8.1 Introduction

The exponential distribution does not provide a reasonable parametric fit for some practical applications where the underlying hazard rates are non constant, presenting monotone shapes. In recent years, in order to overcome such a problem, new classes of models were introduced based on modifications of the exponential distribution. The advantages from both stochastic models and statistical models are efficiently to estimate the unknown parameters of the distribution the lack involved in real life situations. The antigenic diversity threshold is a particular level of the antigenic diversity of the invading antigens beyond which the human immune system breaks down and a person becomes seropositive. The expected time to seroconversion is derived under the assumption that the antigenic diversity threshold comprises of two components namely the natural antigenic diversity threshold level of human immune system and the threshold components due to use of ART has been discussed by Palanivel et.al. (2009).

The mathematical model is developed to obtain the expected time of breakdown point which in other words be called as the time to cross the threshold level of antigenic diversity of the invading antigens. The details regarding the expected time to cross the so called threshold value can be obtained in Essary et.al., (1973). Sathiyamoorthi and Kannan (1998), Elangovan and Ramajayam (2012), Elangovan and Ramajayam (2014), have applied this concepts in several models. In this paper the expected time and variance to cross the threshold level through two modes of transmission using three parameter Generalized Rayleigh Distribution has been derived. Examples of numerical type are given to strengthen the results.

8.2 Assumptions of the Model

1. A person is exposed to HIV infection. At every epoch of contact with an infected there is some contribution to the antigenic diversity.

2. Anti Retroviral Therapy is administed to the infected.

3. There is a particular level of antigenic diversity of the invading antigens and it is called the antigenic diversity threshold. If antigenic diversity crosses this threshold the seroconversion takes place.

4. The interarrival times between the successive contacts are random variables which are identically independently distributed.

8.3 Notations

\( X_i \) : a continuous random variable denoting the amount of contribution to the antigenic diversity of the invading antigens on contact in its an infected the \( i^{th} \) occasion of \( i = 1, 2, 3, \ldots k \) and \( X' \) \( S \) are i.i.d and \( X_i = X \) for all \( i \).
continuous random variable denoting the threshold level of antigenic diversity which follows three parameter Generalized Rayleigh distribution.

The probability density functions (p.d.f) of $X_i$

$g(.) :$ The k-fold convolution of $g(.)$ i.e., p.d.f. of $\sum_{i=1}^{k} X_i$

$g^*(.) :$ Laplace transform of $g(.)$

$g_k^*(.) :$ Laplace transform of $g_k(.)$

$h(.) :$ The probability density function of random threshold level which has three parameter generalized Rayleigh distribution and $H(.)$ is the corresponding Probability generating functions.

$U :$ a continuous random variable denoting the inter-arrival times between contacts, which has p.d.f $f(.)$ and c.d.f $F(.)$

$f(.) :$ p.d.f. of random variable $U$ with corresponding Probability Generating function.

$V_k(t) : F_k(t) - F_{k+1}(t)$

$F_k(t) :$ Probability that there are exactly ‘$k$’ policies decisions in $(0,t]$

$S(.) :$ The survivor function i.e. $P(T > t)$

$L(t) : 1 - S(t)$

**8.4 Model Description and Results**

The human immune systems fail as and when the total contribution to antigenic diversity of the invading antigens due to successive exposures is likely to fail when total exceeds the threshold. The c.d.f of the random threshold is given by
\[ H(x) = \left[1 - e^{-\lambda_1(x-\mu_1)^2}\right]^{\alpha} \left[1 - e^{-\lambda_2(x-\mu_2)^2}\right]^{\alpha}; \]
\[ = 1 - e^{-\lambda_2(x-\mu_2)^2} - e^{-\lambda_1(x-\mu_1)^2} + e^{-(\lambda_1(x-\mu_1)^2 + \lambda_2(x-\mu_2)^2)} \]
\[ \bar{H}(X) = e^{-\lambda_1(x-\mu_1)^2} + e^{-\lambda_2(x-\mu_2)^2} - e^{-(\lambda_1(x-\mu_1)^2 + \lambda_2(x-\mu_2)^2)} \]

Now \( P(\sum_{i=1}^{k} x_i < y) = \int_{0}^{\infty} g_k(x)\bar{H}(X)dx \) \quad \ldots \quad (8.1)

\[ = \int_{0}^{\infty} g_k(x)\left[e^{-\lambda_1(x-\mu_1)^2} + e^{-\lambda_2(x-\mu_2)^2} - e^{-(\lambda_1(x-\mu_1)^2 + \lambda_2(x-\mu_2)^2)}\right]dx \]

The survival function \( S(t) \) is

\[ S(t) = P(T > t) \]
\[ = \sum_{k=0}^{\infty} P(\text{there are exactly } k \text{ contacts } (0, t)) \]

* \( P(\text{the total contribution antigenic diversity due to the contacts does not exceed threshold}) \)

It is also known from renewal process that

\[ P(\text{exactly } k \text{ policy contacts in } (0, t)) = F_k(t) - F_{k+1}(t) \quad \text{with} \quad F_0(t) = 1 \]

\[ P(T > t) = \sum_{k=0}^{\infty} V_k(t)P\left(\sum_{i=1}^{k} x_i < y\right) \]

\[ = \sum_{k=0}^{\infty} \left[ F_k(t) - F_{k+1}(t) \right] [g^*\lambda_1(1 - \mu_1)^2]^k \]
\[ + \sum_{k=0}^{\infty} \left[ F_k(t) - F_{k+1}(t) \right] [g^*\lambda_2(1 - \mu_2)^2]^k \]
\[ - \sum_{k=0}^{\infty} \left[ F_k(t) - F_{k+1}(t) \right] [g^*(\lambda_1(1 - \mu_1)^2 + \lambda_2(1 - \mu_2)^2)]^k \quad \ldots \quad (8.2) \]
\[ P(T < t) = L(t) \]

Hence \( L(t) = 1 - S(t) \)

\[
= 1 - \left\{ \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] [g^* \lambda_1 (1 - \mu_1)^2]^k + \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] [g^* \lambda_2 (1 - \mu_2)^2]^k - \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] [g^* (\lambda_1 (1 - \mu_1)^2 + \lambda_2 (1 - \mu_2)^2)]^k \right\}
\]

\[
= 1 - \left\{ \frac{1 - g^* (\lambda_1 + \lambda_1 \mu_1^2 - 2\lambda_1 \mu_1)}{[1 - g^* (\lambda_1 + \lambda_1 \mu_1^2 - 2\lambda_1 \mu_1)] f^* (s)} + \frac{1 - g^* (\lambda_2 + \lambda_2 \mu_2^2)}{[1 - g^* (\lambda_2 + \lambda_2 \mu_2^2)] f^* (s)} \right\}
\]

On simplification

Let the random variable \( U \) denoting inter arrival time which follows exponential with parameter \( c \). Now \( f^* (s) = \left( \frac{c}{c+s} \right) \), and substituting in the above equation (3) we get,

\[
= 1 - \left\{ \frac{c [1 - g^* (\lambda_1 + \lambda_1 \mu_1^2 - 2\lambda_1 \mu_1)]}{[c + s - g^* (\lambda_1 + \lambda_1 \mu_1^2 - 2\lambda_1 \mu_1)c]} + \frac{c [1 - g^* (\lambda_2 + \lambda_2 \mu_2^2 - 2\lambda_2 \mu_2)]}{[c + s - g^* (\lambda_2 + \lambda_2 \mu_2^2 - 2\lambda_2 \mu_2)c]} \right\}
\]

\[
E(T) = \frac{-d}{ds} l^* (s) \quad \text{Given } S = 0
\]

On simplification we get
$E(T) = \frac{1}{c[1 - g^*(\lambda_1 + \lambda_1 \mu_1^2 - 2\lambda_1 \mu_1)]} + \frac{1}{c[1 - g^*(\lambda_2 + \lambda_2 \mu_2^2 - 2\lambda_2 \mu_2)]}$

$$= \frac{1}{c[1 - g^*(\lambda_1 + \lambda_1 \mu_1^2 - 2\lambda_1 \mu_1 + \lambda_2 + \lambda_2 \mu_2^2 - 2\lambda_2 \mu_2)]} \quad \ldots (8.3)$$

Let $g^*(\cdot) \sim \exp(\mu)$, $g^*(\lambda_1) = \frac{\mu_1}{\mu_1 + \lambda_1}$, $g^*(\lambda_2) = \frac{\mu_2}{\mu_2 + \lambda_2}$, $g^*(\lambda_1 \mu_1) = \frac{\mu_1}{\mu_1 + \lambda_1 \mu_1^2}$

$g^*(\lambda_2 \mu_2^2) = \frac{\mu_2}{\mu_2 + \lambda_2 \mu_2^2}$, $g^*(2\mu_3 \lambda_1) = \frac{\mu_1}{\mu_1 + 2\mu_1 \lambda_1}$, $g^*(2\mu_2 \lambda_2) = \frac{\mu_2}{\mu_2 + 2\mu_2 \lambda_2}$

$$E(T) = \frac{1}{c \left[ 1 - \left( \frac{\mu_1}{\mu_1 + \lambda_1} + \frac{\mu_1}{\mu_1 + \lambda_1 \mu_1^2} - \frac{\mu_1}{\mu_1 + 2\mu_1 \lambda_1} \right) \right]}$$

$$+ \frac{1}{c \left[ 1 - \left( \frac{\mu_2}{\mu_2 + \lambda_2} + \frac{\mu_2}{\mu_2 + \lambda_2 \mu_2^2} - \frac{\mu_2}{\mu_2 + 2\mu_2 \lambda_2} \right) \right]}$$

$$- \frac{1}{c \left[ 1 - \left( \frac{\mu_1}{\mu_1 + \lambda_1} + \frac{\mu_1}{\mu_1 + \lambda_1 \mu_1^2} - \frac{\mu_1}{\mu_1 + 2\mu_1 \lambda_1} + \frac{\mu_2}{\mu_2 + \lambda_2} + \frac{\mu_2}{\mu_2 + \lambda_2 \mu_2^2} - \frac{\mu_2}{\mu_2 + 2\mu_2 \lambda_2} \right) \right]}$$

Let,

$I_1 = \frac{1}{c \left[ 1 - \left( \frac{\mu_1}{\mu_1 + \lambda_1} + \frac{\mu_1}{\mu_1 + \lambda_1 \mu_1^2} - \frac{\mu_1}{\mu_1 + 2\mu_1 \lambda_1} \right) \right]}$  

$$= \frac{\lambda_1^2 + \mu_1^2 + 2\mu_1^3 + \lambda_1 \mu_3 + 2\lambda_1^2 \lambda_2^2 + \mu_4 + 2\lambda_1 \mu_4 + 2\lambda_1 \mu_1^2}{c \left[ \mu_1^2 + 2\lambda_1^2 \mu_1^3 - 2\mu_1^3 + 2\lambda_1 \mu_1^3 + \mu_4 \right]} \quad \ldots (8.4)$

$I_2 = \frac{1}{c \left[ 1 - \left( \frac{\mu_2}{\mu_2 + \lambda_2} + \frac{\mu_2}{\mu_2 + \lambda_2 \mu_2^2} - \frac{\mu_2}{\mu_2 + 2\mu_2 \lambda_2} \right) \right]}$  

$$= \frac{\lambda_2^2 + \mu_2^2 + 2\mu_2^3 + \lambda_2 \mu_3 + 2\lambda_2^2 \lambda_2^2 + \mu_4 + 2\lambda_2 \mu_4 + 2\lambda_2 \mu_2^2}{c \left[ \mu_2^2 + 2\lambda_2^2 \mu_2^3 - 2\mu_2^3 + 2\lambda_2 \mu_2^3 + \mu_4 \right]} \quad \ldots (8.5)$
\[ I_3 = - \frac{1}{c \left[ 1 - \left( \frac{\mu_1}{\mu_1 + \lambda_1} - \frac{\mu_1}{\mu_1 + \lambda_2} - \frac{\mu_2}{\mu_2 + \lambda_1} - \frac{\mu_2}{\mu_2 + \lambda_2} \right) \right]} \] ... (8.6)

On simplification we get

\[ T_1 = \lambda_1 \lambda_2 + \lambda_1 \mu_2 + \lambda_1 \lambda_2^2 \mu_2 + \lambda_1 \lambda_2 \mu_2^2 + \lambda_1 \lambda_2^2 \mu_2 + \lambda_1 \lambda_2^2 \mu_2 + 2 \lambda_1 \lambda_2 \mu_2 + 2 \lambda_1 \lambda_2^3 \mu_2 \]

\[ + 2 \lambda_2^2 \lambda_1 \mu_2^2 + \mu_1 \lambda_2 + \mu_1 \mu_2 + \mu_1 \lambda_2 \mu_2^2 + \mu_1 \lambda_2 \mu_2^2 + 2 \mu_1 \lambda_2^2 + 2 \mu_1 \mu_2 \lambda_2 \]

\[ + 2 \mu_1 \mu_2 \lambda_2^2 \mu_2^3 + 2 \mu_1 \mu_2 \lambda_2 \mu_2^2 + \lambda_1 \lambda_2 \mu_1 + \mu_1 \mu_2 \lambda_2 \lambda_2^2 + \mu_1 \mu_2 \lambda_2 \lambda_2^2 + \lambda_1 \lambda_2 \mu_1 + 2 \lambda_1 \lambda_2^3 \mu_1 \]

\[ + 2 \mu_1 \mu_2 \lambda_2 \lambda_2 \lambda_1^2 + 2 \mu_1 \mu_2 \lambda_1 \lambda_2 \lambda_2^2 + 2 \lambda_1 \mu_2 \lambda_1 \lambda_2 \mu_2^2 + \lambda_1 \lambda_2 \mu_1 + \lambda_1 \mu_2 \mu_1^2 \]

\[ + \lambda_1 \mu_2 \mu_1^2 \lambda_2 \lambda_2^2 + \lambda_1 \lambda_2 \mu_2^2 \mu_1^2 + 2 \mu_1 \lambda_1 \lambda_2 \lambda_2^2 + 2 \mu_1 \lambda_1 \mu_2 \lambda_2 + 2 \mu_1 \lambda_1 \lambda_2^3 \mu_2 \]

\[ + 4 \lambda_1 \lambda_2 \mu_2 + 2 \mu_1 \lambda_2 \lambda_2 \mu_2^2 + 2 \lambda_1 \lambda_2 \mu_2 + 2 \lambda_1 \lambda_2 \mu_2 + 2 \lambda_1 \lambda_2 \mu_2 + 2 \lambda_1 \lambda_2 \mu_2 \]

\[ + 4 \lambda_1 \lambda_2 \mu_2 + 4 \lambda_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_2 \lambda_2 \mu_2 + 2 \lambda_1 \lambda_2 \mu_1 + 2 \lambda_1 \mu_2 \mu_1 + 2 \lambda_1 \mu_2 \mu_1 \]

\[ + 2 \lambda_1 \mu_1 \lambda_2 \mu_2^2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 \]

\[ + 2 \lambda_1 \lambda_2 \mu_2 + 2 \lambda_1 \mu_2 \lambda_2 \mu_2^2 + 2 \lambda_1 \mu_1 \lambda_2 \mu_2 + 2 \lambda_1 \mu_2 \lambda_2 \mu_2 + 2 \lambda_1 \mu_2 \lambda_2 \mu_2 + 2 \lambda_1 \mu_2 \lambda_2 \mu_2 \]

\[ + 2 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 \]

\[ + 2 \lambda_1 \mu_1 \lambda_2 \mu_2 + 2 \lambda_1 \mu_1 \lambda_2 \mu_2 + 2 \lambda_1 \mu_1 \lambda_2 \mu_2 + 2 \lambda_1 \mu_1 \lambda_2 \mu_2 + 2 \lambda_1 \mu_1 \lambda_2 \mu_2 \]

\[ + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 \]
\[ T_2 = \lambda_1\lambda_2 + 2\lambda_2^2\lambda_1\mu_2 + \lambda_1\lambda_2\mu_2^2 + 2\lambda_1\lambda_2\mu_2 + 2\lambda_1^2\lambda_2\mu_1 
+ 2\lambda_1^2\lambda_2\mu_1\mu_2 + 2\lambda_1^2\lambda_2\mu_1\mu_2 + 4\lambda_1^2\lambda_2\mu_1\mu_2 + 4\lambda_1^2\lambda_2\mu_1\mu_2 + 2\lambda_2^2\lambda_1^2\mu_1\mu_2^2 
+ \lambda_1\lambda_2\mu_1^2 + \lambda_1\lambda_2\mu_1^2\mu_2^2 + 2\lambda_1\mu_1^2\lambda_2^2 + 2\lambda_1\lambda_2\mu_1^2\mu_2 + 2\lambda_1\lambda_2^3\mu_1^2\mu_2 
+ 2\lambda_1\lambda_2\mu_1^2 + 4\lambda_2^2\lambda_1\mu_1 + 4\lambda_1\mu_1\lambda_2^3\mu_2 + 2\lambda_1^3\mu_1\lambda_2 + 2\lambda_1^3\mu_1\mu_2 + 
4\lambda_1^3\lambda_2^2\mu_1\mu_2 + 2\lambda_1^3\lambda_2\mu_1\mu_2^2 + 4\lambda_1^3\lambda_2^2\mu_1 + 4\lambda_1^3\mu_1\lambda_2^2 + 4\lambda_1^3\lambda_2^3\mu_1\mu_2 
+ 2\lambda_1^2\mu_1^2\mu_2 - 2\lambda_1^2\lambda_2^2\mu_2^2 - 2\lambda_1\mu_1\lambda_2 - 4\lambda_1\mu_1\mu_2 - 4\lambda_1\lambda_2^2\mu_1\mu_2 - 4\lambda_1\lambda_2\mu_1\mu_2^2 
- 12\lambda_1\mu_1\lambda_2^2 - 12\lambda_1\lambda_2\mu_1\mu_2 - 8\lambda_1\lambda_2^3\mu_1\mu_2 - 8\lambda_1\mu_1\lambda_2^3\mu_2 - 2\lambda_1^2\mu_1^2\mu_2 
+ \lambda_1^2\mu_1\mu_2 + 2\lambda_1^2\lambda_2^2\mu_1 - \mu_1\mu_2 - 2\lambda_1^2\mu_2 - 2\mu_1\mu_2\lambda_1^3 - 2\mu_1^2\mu_2\lambda_1^2 - 2\mu_1\lambda_2^2 
- 2\lambda_2^2\lambda_1\mu_1^2 - 4\lambda_2^2\lambda_1^2 - 4\lambda_2^2\lambda_1^3\mu_1 - 4\lambda_2^2\lambda_1^2\mu_1^2 - 4\lambda_2^2\lambda_1\mu_2 - 4\lambda_1\mu_2\lambda_2 
- 4\mu_1\mu_2\lambda_1^2\lambda_2 - 4\lambda_2\lambda_1\mu_1^2\mu_2 - 8\lambda_1^2\lambda_2\mu_2 - 8\lambda_1^3\mu_1\mu_2\lambda_2 - 8\lambda_2\mu_2\lambda_1^2\lambda_1^2 
- 2\lambda_2^2\mu_2^2\mu_1 - 4\lambda_2^2\mu_2^2\lambda_1^2 - 4\lambda_2^2\mu_2^2\lambda_1^2\mu_1^2 + \lambda_2^2\mu_1\mu_2 + \lambda_2^2\lambda_1^2\mu_2\mu_1 
+ 2\lambda_1\lambda_2\mu_1^2\mu_2 + 2\lambda_2^2\lambda_1^2\mu_2 + 2\lambda_2^2\lambda_2^2\mu_1^2 
I_3 = -\frac{T_1}{c(T_2)} \quad \ldots (8.7)

Substituting (8.5), (8.6), (8.7) in the equation (8.4) we get

\[ E(T) = I_1 + I_2 - I_3 \]

\[ E(T^2) = \frac{d^2}{ds^2} I^*(s) \quad \text{Given S}=0 \]
\[ E(T^2) = \frac{2}{c^2 \left[ 1 - \left( \frac{\mu_1}{\mu_1 + \lambda_1} + \frac{\mu_1^2}{\mu_1 + \lambda_1 \mu_1^2} - \frac{\mu_1^3}{\mu_1 + 2\mu_1 \lambda_1} \right) \right]^2} \]

\[ + \frac{2}{c^2 \left[ 1 - \left( \frac{\mu_2}{\mu_2 + \lambda_2} + \frac{\mu_2^2}{\mu_2 + \lambda_2 \mu_2^2} - \frac{\mu_2^3}{\mu_2 + 2\mu_2 \lambda_2} \right) \right]^2} \]

\[ - \frac{2}{c^2 \left[ 1 - \left( \frac{\mu_1}{\mu_1 + \lambda_1} + \frac{\mu_1^2}{\mu_1 + \lambda_1 \mu_1^2} - \frac{\mu_1^3}{\mu_1 + 2\mu_1 \lambda_1} \right) \right]^2} \]

Let,

\[ J_1 = \frac{2}{c^2 \left[ 1 - \left( \frac{\mu_1}{\mu_1 + \lambda_1} + \frac{\mu_1^2}{\mu_1 + \lambda_1 \mu_1^2} - \frac{\mu_1^3}{\mu_1 + 2\mu_1 \lambda_1} \right) \right]^2} \] ... (8.8)

\[ J_2 = \frac{2}{c^2 \left[ 1 - \left( \frac{\mu_2}{\mu_2 + \lambda_2} + \frac{\mu_2^2}{\mu_2 + \lambda_2 \mu_2^2} - \frac{\mu_2^3}{\mu_2 + 2\mu_2 \lambda_2} \right) \right]^2} \]

\[ J_3 = - \frac{2}{c^2 \left[ 1 - \left( \frac{\mu_1}{\mu_1 + \lambda_1} + \frac{\mu_1^2}{\mu_1 + \lambda_1 \mu_1^2} - \frac{\mu_1^3}{\mu_1 + 2\mu_1 \lambda_1} \right) \right]^2} \] ... (8.10)

On simplification we get
$$S_1 = \lambda_1 \lambda_2 + \lambda_1 \mu_2 + \lambda_1 \lambda_2^2 \mu_2 + \lambda_1 \lambda_2 \mu_2^2 + \lambda_1 2 \lambda_2^2 + 2 \lambda_1 \lambda_2 \mu_2 + 2 \lambda_1 \lambda_2^3 \mu_2$$

$$+2 \lambda_2^2 \lambda_1 \mu_2^2 + \mu_1 \lambda_2 + \mu_1 \mu_2 + \mu_1 \lambda_2^2 \mu_2 + \mu_1 \lambda_2 \mu_2^2 + 2 \mu_1 \lambda_2^2 + 2 \mu_1 \mu_2 \lambda_2$$

$$+2 \mu_1 \mu_2 \lambda_2^3 + 2 \mu_1 \mu_2^2 \lambda_2^2 + \lambda_1^2 \lambda_2 \mu_1 + \mu_1 \mu_2 \lambda_1^2 + \mu_1 \mu_2 \lambda_1 \lambda_2^2 + \lambda_1^2 \mu_1 \lambda_2 \mu_2$$

$$+\lambda_1^2 \lambda_2^2 \mu_1 + 2 \mu_1 \mu_2 \lambda_2 \lambda_1^2 + 2 \mu_1 \mu_2 \lambda_1^2 + 2 \lambda_1^2 \mu_1 \lambda_2 \mu_2^2 + \lambda_1 \lambda_2 \mu_1^2 + \lambda_1 \mu_2 \lambda_1^2$$

$$+\lambda_1 \mu_2 \lambda_1^2 + 2 \mu_1 \lambda_2^2 \mu_1 + 2 \mu_1^2 \lambda_1 \lambda_2^2 + 2 \mu_1^2 \lambda_1 \mu_2 + 2 \mu_1^2 \lambda_1 \lambda_2 \mu_2$$

$$+\lambda_1 \mu_2 \lambda_1 \lambda_2 + 2 \lambda_1 \lambda_2 \mu_1 + 2 \lambda_1 \mu_2 \lambda_1 + 2 \lambda_1 \mu_2 \lambda_2 + 2 \lambda_1 \lambda_2^2 \mu_2$$

$$+4 \lambda_1 \mu_1 \lambda_2 \mu_2^2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2^2 + 2 \lambda_1^2 \mu_1 \lambda_2 + 2 \lambda_1 \mu_2 \mu_2 + 2 \lambda_1 \lambda_2 \mu_2$$

$$+4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 2 \lambda_1 \mu_1 \mu_2 + 2 \lambda_1 \mu_2 \lambda_1 + 2 \lambda_1 \lambda_2^2 \mu_2$$

$$+2 \lambda_1 \mu_1 \lambda_2 \mu_2^2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2^2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2^2 \mu_2$$

$$+2 \lambda_1^3 \mu_1 \lambda_2 + 2 \lambda_1 \mu_1 \mu_2 + 2 \lambda_1 \mu_1 \lambda_2 + 2 \lambda_1 \mu_1 \lambda_2 + 2 \lambda_1 \mu_1 \lambda_2$$

$$+4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 2 \lambda_1 \mu_1 \mu_2 + 2 \lambda_1 \mu_1 \lambda_2$$

$$+2 \lambda_1 \mu_1 \lambda_2 \mu_2^2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2^2 + 2 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1^2 \mu_1 \lambda_2 \mu_2$$

$$+4 \lambda_1 \mu_1 \lambda_2 \mu_2^2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2 + 4 \lambda_1 \mu_1 \lambda_2 \mu_2$$

$$S_2 = \lambda_1 \lambda_2 + 2 \lambda_2^2 \lambda_1 \mu_2 + \lambda_1 \lambda_2 \mu_2^2 + 2 \lambda_1 \lambda_2 \mu_2 + 2 \lambda_1 \lambda_2^3 \mu_2 + 2 \lambda_1 \lambda_2 \mu_1$$

$$+2 \lambda_1 \lambda_2 \mu_1 \mu_2 + 2 \lambda_1 \lambda_2 \mu_1 \mu_2 + 4 \lambda_1 \lambda_2 \mu_1 \mu_2 + 4 \lambda_1 \lambda_2 \mu_1 \mu_2 + 2 \lambda_1 \lambda_2 \mu_1 \mu_2$$

$$+\lambda_1 \lambda_2 \mu_1 + \lambda_1 \lambda_2 \mu_1 + 2 \lambda_1 \mu_1 \lambda_2 + 2 \lambda_1 \lambda_2 \mu_1 + 2 \lambda_1 \mu_1 \mu_2$$

$$+2 \lambda_1 \lambda_2 \mu_1 + 4 \lambda_1 \lambda_2 \mu_1 + 4 \lambda_1 \lambda_2 \mu_1 + 2 \lambda_1 \lambda_2 \mu_1 + 2 \lambda_1 \lambda_2 \mu_1$$

$$+4 \lambda_1 \lambda_2 \mu_1 + 2 \lambda_1 \lambda_2 \mu_1 + 4 \lambda_1 \lambda_2 \mu_1 + 4 \lambda_1 \lambda_2 \mu_1 + 4 \lambda_1 \lambda_2 \mu_1 + 4 \lambda_1 \lambda_2 \mu_1$$
\[+2\lambda_1^2\mu_1^2\mu_2 - 2\lambda_1^2\lambda_2\mu_2^2 - 2\lambda_1\mu_1\lambda_2 - 4\lambda_1\mu_1\mu_2 - 4\lambda_1\lambda_2\mu_1\mu_2^2 - 2\lambda_1^2\mu_1^2\mu_2^2\]

\[-12\lambda_1\mu_1\lambda_2^2 - 12\lambda_1\lambda_2\mu_1\mu_2 - 8\lambda_1\lambda_2^3\mu_1\mu_2 - 8\lambda_1\mu_1\lambda_2^3\mu_2^2 - 2\lambda_1^2\mu_1^2\mu_2^2\]

\[+\lambda_1^2\mu_1\mu_2 + 2\lambda_1^2\lambda_2^2\mu_1 - \mu_1\mu_2 - 2\lambda_1^2\mu_2 - 2\lambda_1^2\mu_2\lambda_1^3 - 2\mu_1^2\mu_2\lambda_1^2 -
\]

\[2\mu_1\lambda_2^2 - 2\lambda_2^2\lambda_1\mu_1^2 - 4\lambda_2^2\lambda_1^2\mu_1 - 4\lambda_2^2\lambda_1^3\mu_1 - 4\lambda_2^2\lambda_1^2\mu_1^2 - 4\lambda_2^2\lambda_1\mu_2\]

\[-4\mu_1\mu_2\lambda_2 - 4\mu_1\mu_2\lambda_1^2\lambda_2 - 4\mu_2\lambda_1\mu_1^2\mu_2 - 8\lambda_1^2\lambda_2^2\mu_2 - 8\lambda_1^3\mu_1\mu_2\lambda_2\]

\[-8\lambda_2^2\mu_2\mu_1^2\lambda_1 - 2\lambda_2^2\mu_2^2\mu_1 - 4\lambda_2^2\mu_2^2\lambda_1^2 - 4\lambda_2^2\mu_2^2\lambda_1^2\mu_1^2 + \lambda_2^2\mu_1\mu_2\]

\[+\lambda_2^2\lambda_1^2\mu_2\mu_1 + 2\lambda_1\lambda_2^2\mu_1^2\mu_2 + 2\lambda_2^2\lambda_1^2\mu_2 + 2\lambda_2^2\lambda_1^2\mu_2\mu_1^2\]  \quad \ldots (8.11)

\[J_3 = -\frac{2(s_1)^2}{c^2(s_2)^2} \quad \ldots (8.12)\]

Substituting (8.9), (8.10), (8.11) in the equation (8.8) we get

\[E(T^2) = J_1 + J_2 - J_3 \quad \ldots (8.13)\]

\[V(T) = E(T^2) - [E(T)]^2\]

On simplification we get,

\[= (J_1 + J_2 - J_3) - (I_1 + I_2 - I_3)^2 \quad \ldots (8.14)\]

8.5 Numerical Examples

The behaviour of \(E(T)\) and \(V(T)\) due to the changes in the different parameters associated with the distribution of the random variables in the model is explained by taking the numerical examples.
Table 1: Variation in $E(T)$ and $V(T)$ for changes in the inter arrival times between successive Contacts, keeping $\lambda_2 = 0.2$, $\mu_1 = 0.4$, $\mu_2 = 0.3$ fixed.

<table>
<thead>
<tr>
<th>c</th>
<th>$\lambda_1 = 0.5$</th>
<th>$\lambda_1 = 1.0$</th>
<th>$\lambda_1 = 1.5$</th>
<th>$\lambda_1 = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E(T)$</td>
<td>$V(T)$</td>
<td>$E(T)$</td>
<td>$V(T)$</td>
</tr>
<tr>
<td>1</td>
<td>11.283</td>
<td>69.010</td>
<td>12.339</td>
<td>56.713</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>13.255</td>
<td>43.024</td>
</tr>
<tr>
<td>2</td>
<td>5.6414</td>
<td>17.252</td>
<td>6.1698</td>
<td>14.178</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>6.6277</td>
<td>10.756</td>
</tr>
<tr>
<td>3</td>
<td>3.7612</td>
<td>7.6678</td>
<td>4.1132</td>
<td>6.3015</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>4.4184</td>
<td>4.7805</td>
</tr>
<tr>
<td>4</td>
<td>2.8207</td>
<td>4.3131</td>
<td>3.0849</td>
<td>3.5446</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3.3138</td>
<td>2.6890</td>
</tr>
<tr>
<td>5</td>
<td>2.2566</td>
<td>2.7604</td>
<td>2.4679</td>
<td>2.2685</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.6510</td>
<td>1.7209</td>
</tr>
<tr>
<td>6</td>
<td>1.8805</td>
<td>1.9169</td>
<td>2.0566</td>
<td>1.5753</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.2092</td>
<td>1.1951</td>
</tr>
<tr>
<td>7</td>
<td>1.6118</td>
<td>1.4083</td>
<td>1.7628</td>
<td>1.1574</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.8936</td>
<td>0.8780</td>
</tr>
<tr>
<td>8</td>
<td>1.4103</td>
<td>1.0782</td>
<td>1.5424</td>
<td>0.8861</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.6569</td>
<td>0.6722</td>
</tr>
<tr>
<td>9</td>
<td>1.2536</td>
<td>0.8519</td>
<td>1.3710</td>
<td>0.7001</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.4728</td>
<td>0.5311</td>
</tr>
<tr>
<td>10</td>
<td>1.1283</td>
<td>0.6901</td>
<td>1.2339</td>
<td>0.5671</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.3255</td>
<td>0.4302</td>
</tr>
</tbody>
</table>

Fig.1a. Changes of $E(T)$ due to the variations in $c$
Fig. 1b. Changes of $V(T)$ due to the variations in $c$

The inter arrival times between contacts taken to be a random variable $U$ that follows exponential distribution in the parameter $c$. So $(U) = \frac{1}{c}$. As $c$ increases $E(U)$ decreases. Hence the frequency of contacts increases. Therefore $E(T)$ and $V(T)$ both decrease. This is so for different values of $\lambda_1$ when the other parameter kept fixed.

Table 2: Variation in $E(T)$ and $V(T)$ for changes in the inter arrival times between successive Contacts, keeping $\lambda_1 = 0.4$, $\lambda_2 = 0.2$, $\mu_2 = 0.3$ fixed.

<table>
<thead>
<tr>
<th>C</th>
<th>$\mu_1 = 0.5$</th>
<th>$\mu_1 = 1.0$</th>
<th>$\mu_1 = 1.5$</th>
<th>$\mu_1 = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E(T)$</td>
<td>$V(T)$</td>
<td>$E(T)$</td>
<td>$V(T)$</td>
</tr>
<tr>
<td>1</td>
<td>11.057</td>
<td>74.280</td>
<td>10.447</td>
<td>78.309</td>
</tr>
<tr>
<td>4</td>
<td>2.7642</td>
<td>4.6425</td>
<td>2.8618</td>
<td>4.8943</td>
</tr>
<tr>
<td>5</td>
<td>2.2114</td>
<td>2.9712</td>
<td>2.2894</td>
<td>3.1323</td>
</tr>
<tr>
<td>6</td>
<td>1.8428</td>
<td>2.0633</td>
<td>1.9078</td>
<td>2.1752</td>
</tr>
<tr>
<td>7</td>
<td>1.5795</td>
<td>1.5159</td>
<td>1.6353</td>
<td>1.5981</td>
</tr>
<tr>
<td>8</td>
<td>1.3821</td>
<td>1.1606</td>
<td>1.4309</td>
<td>1.2235</td>
</tr>
<tr>
<td>9</td>
<td>1.2285</td>
<td>0.9170</td>
<td>1.2719</td>
<td>0.9667</td>
</tr>
<tr>
<td>10</td>
<td>1.1057</td>
<td>0.7428</td>
<td>1.1447</td>
<td>0.7830</td>
</tr>
</tbody>
</table>
The value of $E(T)$ and $V(T)$ both decrease when $c$ which is the parameter of the exponential distribution of inter contacts time is increases. The other parameter namely $\mu_1$ is given different values and other parameters $\lambda_1, \lambda_2, \mu_2$ are kept fixed. This is seen in fig.2a and 2b.
Table 3: Variation in $E(T)$ and $V(T)$ for changes in the inter arrival times between successive Contacts, keeping $\lambda_1 = 0.4$, $\mu_1 = 0.3$, $\mu_2 = 0.2$ fixed.

<table>
<thead>
<tr>
<th>C</th>
<th>$\lambda_2 = 0.1$</th>
<th>$\lambda_2 = 0.2$</th>
<th>$\lambda_2 = 0.3$</th>
<th>$\lambda_2 = 0.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>E(T)</td>
<td>V(T)</td>
<td>E(T)</td>
<td>V(T)</td>
</tr>
<tr>
<td>1</td>
<td>10.221</td>
<td>154.58</td>
<td>11.661</td>
<td>184.03</td>
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<tr>
<td>2</td>
<td>4.1105</td>
<td>38.646</td>
<td>5.8309</td>
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<tr>
<td>6</td>
<td>1.7035</td>
<td>4.2940</td>
<td>1.9436</td>
<td>5.1121</td>
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<tr>
<td>7</td>
<td>1.4601</td>
<td>3.1548</td>
<td>1.6659</td>
<td>3.7558</td>
</tr>
<tr>
<td>8</td>
<td>1.2776</td>
<td>2.4154</td>
<td>1.4577</td>
<td>2.8755</td>
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<tr>
<td>9</td>
<td>1.1356</td>
<td>1.9084</td>
<td>1.2957</td>
<td>2.2720</td>
</tr>
<tr>
<td>10</td>
<td>1.0221</td>
<td>1.5458</td>
<td>1.1661</td>
<td>1.8403</td>
</tr>
</tbody>
</table>

![Fig.3 a. Changes of $E(T)$ due to the variations in $c$](image_url)
The value of \( E(T) \) and \( V(T) \) both decrease when \( c \) which is the parameter of the exponential distribution of inter contacts time is increases. The other parameter namely \( \lambda_2 \) is given different values and other parameters \( \lambda_1, \mu_1, \mu_2 \) are kept fixed. This is seen in fig.3a and 3b.

### Table 4: Variation in \( E(T) \) and \( V(T) \) for changes in the inter arrival times between successive Contacts, keeping \( \lambda_1 = 0.4, \lambda_2 = 0.2, \mu_1 = 0.3 \) fixed.

<table>
<thead>
<tr>
<th>C</th>
<th>( \mu_2 = 0.1 )</th>
<th>( \mu_2 = 0.2 )</th>
<th>( \mu_2 = 0.3 )</th>
<th>( \mu_2 = 0.4 )</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>( E(T) )</td>
<td>( V(T) )</td>
<td>( E(T) )</td>
<td>( V(T) )</td>
</tr>
<tr>
<td>1</td>
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<td>12.305</td>
<td>1345.2</td>
</tr>
<tr>
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<td>7.5775</td>
<td>46.009</td>
<td>6.1527</td>
<td>336.29</td>
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<td>5.0517</td>
<td>20.448</td>
<td>4.1018</td>
<td>149.46</td>
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<tr>
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<td>11.502</td>
<td>3.0763</td>
<td>84.074</td>
</tr>
<tr>
<td>5</td>
<td>3.0310</td>
<td>7.3614</td>
<td>2.4611</td>
<td>53.807</td>
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<td>2.5258</td>
<td>5.1121</td>
<td>2.0509</td>
<td>37.366</td>
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<tr>
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<td>2.1650</td>
<td>3.7558</td>
<td>1.7579</td>
<td>27.453</td>
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<tr>
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<td>2.8755</td>
<td>1.5381</td>
<td>21.018</td>
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<tr>
<td>9</td>
<td>1.6839</td>
<td>2.2720</td>
<td>1.3672</td>
<td>16.607</td>
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<tr>
<td>10</td>
<td>1.5155</td>
<td>1.8403</td>
<td>1.2305</td>
<td>13.452</td>
</tr>
</tbody>
</table>
The value of $E(T)$ and $V(T)$ both decrease when $c$ which is the parameter of the exponential distribution of inter contacts time is increases. The other parameter namely $\mu_2$ is given different values and other parameters $\lambda_1, \lambda_2, \mu_1, \mu_2$ are kept fixed. This is seen in fig.3a and 3b.
8.6 Conclusion

It is observed that from table 1 the parameter denoting the antigenic diversity threshold $\lambda_1$ increases and the threshold parameter $\mu_1$, $\mu_2$ are kept fixed, the simulated results shows that as the inter arrival time follows exponential distribution $c$ takes the value 1,2,...,10, the expected time to cross the antigenic diversity threshold decreases and variance also decreases which is depicted in fig.1a and fig.1b. This is due to fact that the successive contacts between the infected partners increases expected time to cross the antigenic diversity threshold is decreases.

From table 2 it is observed that the threshold parameter $\mu_1$ takes values $\mu_1 = 0.5, 1.0, 1.5$ and 2.0 keeping $\lambda_1 = 0.4, \lambda_2=0.2$ and $\mu_2 = 0.3$ are kept fixed, the results shows that as inter arrival time follows exponential distribution $c$ increases the expected time to cross the antigenic diversity threshold and its variance decreases as depicted in fig.2a and fig.2b. From table 3 it is observed that the threshold parameter $\mu_1 = 0.3$ and $\mu_2 = 0.2$ and the parameter of the antigenic diversity threshold $\lambda_1 = 0.4$ are kept fixed, as the inter contacts time $c$ increases the expected time the antigenic diversity and it variance decrease. From table 4 it is observed that the changes in the inter arrival time between successive contacts and threshold parameter $\mu_2$ increases, keeping the antigenic diversity threshold parameters $\lambda_1, \lambda_2$ and threshold parameter $\mu_1$ kept fixed, then $E(T)$ and $V(T)$ show a decrease as depicted in fig.4a and 4b. To analyze HIV/AIDS epidemiological data, many parametric distributions have been assumed for the HIV infection and seroconversion without due regard to the dynamics of the HIV epidemic and the biological and clinical features of the HIV. Results of practical utility can be achieved by collecting real life data by assuming
appropriate distributions and test for the goodness of fit can be used to validate the model. The major use of mathematical models of the transmission dynamics of HIV at present is to focus attention on the epidemiological parameters that need to be measured to predict future trends and to help to assess how different methods of control will influence the incidence of AIDS.