Weir [66]:

**Problem (PS):** Minimize $f(z)$

subject to

$h(z) = 0$

$g(z) \leq 0$

and

**Problem (Mix DS):** Maximize $f(z) + \sum_{i \in I_0} \mu^i h^i(z) + \sum_{j \in J_0} \lambda^j g^j(z)$

subject to

$f(z) + \mu^T h_z(z) + \lambda^T g_z(z) = 0$

$\sum_{i \in I_0} \mu^i h^i(z) + \sum_{j \in J_0} \lambda^j g^j(z) \geq 0$, $\alpha = 1, 2, \ldots, r$

$\lambda \geq 0$,

where $z = (x, u)$. 