CHAPTER V

EXPECTED TIME TO SEROCONVERSION OF HIV INFECTED WHEN BOTH ANTIGENIC DIVERSITY THRESHOLD AND VIRULENCE THRESHOLD SATISFY SCBZ PROPERTY

5.1 Introduction

In the present chapter a Stochastic model to derive the expected time to seroconversion under the assumption that both the antigenic diversity threshold and the virulence threshold are such that they are random variables having the probability distribution which satisfy the SCBZ property discussed by Raja Rao and Talwalkar (1990).

In doing so it assumed that the occurrence of the Seroconversion takes place if either the cumulative antigenic diversity of the invading antigens crosses the so called antigenic diversity threshold or the cumulative level of virulence crosses the virulence threshold level. In doing so the shock model and cumulative damage process due to Eassary et.al (1973) has been applied.

* This work was already presented in the National Science Day, Organized by the Annamalai University 28th February, 2015
5.2 Assumptions

1. A person is exposed to sexual contacts with an infected partner and on each occasion of contact the transmission of HIV takes place.

2. The mode of transmission of HIV on successive occasions results in the contribution to the antigenic diversity of the invading antigens. Also there is increase in the virulence of the invading antigens.

3. As and when the total antigenic diversity crosses a particular level called the antigenic diversity threshold, and then the seroconversion takes place. Similarly if the total virulence of the invading antigens crosses the virulence threshold, then the seroconversion will occur.

4. The crossing of both antigenic diversity threshold and virulence threshold simultaneously is considered to be an impossible event.

5. The two thresholds are random variables and are mutually independent.

6. Both the thresholds satisfy the SCBZ property.

5.3 Notations

\( X_i \) : a random variable denoting the contribution to antigenic diversity on the \( i^{th} \) contact \( i = 1, 2, 3, \ldots, k \) and with probability density function \( g(.) \) with cumulative distribution function \( G(.) \).
\(Y_i\) : the increase in the virulence due to the \(i^{th}\) contact, \(i = 1,2,3, \ldots, k\) with probability density function \(q(.)\) and cumulative distribution function \(Q(.)\).

\(Z_1\) : a random variable denoting antigenic threshold and has probability density function \(h(.)\) and cumulative distribution function \(H(.)\).

\(Z_2\) : a random variable denoting the virulence threshold with probability density function \(m(.)\) and cumulative distribution function \(M(.)\).

\(U_i\) : a random variable denoting the interarrival times between contacts \(i = 1,2,3, \ldots, k\) with probability density function of \(f(.)\) and cumulative distribution function \(F(.)\).

\(I^\theta(s)\) : Laplace transform of \(l(t)\)

\(T\) : time to seroconversion

### 5.4 Results

The survivor function \(S(t)\) is given by

\[
S(t) = P[T > t] = P\left[\text{The antigenic diversity as well as the virulence due to} \right.
\]

\[\begin{array}{c}
\text{"}k\text{" successive contacts do not cross the respective thresholds}\n\end{array}\]

Hence,

\[
= P\left[\sum_{i=1}^{k} X_i < Z_1\right] \cdot P\left[\sum_{i=1}^{k} Y_i < Z_2\right]
\]
= \Pr \{\text{There are } k \text{ contacts in } (0, t) \text{ and the total antigenic diversity as well as total virulence does not cross the respective thresholds}\}

\[ S(t) = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] \left[ \int_0^\infty g_k(x)H(x)dx \right] \left[ \int_0^\infty q_k(y)M(y)dy \right] \]

Now, the random variable \( Z_1 \) and \( Z_2 \) denoting the antigenic diversity threshold and virulence threshold respectively both undergo parametric changes in their respective probability distributions, since both the thresholds satisfy the SCBZ property.

Hence,

Now, the p.d.f. of \( Z_1 \) is given by

\[ h(z_1) = \theta_1 e^{-\theta_1 z_1} \text{if } z_1 \leq \tau_0 \]

\[ = \theta_2 e^\tau_0(\theta_2-\theta_1) e^{-\theta_2 z_1} \text{if } z_1 > \tau_0 \]

where \( \tau_0 \) is itself a random variable which follows \( \exp(\eta) \).

It can be proved that

\[ h(z_1) = \frac{(\theta_1 - \theta_2)(\eta + \theta_2)}{(\eta + \theta_1 - \theta_2)} e^{-z_1(\theta_1+\eta)} + \frac{\theta_2 \eta}{\theta_1 + \eta - \theta_2} e^{-\theta_2 z_1} \]

Also it can be seen that

\[ H(z_1) = 1 - p_1 e^{-z_1(\theta_1+\eta)} - q_1 e^{-\theta_2 z_1} \]

where \( p_1 = \frac{(\theta_1 - \theta_2)}{(\eta + \theta_1 - \theta_2)} \) and \( q_1 = \frac{\eta}{\theta_1 + \eta - \theta_2} \)

So it is seen that \( H(x) = p_1 e^{-x(\theta_1+\eta)} + q_1 e^{-\theta_2 x} \) \hspace{1cm} \ldots (5.1)
Now, since the virulence threshold is also satisfying the SCBZ property we have
\[ Z_2 \sim m(. \text{ as p.d.f. and } M(. \text{ is the c.d.f.}) \]

Since \( Z_2 \) satisfies the SCBZ property we have
\[ m(Z_2) = \lambda_1 e^{-\lambda_1 z_2} \text{ if } z_2 \leq \tau_0 \]
\[ = \lambda_2 e^{\tau_0(\lambda_2 - \lambda_1)} e^{-\lambda_2 z_2} \text{ if } z_2 > \tau_0 \]

where \( \tau_0 \sim \text{exp } (\gamma) \)

As above it can be shown that
\[ \overline{M}(\gamma) = p_2 e^{-\gamma(\lambda_1 + \alpha)} + q_2 e^{-\lambda_2 \gamma} \quad \text{(5.2)} \]

where \( p_2 = \frac{(\lambda_1 - \lambda_2)}{\lambda_1 + \gamma - \lambda_2} \quad q_2 = \frac{\gamma}{\gamma + \lambda_1 - \lambda_2} \)

Now,
\[ S(t) = \Pr \{ \text{that there are exactly } k \text{ contacts in } (0, t) \text{ and the antigenic diversity, virulence developed does not cross the respective threshold levels} \}

\[ S(t) = \sum_{k=0}^{\infty} \left[ F_k(t) - F_{k+1}(t) \right] \left[ \int_{0}^{\infty} g_k(x) \overline{H}(x) \, dx \right] \left[ \int_{0}^{\infty} g_k(y) \overline{M}(y) \, dy \right] \]

\[ = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)]. \]

\[ \left\{ \int_{0}^{\infty} g_k(x) p_1 e^{-x(\theta_1 + \eta)} + q_1 e^{-\theta_2 x} \, dx \right\} \left\{ \int_{0}^{\infty} g_k(y) p_2 e^{-y(\lambda_1 + \alpha)} + q_2 e^{-\lambda_2 y} \, dy \right\} \]

where \( F_k(t) - F_{k+1}(t) \) denotes the probability that there are exactly \( k \) contacts in \((0, t)\) as per renewal theory.

Hence it is seen that
\[ S(t) = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)]. \]
\[
\left[ \int_{0}^{\infty} g_k(x) p_1 e^{-x(\theta_1 + \eta)} + q_1 e^{-\theta_2 x} \, dx \right] \left[ \int_{0}^{\infty} g_k(y) p_2 e^{-y(\lambda_1 + \alpha)} + q_2 e^{-\lambda_2 y} \, dy \right] \\
= \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)]. \\
= \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] p_1 [g^*(\theta_1 + \eta)]^k + \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] q_1 [g^*(\theta_2)]^k \\
+ \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] p_2 [g^*(\lambda_1 + \alpha)]^k + \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] q_2 [g^*(\lambda_2)]^k \\
= T_1 + T_2 + T_3 + T_4 \\
\text{... (5.3)}
\]

Now,

\[T_1 = p_1 \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)][g^*(\theta_1 + \eta)]^k\]

\[= p_1 [1 - g^*(\theta_1 + \eta)] \sum_{k=1}^{\infty} F_k(t) g^*(\theta_1 + \eta)^{k-1}\]

\[T_2 = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] q_1 [g^*(\theta_2)]^k\]

\[= q_1 [1 - g^*(\theta_2)] \sum_{k=1}^{\infty} F_k(t) g^*(\theta_2)^{k-1}\]

\[T_3 = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] p_2 [g^*(\lambda_1 + \alpha)]^k\]

\[= p_2 [1 - g^*(\lambda_1 + \alpha)] \sum_{k=1}^{\infty} F_k(t) g^*(\lambda_1 + \alpha)^{k-1}\]
and

\[ T_4 = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] q_2 [g^*(\lambda_1)]^k \]

\[ = q_2 [1 - g^*(\lambda_2)] \sum_{k=1}^{\infty} F_k(t) g^*(\lambda_2)^{k-1} \]

Now, since \( L(t) = 1 - S(t) \) and taking the Laplace transform we get,

\[ L^*(s) = p_1[1 - g^*(\theta_1 + \eta)] f^*(s) \sum_{k=1}^{\infty} [f^*(s) g^*(\theta_1 + \eta)]^k \]

\[ + q_1[1 - g^*(\theta_2)] f^*(s) \sum_{k=1}^{\infty} [f^*(s) g^*(\theta_2)]^k \]

\[ + p_2[1 - g^*(\lambda_1 + \alpha)] f^*(s) \sum_{k=1}^{\infty} [f^*(s) g^*(\lambda_1 + \alpha)]^k \]

\[ + q_2[1 - g^*(\lambda_2)] f^*(s) \sum_{k=1}^{\infty} [f^*(s) g^*(\lambda_2)]^k \]

\[ = A + B + C + D \quad \text{on simplification} \quad \ldots \quad (5.4) \]

where \( A = \frac{p_1[1 - g^*(\theta_1 + \eta)] f^*(s)}{[1 - f^*(s) g^*(\theta_1 + \eta)]} \)

Similarly,

\[ B = \frac{q_1[1 - g^*(\theta_2)] f^*(s)}{[1 - f^*(s) g^*(\theta_2)]} \]
\[ C = \frac{p_2[1 - g^*(\lambda_1 + \alpha)]f^*(s)}{[1 - f^*(s)g^*(\lambda_1 + \alpha)]} \]

\[ D = \frac{q_2[1 - g^*(\lambda_2)]f^*(s)}{[1 - f^*(s)g^*(\lambda_2)]} \]

Now \( E(T) = \text{Expected time to seroconversion} \)

\[-\frac{d}{ds} L^*(s) \text{ Given } s = 0\]

\[ = -\left[ \frac{dA}{ds} + \frac{dB}{ds} + \frac{dC}{ds} + \frac{dD}{ds} \right]_{s=0} \]

Let us assume that

\( f(.) \sim \exp(\beta), g(.) \sim \exp(\gamma), q(.) \sim \exp(\delta) \)

\[ \therefore f^*(s) = \frac{\beta}{\beta + s}, g^*(\theta_2) = \frac{\gamma}{\gamma + \theta_2} g^*(\theta_1 + \eta) = \frac{\gamma}{\gamma + \theta_1 + \eta} \]

\[ q^*(\lambda_1 + \alpha) = \frac{\delta}{\delta + \lambda_1 + \alpha}, q^*(\lambda_2) = \frac{\delta}{\delta + \lambda_2} \]

Now \(-\frac{dA}{ds}_{s=0} \)

Now, it is seen that

\[ A = \frac{p_1[1 - g^*(\theta_1 + \eta)]f^*(s)}{[1 - f^*(s)g^*(\theta_1 + \eta)]} \]

\[ A = p_1 \left[ \frac{1 - \frac{\gamma}{\gamma + \theta_1 + \eta}}{1 - \left( \frac{\beta}{\beta + s} \right) \left( \frac{\gamma}{\gamma + \theta_1 + \eta} \right)} \right] \cdot p_1 \left[ \frac{\theta_1 + \eta}{\gamma + \theta_1 + \eta} \right] \cdot \beta (\beta + s)^{-1} \]

\[ = p_1 \left[ \frac{1 - \frac{\gamma}{\gamma + \theta_1 + \eta}}{1 - \left( \frac{\beta}{\beta + s} \right) \left( \frac{\gamma}{\gamma + \theta_1 + \eta} \right)} \right] \cdot p_1 \left[ \frac{\theta_1 + \eta}{\gamma + \theta_1 + \eta} \right] \cdot \beta (\beta + s)^{-1} \]
\[
B = \frac{q_1[1 - g^*(\theta_2)]f^*(s)}{[1 - f^*(s)g^*(\theta_2)]}
\]

\[
B = \frac{q_1[1 - \gamma/\gamma + \theta_2] \left( \frac{\beta}{\beta + s} \right) \left[ \frac{\beta \theta_2 + \beta \gamma + s(\theta_2 + \gamma) - \beta \beta}{(\beta + s)(\theta_2 + \gamma)} \right]}{1 - \left( \frac{\beta}{\beta + s} \right) \left( \frac{\gamma}{\gamma + \theta_2} \right)}
\]

\[
B = \frac{q_1[\theta_2] \beta(\beta + s)^{-1} \times (\beta + s)(\theta_2 + \gamma)}{(\theta_2 + \gamma) \beta \theta_2 + s(\theta_2 + \gamma)}
\]

\[
B = \frac{\beta q_1}{\beta \theta_2 + s(\theta_2 + \gamma)}
\]

\[
C = \frac{p_2[1 - g^*(\lambda_1 + \alpha)]f^*(s)}{[1 - f^*(s)g^*(\lambda_1 + \alpha)]}
\]

\[
C = \frac{p_2[1 - \delta/\delta + \lambda_1 + \alpha] \left( \frac{\beta}{\beta + s} \right) \left[ \frac{\beta \lambda_1 + \beta \alpha + \beta \delta + s(\delta + \lambda_1 + \alpha) - \beta \delta}{(\beta + s)(\delta + \lambda_1 + \alpha)} \right]}{1 - \left( \frac{\beta}{\beta + s} \right) \left( \frac{\delta}{\delta + \lambda_1 + \alpha} \right)}
\]

\[
C = \frac{p_2[\lambda_1 + \alpha] \times \beta(\beta + s)^{-1}(\beta + s)(\delta + \lambda_1 + \alpha)}{(\delta + \lambda_1 + \alpha) \beta(\lambda_1 + \alpha) + s(\delta + \lambda_1 + \alpha)}
\]

\[
C = \frac{\beta p_2[\lambda_1 + \alpha]}{\beta(\lambda_1 + \alpha) + s(\delta + \lambda_1 + \alpha)}
\]

\[
D = \frac{q_2[1 - g^*(\lambda_2)]f^*(s)}{[1 - f^*(s)g^*(\lambda_2)]}
\]

\[
D = \frac{q_2[1 - \gamma/\gamma + \lambda_2] \left( \frac{\beta}{\beta + s} \right) \left[ \frac{\beta \lambda_1 + \beta \alpha + \beta \delta + s(\delta + \lambda_1 + \alpha) - \beta \delta}{(\beta + s)(\delta + \lambda_1 + \alpha)} \right]}{1 - \left( \frac{\beta}{\beta + s} \right) \left( \frac{\gamma}{\gamma + \lambda_2} \right)}
\]

\[
D = \frac{q_2[\lambda_2] \beta(\beta + s)^{-1} \times (\beta + s)(\lambda_2 + \gamma)}{(\lambda_2 + \gamma) \beta \lambda_1 + s(\delta + \lambda_1 + \alpha)}
\]

\[
D = \frac{\beta q_2[\lambda_2]}{\beta(\lambda_1 + \alpha) + s(\delta + \lambda_1 + \alpha)}
\]
\[ q_2 \left[ 1 - \frac{\delta}{\delta + \lambda_2} \right] \left( \frac{\beta}{\beta + s} \right) = q_2 \left[ \frac{\lambda_2}{\delta + \lambda_2} \right] \frac{\beta (\beta + s)^{-1}}{\left[ 1 - \left( \frac{\beta}{\beta + s} \right) \left( \frac{\delta}{\delta + \lambda_2} \right) \right]}
\]

\[ = \frac{\beta q_2 \lambda_2}{\beta \lambda_2 + s(\delta + \lambda_2)} \]

\[ L^*(s) = \frac{\beta p_1[\theta_1 + \eta]}{\beta(\theta_1 + \eta) + s(\theta_1 + \eta)} + \frac{\beta q_1}{\beta \theta_2 + s(\theta_2 + \gamma)} + \frac{\beta p_2[\lambda_1 + \alpha]}{\beta(\lambda_1 + \alpha) + s(\lambda_1 + \alpha)} + \frac{\beta q_2 \lambda_2}{\beta \lambda_2 + s(\delta + \lambda_2)} \]

\[ E(T) = \left[ -\frac{d}{ds} L^*(s) \right]_{s=0} \]

\[ E(T^2) = \left[ -\frac{d^2}{ds} L^*(s) \right]_{s=0} \]

\[ V(T) = E(T^2) - [E(T)]^2 \]

Let \( T_1 = \frac{\beta p_1[\theta_1 + \eta]}{\beta(\theta_1 + \eta) + s(\theta_1 + \eta)} \)

\[ = \beta p_1[\theta_1 + \eta][\beta(\theta_1 + \eta) + s(\theta_1 + \eta)]^{-1} \]

\[ \frac{dT_1}{ds} = \beta p_1[\theta_1 + \eta](-1)[\beta(\theta_1 + \eta) + s(\theta_1 + \eta)]^{-2}(\gamma + \theta_1 + \eta) \]

\[ T_2 = \frac{\beta q_1}{\beta \theta_2 + s(\theta_2 + \gamma)} \]

\[ = \beta q_1[\beta \theta_2 + s(\theta_2 + \gamma)]^{-1} \]

\[ \frac{dT_2}{ds} = \beta q_1(-1)[\beta \theta_2 + s(\theta_2 + \gamma)]^{-2}(\theta_2 + \gamma) \]
\[ T_3 = \frac{\beta p_2[\lambda_1 + \alpha]}{\beta(\lambda_1 + \alpha) + s(\delta + \lambda_1 + \alpha)} \]

\[ = \beta p_2[\lambda_1 + \alpha][\beta(\lambda_1 + \alpha) + s(\delta + \lambda_1 + \alpha)]^{-1} \quad \ldots (5.8) \]

\[ \frac{dT_3}{ds} = \beta p_2[\lambda_1 + \alpha](-1)[\beta(\lambda_1 + \alpha) + s(\delta + \lambda_1 + \alpha)]^{-2}(\delta + \lambda_1 + \alpha) \]

\[ T_4 = \frac{\beta q_2 \lambda_2}{\beta \lambda_2 + s(\delta + \lambda_2)} \]

\[ = \beta q_2 \lambda_2[\beta \lambda_2 + s(\delta + \lambda_2)]^{-1} \quad \ldots (5.9) \]

\[ \frac{dT_4}{ds} = \beta q_2 \lambda_2(-1)[\beta \lambda_2 + s(\delta + \lambda_2)]^{-2}(\delta + \lambda_2) \]

Substituting (5.6), (5.7), (5.8), (5.9) in (5.5) we get,

\[ \frac{dl'(s)}{ds} = \beta p_1[\theta_1 + \eta](\theta_1 + \eta)(-1)[\beta(\theta_1 + \eta) + s(\gamma + \theta_1 + \eta)]^{-2}(\gamma + \theta_1 + \eta) \]

\[ + \beta q_1(-1)[\beta \theta_2 + s(\theta_2 + \gamma)]^{-2}(\theta_2 + \gamma) \]

\[ + \beta p_2[\lambda_1 + \alpha](-1)[\beta(\lambda_1 + \alpha) + s(\delta + \lambda_1 + \alpha)]^{-2}(\delta + \lambda_1 + \alpha) \]

\[ + \beta q_2 \lambda_2(-1)[\beta \lambda_2 + s(\delta + \lambda_2)]^{-2}(\delta + \lambda_2) \]

\[ \left[-\frac{dl'(s)}{ds}\right]_{s=0} = \beta p_1[\theta_1 + \eta](-1)[\beta(\theta_1 + \eta)]^{-2}(\gamma + \theta_1 + \eta) \]

\[ + \beta q_1(-1)[\beta \theta_2 + \gamma]^{-2}(\theta_2 + \gamma) + \beta p_2[\lambda_1 + \alpha](-1)[\beta(\lambda_1 + \alpha)]^{-2}(\delta + \lambda_1 + \alpha) \]

\[ + \beta q_2 \lambda_2(-1)[\beta \lambda_2]^{-2}(\delta + \lambda_2) \]

\[ = \frac{\beta p_1[\theta_1 + \eta](\gamma + \theta_1 + \eta)}{[\beta(\theta_1 + \eta)]^2} + \frac{\beta q_1(\theta_2 + \gamma)}{[\beta \theta_2]^2} \]
\[
\frac{d^2l^2(s)}{ds^2} = \beta p_1[\theta_1 + \eta](-1)(-2)[\beta(\theta_1 + \eta) + s(\gamma + \theta_1 + \eta)]^{-3} (\gamma + \theta_1 + \eta)^2 \\
+ \beta q_1(-1)(-2)[\beta \theta_2 + s(\theta_2 + \gamma)]^{-3}(\theta_2 + \gamma)^2 \\
+ \beta p_2[\lambda_1 + \alpha](-1)(-2)[\beta(\lambda_1 + \alpha) + s(\delta + \lambda_1 + \alpha)]^{-3}(\delta + \lambda_1 + \alpha)^2 \\
+ \beta q_2 \lambda_2(-1)(-2)[\beta \lambda_2 + s(\delta + \lambda_2)]^{-3}(\delta + \lambda_2)^2
\]

\[
\left[ \frac{d^2l^2(s)}{ds^2} \right]_{s=0} = 2\beta p_1[\theta_1 + \eta][\beta(\theta_1 + \eta)]^{-3} (\gamma + \theta_1 + \eta)^2 \\
+ 2\beta q_1[\beta \theta_2]^{-3}(\theta_2 + \gamma)^2 \\
+ 2\beta p_2[\lambda_1 + \alpha][\beta(\lambda_1 + \alpha)]^{-3}(\delta + \lambda_1 + \alpha)^2 \\
+ 2\beta q_2 \lambda_2[\beta \lambda_2]^{-3}(\delta + \lambda_2)^2
\]

\[
= \frac{2\beta p_1[\theta_1 + \eta](\gamma + \theta_1 + \eta)^2}{[\beta(\theta_1 + \eta)]^3} + \frac{2\beta q_1(\theta_2 + \gamma)^2}{[\beta \theta_2]^3} \\
+ \frac{2\beta p_2[\lambda_1 + \alpha](\delta + \lambda_1 + \alpha)^2}{[\beta(\lambda_1 + \alpha)]^3} + \frac{2\beta q_2 \lambda_2(\delta + \lambda_2)^2}{[\beta \lambda_2]^3}
\]
\[ E(T^2) = \frac{2p_1(\gamma + \theta_1 + \eta)^2}{\beta^2(\theta_1 + \eta)^2} + \frac{2q_1(\theta_2 + \gamma)^2}{\beta^2\theta_2^3} \]
\[ + \frac{2p_2(\lambda_1 + \alpha)(\delta + \lambda_1 + \alpha)^2}{\beta^2(\lambda_1 + \alpha)^3} + \frac{2q_2\lambda_2(\delta + \lambda_2)^2}{\beta^2\lambda_2^3} \]...

\[ V(T) = E(T^2) - [E(T)]^2 \]
\[ = \left[ \frac{2p_1(\gamma + \theta_1 + \eta)^2}{\beta^2(\theta_1 + \eta)^2} + \frac{2q_1(\theta_2 + \gamma)^2}{\beta^2\theta_2^3} + \frac{2p_2(\lambda_1 + \alpha)(\delta + \lambda_1 + \alpha)^2}{\beta^2(\lambda_1 + \alpha)^3} \right. \]
\[ + \left. \frac{2q_2\lambda_2(\delta + \lambda_2)^2}{\beta^2\lambda_2^3} \right] \]
\[ - \left[ \frac{p_1(\gamma + \theta_1 + \eta)}{\beta(\theta_1 + \eta)} + \frac{q_1(\theta_2 + \gamma)}{\beta\theta_2^2} + \frac{p_2(\lambda_1 + \alpha)(\delta + \lambda_1 + \alpha)}{\beta(\lambda_1 + \alpha)^2} + \frac{q_2\lambda_2(\delta + \lambda_2)}{\beta\lambda_2^2} \right]^2 \]...

### 5.5 Numerical Illustration.

The behaviour of \( E(T) \) and \( V(T) \) due to the changes in the different parameters associated with the distribution of the random variables in the model is explained by taking a numerical example.

**Table 1: Changes of \( E(T) \) and \( V(T) \) due to the variations in \( \gamma \)**

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( E(T) )</th>
<th>( V(T) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>6.0662</td>
<td>28.1175</td>
</tr>
<tr>
<td>1.0</td>
<td>6.4324</td>
<td>28.2655</td>
</tr>
<tr>
<td>1.5</td>
<td>6.7986</td>
<td>28.8538</td>
</tr>
<tr>
<td>2.0</td>
<td>7.1648</td>
<td>29.9826</td>
</tr>
<tr>
<td>2.5</td>
<td>7.5310</td>
<td>31.3518</td>
</tr>
</tbody>
</table>
Fig. 1: Changes of $E(T)$ and $V(T)$ due to the variations in $\gamma$

Table 2: Changes of $E(T)$ and $V(T)$ due to the variations in $\theta_1$

$\theta_2 = 2.5, \, p_1 = 0.2, \, p_2 = 0.8, \, q_1 = 1.4, \, q_2 = 0.4, \gamma = 1.5,$

$\eta = 1.4, \, \beta = 1.4, \, \lambda_1 = 1.0, \, \lambda_2 = 0.6, \, \alpha = 0.5, \, \delta = 1.2$

<table>
<thead>
<tr>
<th>$\theta_1$</th>
<th>$E(T)$</th>
<th>$V(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>6.7877</td>
<td>28.6614</td>
</tr>
<tr>
<td>1.0</td>
<td>6.6220</td>
<td>28.5069</td>
</tr>
<tr>
<td>1.5</td>
<td>6.5789</td>
<td>28.4479</td>
</tr>
<tr>
<td>2.0</td>
<td>6.3484</td>
<td>28.2265</td>
</tr>
<tr>
<td>2.5</td>
<td>6.1258</td>
<td>28.0213</td>
</tr>
</tbody>
</table>
Fig. 2: Changes of $E(T)$ and $V(T)$ due to the variations in $\theta_1$

Table 3: Changes of $E(T)$ and $V(T)$ due to the variations in $\theta_2$

\[
\begin{align*}
\theta_1 &= 1.5, \quad p_1 = 0.2, \quad p_2 = 0.8, \quad q_1 = 1.4, \quad q_2 = 0.4, \quad \gamma = 1.5, \\
\eta &= 1.4, \quad \beta = 1.4, \quad \lambda_1 = 1.0, \quad \lambda_2 = 0.6, \quad \alpha = 0.5, \quad \delta = 1.2
\end{align*}
\]

<table>
<thead>
<tr>
<th>$\theta_2$</th>
<th>$E(T)$</th>
<th>$V(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>8.7307</td>
<td>18.3139</td>
</tr>
<tr>
<td>1.0</td>
<td>7.3307</td>
<td>25.4001</td>
</tr>
<tr>
<td>1.5</td>
<td>6.6027</td>
<td>28.4765</td>
</tr>
<tr>
<td>2.0</td>
<td>6.1641</td>
<td>30.1774</td>
</tr>
<tr>
<td>2.5</td>
<td>5.8736</td>
<td>31.2571</td>
</tr>
</tbody>
</table>
Fig. 3: Changes of $E(T)$ and $V(T)$ due to the variations in $\theta_2$

Table 4: Changes of $E(T)$ and $V(T)$ due to the variations in $\eta$

$$\theta_1 = 1.5, \theta_2 = 2.5, p_2 = 0.8, q_1 = 1.4, q_2 = 0.4\gamma = 1.5,$$

$$p_1 = 0.2, \beta = 1.4, \lambda_1 = 1.0, \lambda_2 = 0.6, \alpha = 0.5, \delta = 1.2$$

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>$E(T)$</th>
<th>$V(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>6.872</td>
<td>28.7172</td>
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<tr>
<td>1.0</td>
<td>6.712</td>
<td>28.6003</td>
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<td>1.5</td>
<td>6.472</td>
<td>28.5416</td>
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<tr>
<td>2.0</td>
<td>6.243</td>
<td>28.4245</td>
</tr>
<tr>
<td>2.5</td>
<td>6.022</td>
<td>28.2213</td>
</tr>
</tbody>
</table>
Fig 4: Changes of $E(T)$ and $V(T)$ due to the variations in $\eta$

Table 5: Changes of $E(T)$ and $V(T)$ due to the variations in $\beta$

$\theta_1 = 1.5$, $\theta_2 = 2.5$, $p_2 = 0.8$, $q_1 = 1.4$, $q_2 = 0.4\gamma = 1.5$, $p_1 = 0.2$, $\eta = 1.4$, $\lambda_1 = 1.0$, $\lambda_2 = 0.6$, $\alpha = 0.5$, $\delta = 1.2$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$E(T)$</th>
<th>$V(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.2631</td>
<td>18.206</td>
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<tr>
<td>1.0</td>
<td>2.6315</td>
<td>4.5516</td>
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<tr>
<td>1.5</td>
<td>1.7543</td>
<td>2.0229</td>
</tr>
<tr>
<td>2.0</td>
<td>1.3157</td>
<td>1.1379</td>
</tr>
<tr>
<td>2.5</td>
<td>1.0526</td>
<td>0.7282</td>
</tr>
</tbody>
</table>
Fig.5: Changes of $E(T)$ and $V(T)$ due to the variations in $\beta$

Table 6: Changes of $E(T)$ and $V(T)$ due to the variations in $\lambda_1$

$\theta_1 = 1.5$, $\theta_2 = 2.5$, $q_1 = 1.4$, $\beta = 0.4$, $q_2 = 0.4\gamma = 1.5$, $p_1 = 0.2$, $\eta = 1.4$, $p_2 = 0.2$, $\lambda_2 = 0.6$, $\alpha = 0.5$, $\delta = 1.2$

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$E(T)$</th>
<th>$V(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>6.8789</td>
<td>30.1106</td>
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<tr>
<td>1.0</td>
<td>6.6789</td>
<td>28.8222</td>
</tr>
<tr>
<td>1.5</td>
<td>6.5789</td>
<td>28.4479</td>
</tr>
<tr>
<td>2.0</td>
<td>6.3189</td>
<td>28.3098</td>
</tr>
<tr>
<td>2.5</td>
<td>6.2789</td>
<td>28.2537</td>
</tr>
</tbody>
</table>
Fig 6: Changes of $E(T)$ and $V(T)$ due to the variations in $\lambda_1$

Table 7: Changes of $E(T)$ and $V(T)$ due to the variations in $\alpha$

$\theta_1 = 1.5, \theta_2 = 2.5, q_1 = 1.4, \beta = 0.4, q_2 = 0.4\gamma = 1.5,$

$p_1 = 0.2, \eta = 1.4, p_2 = 0.2, \lambda_2 = 0.6, \lambda_3 = 1.5, \delta = 1.2$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$E(T)$</th>
<th>$V(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>6.9789</td>
<td>28.4479</td>
</tr>
<tr>
<td>1.0</td>
<td>6.7189</td>
<td>28.3098</td>
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<tr>
<td>1.5</td>
<td>6.6789</td>
<td>28.2537</td>
</tr>
<tr>
<td>2.0</td>
<td>6.4503</td>
<td>28.1313</td>
</tr>
<tr>
<td>2.5</td>
<td>6.3289</td>
<td>28.0241</td>
</tr>
</tbody>
</table>
Fig. 7: Changes of $E(T)$ and $V(T)$ due to the variations in $\alpha$

Table 8: Changes of $E(T)$ and $V(T)$ due to the variations in $\lambda_2$

$\theta_1 = 1.5, \theta_2 = 2.5, q_1 = 1.4, \beta = 0.4, q_2 = 0.4\gamma = 1.5,$

$p_1 = 0.2, \eta = 1.4, p_2 = 0.2, \alpha = 0.5, \lambda_1 = 1.5, \delta = 1.2$

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$E(T)$</th>
<th>$V(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>6.9789</td>
<td>35.8248</td>
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<tr>
<td>1.0</td>
<td>5.7789</td>
<td>17.5342</td>
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<tr>
<td>1.5</td>
<td>5.3789</td>
<td>13.9973</td>
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<tr>
<td>2.0</td>
<td>5.1789</td>
<td>12.7089</td>
</tr>
<tr>
<td>2.5</td>
<td>5.0589</td>
<td>12.0894</td>
</tr>
</tbody>
</table>
Fig. 8: Changes of E(T) and V(T) due to the variations in $\lambda_2$

Table 9: Changes of E(T) and V(T) due to the variations in $\delta$

$\theta_1 = 1.5, \theta_2 = 2.5, q_1 = 1.4, \beta = 0.4, q_2 = 0.4, \gamma = 1.5,$

$p_1 = 0.2, \eta = 1.4, p_2 = 0.2, \alpha = 0.5, \lambda_1 = 1.5, \lambda_2 = 0.6$

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>E(T)</th>
<th>V(T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.2372</td>
<td>13.6131</td>
</tr>
<tr>
<td>1.0</td>
<td>6.1955</td>
<td>23.1254</td>
</tr>
<tr>
<td>1.5</td>
<td>7.1539</td>
<td>38.0578</td>
</tr>
<tr>
<td>2.0</td>
<td>8.1122</td>
<td>58.4104</td>
</tr>
<tr>
<td>2.5</td>
<td>9.0705</td>
<td>84.1832</td>
</tr>
</tbody>
</table>
5.6 Conclusion

On the basis of the numerical example worked out for this model the following conclusions can be drawn.

1. If $\gamma$ which is the parameter of the density function of $g(.)$ of the random variable $x_i$ which is the contribution to antigenic diversity shows an increase, then the mean contribution per contact is $\frac{1}{\gamma}$ due to fact that $g(.)$ has exponential distribution. Therefore $E(x) = \frac{1}{\gamma}$ decreases as $\gamma$ increases. So $E(T)$ increases and also the variance $V(T)$. It is seen from table 1.

2. $\theta_1$ is the parameter of the distribution of a random variable $Z_1$ which denotes the antigenic diversity threshold. It satisfies the SCBZ property and the parameter $\theta_1$ is prior to the truncation point $\tau_0$. It follows exponential distribution also. As $\theta_1$ increases $E(T)$ and also the $V(T)$ show a very small decrease. Table 2 gives the picture.
3. When \( \theta_2 \) the parameter of the distribution \( Z_1 \) beyond the truncation point \( \tau_0 \) increases a small decrease in \( E(T) \) is noted. But the \( V(T) \) shows an increase. It is observed in table 3.

4. When the values of \( \eta \) which is the parameter of the random variable \( \tau_0 \) which is truncation point follows exponential distribution with parameter \( \eta \), then an increase in \( \eta \) results in a small decrease in both \( E(T) \) and \( V(T) \) as observed from table 4.

5. The random variable \( U_i \) denotes the interarrival times between successive contacts. It follows exponential distribution with parameter \( \beta \). So the mean interarrival time is \( E(U) = \frac{1}{\beta} \). It implies that as \( \beta \) increases \( E(U) \) decreases so the interarrival time between contacts are smaller. Hence there will be more contribution to antigenic diversity and virulence. Therefore as \( \beta \) increases, both \( E(T) \) and \( V(T) \) decrease as observed from table 5.

6. The virulence threshold satisfies the SCBZ property and follows exponential distribution. The parameter is \( \lambda_1 \) prior to the truncation point \( \tau_0 \) and it is \( \lambda_2 \) after the truncation point. As \( \lambda_1 \) increases both \( E(T) \) and \( V(T) \) show a small decrease. It is observed in table 6. Similarly as \( \lambda_2 \) increases \( E(T) \) decreases and also \( V(T) \) as observed in table 7.

7. If \( \alpha \) which is the parameter of the truncation point of the virulence threshold which satisfies the SCBZ property, and if this random variable has exponential distribution with parameter \( \alpha \) then \( E(T) \) and \( V(T) \) shows a small decrease. Table 8 depicts the situation.
8. The random variable denoting virulence threshold has $\lambda_2$ as the parameter after the truncation point. If it increases both $E(T)$ and $V(T)$ decrease as observed in table 9.