In this chapter, mathematical modelling for the prediction of the quality characteristics including: surface roughness, dimensional accuracy, hardness and wear of the composite castings obtained through FDMAIC process has been carried out. Mathematical models presented in present chapter were developed by using the results of the final experimentation (given in chapter 5) conducted as per Taguchi L18 DOE technique. For the development of mathematical models, Buckingham’s $\pi$-theorem based approach has been used.

### 8.1 Buckingham $\Pi$-Theorem

Buckingham’s $\pi$ - theorem is one of the most powerful tools for dimensional analysis of the quality characteristics [255]. Till date, various researchers have employed this approach for dimensionless analysis of the output quality parameters that demonstrate the effectiveness of this approach for generating complex analytic equations having larger number of process variable [256]. Nowadays, dimensionless analysis can be used for reducing the influence of the process variable by generating mathematical equations [257].

According to Buckingham’s $\pi$ theorem, it is feasible to bring together the various process parameters existing in the problem into number of dimensionless products ($\pi_i$) [258]. Each term is called $\pi$ - term.

Mathematically [252-253];

Let $Z_1, Z_2, Z_3, \ldots, Z_n$ are the parameters involved in a physical problem. If $Z_1$ is a dependent parameter whereas $Z_2, Z_3, Z_n$ are independent parameters on which $Z_1$ depends. Then parameter $Z_1$ can be written as the function of $Z_2, Z_3, \ldots, Z_n$ which can be expressed as:

$$Z_1 = f(Z_2, Z_3, \ldots, Z_n) \quad (8.1)$$

Eqn. 8.1 can be modified as:
\[ f(Z_1, Z_2, Z_3, \ldots, Z_n) = 0 \]  
(8.2)

In case of Buckingham’s \(\pi\)-theorem approach, parameters (such as: \(Z_1, Z_2, Z_3, \ldots, Z_n\)) in Eqn. 8.2 can be replaced with dimensionless groups (such as: \(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-m}\)).

\[ f(\pi_1, \pi_2, \pi_3, \ldots, \pi_{n-m}) = 0 \]  
(8.3)

### 8.2 Mathematical Modeling for Predicting Quality Characteristics of Composite Castings Developed Using Fused Deposition Modelling Assisted Investment Casting Process

The input parameters selected for the present research work namely: filament proportion (A), volume of FDM pattern (B), density of FDM pattern (C), BF cycle time (D), BF media weight (E) and number of IC slurry layers (F) were selected for Buckingham’s \(\pi\) based dimensional analysis for predicting surface roughness, dimensional accuracy, surface hardness and wear of castings by generate mathematical models.

For generating a mathematical equation for above mentioned output properties of the castings, subsequent assumptions have been made:

- It is pre-assumed that all the input parameters of FDM which are not being considered in the present research work are kept at their optimum levels for producing reinforced FDM pattern for IC process.
- The results of Taguchi analysis as well as ANOVA were used for the development of mathematical models for quality characteristics.

Before proceeding for the development of the mathematical equations, all the input parameters were converted into numerical values as shown in Table 8.1. On the basis of the ANOVA analysis conducted for various output properties in Chapter 5, Table 8.2 shows the percentage contribution of the input parameters in output properties such as surface roughness, dimensional accuracy, surface hardness and wear.
Table 8.1 Results of various output parameters.

<table>
<thead>
<tr>
<th>Exp. No.</th>
<th>Parameter ‘A’ in terms of tensile strength, N/mm²</th>
<th>B, mm³</th>
<th>C, N/mm³</th>
<th>D, sec</th>
<th>E, N</th>
<th>Parameter ‘F’ in terms of mould wall thickness, mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21.65</td>
<td>17576</td>
<td>5.12x10⁻⁶</td>
<td>1200</td>
<td>98</td>
<td>11.5</td>
</tr>
<tr>
<td>2</td>
<td>21.65</td>
<td>17576</td>
<td>7.63x10⁻⁶</td>
<td>2400</td>
<td>147</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>21.65</td>
<td>17576</td>
<td>9.163x10⁻⁶</td>
<td>3600</td>
<td>196</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>21.65</td>
<td>27000</td>
<td>5.12x10⁻⁶</td>
<td>1200</td>
<td>147</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>21.65</td>
<td>27000</td>
<td>7.63x10⁻⁶</td>
<td>2400</td>
<td>196</td>
<td>15</td>
</tr>
<tr>
<td>6</td>
<td>21.65</td>
<td>27000</td>
<td>9.163x10⁻⁶</td>
<td>3600</td>
<td>98</td>
<td>11.5</td>
</tr>
<tr>
<td>7</td>
<td>21.65</td>
<td>39304</td>
<td>5.12x10⁻⁶</td>
<td>2400</td>
<td>98</td>
<td>15</td>
</tr>
<tr>
<td>8</td>
<td>21.65</td>
<td>39304</td>
<td>7.63x10⁻⁶</td>
<td>3600</td>
<td>147</td>
<td>11.5</td>
</tr>
<tr>
<td>9</td>
<td>21.65</td>
<td>39304</td>
<td>9.163x10⁻⁶</td>
<td>1200</td>
<td>196</td>
<td>13</td>
</tr>
<tr>
<td>10</td>
<td>21.53</td>
<td>17576</td>
<td>5.12x10⁻⁶</td>
<td>3600</td>
<td>196</td>
<td>13</td>
</tr>
<tr>
<td>11</td>
<td>21.53</td>
<td>17576</td>
<td>7.63x10⁻⁶</td>
<td>1200</td>
<td>98</td>
<td>15</td>
</tr>
<tr>
<td>12</td>
<td>21.53</td>
<td>17576</td>
<td>9.163x10⁻⁶</td>
<td>2400</td>
<td>147</td>
<td>11.5</td>
</tr>
<tr>
<td>13</td>
<td>21.53</td>
<td>27000</td>
<td>5.12x10⁻⁶</td>
<td>2400</td>
<td>196</td>
<td>11.5</td>
</tr>
<tr>
<td>14</td>
<td>21.53</td>
<td>27000</td>
<td>7.63x10⁻⁶</td>
<td>3600</td>
<td>98</td>
<td>13</td>
</tr>
<tr>
<td>15</td>
<td>21.53</td>
<td>27000</td>
<td>9.163x10⁻⁶</td>
<td>1200</td>
<td>147</td>
<td>15</td>
</tr>
<tr>
<td>16</td>
<td>21.53</td>
<td>39304</td>
<td>5.12x10⁻⁶</td>
<td>3600</td>
<td>147</td>
<td>15</td>
</tr>
<tr>
<td>17</td>
<td>21.53</td>
<td>39304</td>
<td>7.63x10⁻⁶</td>
<td>1200</td>
<td>196</td>
<td>11.5</td>
</tr>
<tr>
<td>18</td>
<td>21.53</td>
<td>39304</td>
<td>9.163x10⁻⁶</td>
<td>2400</td>
<td>98</td>
<td>13</td>
</tr>
</tbody>
</table>

Table 8.2 Percentage contribution of input process parameters.

<table>
<thead>
<tr>
<th>Factor</th>
<th>Surface roughness</th>
<th>Dimensional accuracy</th>
<th>Hardness</th>
<th>Wear</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4.16%</td>
<td>0.76%</td>
<td>7.69%</td>
<td>6%</td>
</tr>
<tr>
<td>B</td>
<td>43.84%*</td>
<td>16.95%</td>
<td>8.85%</td>
<td>62%*</td>
</tr>
<tr>
<td>C</td>
<td>3.03%</td>
<td>19.83%</td>
<td>65.75%*</td>
<td>23.9%</td>
</tr>
<tr>
<td>D</td>
<td>6.45%</td>
<td>3.30%</td>
<td>1.03%</td>
<td>0.40%</td>
</tr>
<tr>
<td>E</td>
<td>2.94%</td>
<td>31.71%*</td>
<td>0.8%</td>
<td>0.08%</td>
</tr>
<tr>
<td>F</td>
<td>5.72%</td>
<td>8.97%</td>
<td>14.14%</td>
<td>0.11%</td>
</tr>
<tr>
<td>Error</td>
<td>33.86%</td>
<td>18%</td>
<td>1.74%</td>
<td>8.03%</td>
</tr>
<tr>
<td>Total</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
<td>100%</td>
</tr>
</tbody>
</table>

*Highly contributed factor.
Table 8.3 shows symbols, their units and dimensions of input and output process parameters.

Table 8.3 Symbols, units and dimensions of input and output parameters.

<table>
<thead>
<tr>
<th>S. No.</th>
<th>Parameters</th>
<th>Symbol used</th>
<th>Unit</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Surface Roughness</td>
<td>$R_a$</td>
<td>Mm</td>
<td>$L$</td>
</tr>
<tr>
<td>2.</td>
<td>Dimensional Accuracy</td>
<td>$\Delta d$</td>
<td>Mm</td>
<td>$L$</td>
</tr>
<tr>
<td>3.</td>
<td>Surface Hardness</td>
<td>$H$</td>
<td>$9.807 N/mm^2$</td>
<td>$ML^2 T^{-2}$</td>
</tr>
<tr>
<td>4.</td>
<td>Wear</td>
<td>$L$</td>
<td>Mm</td>
<td>$ML^{-2}$</td>
</tr>
<tr>
<td>5.</td>
<td>A (in terms of tensile strength of the filament)</td>
<td>$P$</td>
<td>N/mm$^2$</td>
<td>$ML^{-1} T^{-2}$</td>
</tr>
<tr>
<td>6.</td>
<td>$B$</td>
<td>$V$</td>
<td>$mm^3$</td>
<td>$L^3$</td>
</tr>
<tr>
<td>8.</td>
<td>$C$</td>
<td>$\rho$</td>
<td>N/mm$^2$</td>
<td>$ML^{-2}$</td>
</tr>
<tr>
<td>9.</td>
<td>$D$</td>
<td>$T$</td>
<td>Sec</td>
<td>$T$</td>
</tr>
<tr>
<td>10.</td>
<td>$E$</td>
<td>$W$</td>
<td>N</td>
<td>M</td>
</tr>
<tr>
<td>11.</td>
<td>$F$</td>
<td>$l$</td>
<td>Mm</td>
<td>$L$</td>
</tr>
</tbody>
</table>

8.2.1 Modeling for Predicting Surface Roughness of Castings

In the present study, surface roughness as first output parameter is kept as a function of all input process parameters as given in Eqn. 8.4.

Now, $R_a = f(F, V, \rho, t, W, l)$ \hspace{1cm} \hspace{1cm} (8.4)

Based on the Table 8.2, parameters A, C and E are the least contributing parameters for surface roughness, so they will directly go in “$\pi$” groups.

\begin{align*}
\pi_1 &= R_a \ (W)^{a_1} \ (\rho)^{b_1} \ (F)^{c_1} \\
\pi_2 &= V \ (W)^{a_2} \ (\rho)^{b_2} \ (F)^{c_2} \\
\pi_3 &= t \ (W)^{a_3} \ (\rho)^{b_3} \ (F)^{c_3} \\
\pi_4 &= l \ (W)^{a_4} \ (\rho)^{b_4} \ (F)^{c_4}
\end{align*} \hspace{1cm} \hspace{1cm} (8.5-8.8)

Now, putting the dimensions (refer Table 8.3) of all the parameters in Eqns. (8.5-8.8) and further equating these equations to zero, we can achieve the final exponent for each basic
dimension as the “π’s” are dimensional groups. So, xi, yi and zi (where i = 1, 2, 3...) can be solved. On solving π1, we get:

\[ \pi_1 = L (M)^{x_1} (ML^{-3})^{y_1} (ML^{-1}T^{-2})^{z_1} \]  

(8.9)

Here,

M: \( x_1 + y_1 + z_1 = 0 \)

L: 1-3\( y_1 - z_1 = 0 \)

T: -2\( z_1 = 0 \)

So,

\( x_1 = -1/3, \ y_1 = 1/3 \) and \( z_1 = 0 \)

Thus, \( \pi_1 = R_a (\rho/W)^{1/3} \)  

(8.10)

Similarly, we get;

\[ \pi_2 = L^3 (M)^{x_2} (ML^{-3})^{y_2} (ML^{-1}T^{-2})^{z_2} \]  

(8.11)

Here,

M: \( x_2 + y_2 + z_2 = 0 \)

L: 3-3\( y_2 - z_2 = 0 \)

T: -2\( z_2 = 0 \)

So,

\( x_2 = -1, \ y_2 = 1 \) and \( z_2 = 0 \)

Thus, \( \pi_2 = V(\rho/W) \)  

(8.12)

Similarly, we get;

\[ \pi_3 = T (M)^{x_3} (ML^{-3})^{y_3} (ML^{-1}T^{-2})^{z_3} \]  

(8.13)
Here,  

M: \( x_3 + y_3 + z_3 = 0 \)  
L: \(-3y_3 - z_3 = 0 \)  
T: \( 1 - 2z_3 = 0 \)  

So,  

\( x_3 = -1/3, \ y_3 = -1/6 \) and \( z_3 = 1/2 \)  

Thus, \( \pi_3 = t \ (W)^{1/3}. \ (\rho)^{1/6}. \ (F)^{1/3} \) \hspace{1cm} (8.14)  

Similarly, we get;  

\( \pi_4 = L \ (M)^{x_4} (ML^{-3})^{y_4} (ML^{-1}T^{-2})^{z_4} \) \hspace{1cm} (8.15)  

Here,  

M: \( x_4 + y_4 + z_4 = 0 \)  
L: \( 1 - 3y_4 - z_4 = 0 \)  
T: \( -2z_4 = 0 \)  

So,  

\( x_4 = -1/3, \ y_4 = 1/3 \) and \( z_4 = 0 \)  

Thus, \( \pi_4 = l(W/\rho)^{1/3} \) \hspace{1cm} (8.16)  

The final relationship can be presented in the form:  

\( \pi_1 = f(\pi_2, \pi_3 \) and \( \pi_4) \)  

Or  

\( R_a(\rho/W)^{1/3} = f \left( \frac{\nu}{W}, \frac{t}{(\rho)^{1/5}(FW)^{3/2}}, \text{and} \left( \frac{W}{\rho} \right)^{1/3} \right) \)
Or

\[ R_a = K \cdot (V \cdot t \cdot l \cdot \rho \bar{\sigma}) / F^{1/3} \cdot W^{2/3} \]  

\[ (8.17) \]

“K” is constant of proportionality.

It has been experimentally found that surface roughness is significantly affected by parameter ‘B’ (refer Table 8.2). Therefore “volume of the pattern” can be assumed to representative for generating the mathematical model. For doing the same, the data from the final experimentation has been plotted in the form of graph as shown in Figure 8.1. Microsoft Excel software was used for generating best suitable curve from available data. A 2\textsuperscript{nd} order polynomial equation has been determined with regression equals to 1.

![Figure 8.1. Surface roughness vs. volume of FDM pattern plot.](image)

So, the final mathematical model for surface roughness is:

\[ R_a = [(6E-12V^2) - 3E-07V - 0.0082] \cdot (t \cdot l \cdot \rho \bar{\sigma}) / F^{1/3} \cdot W^{2/3} \]  

\[ (8.18) \]
8.2.2 Modeling for Predicting Dimensional Accuracy of Castings

In the present study, dimensional accuracy as second output parameter is kept as a function of all input process parameters as given in Eqn. 8.19.

\[ \Delta d = f(F, V, \rho, t, W, l) \]  

(8.19)

Based on the Table 8.2; filament proportion, barrel finishing cycle time and number of IC slurry layers are the least significant in present case so they will directly go in “\( \pi \)” groups.

So, \( \pi_1 = \Delta d (F)^{a_1}(t)^{b_1}(L)^{c_1} \)  

(8.20)

\( \pi_2 = W (F)^{a_2}(t)^{b_2}(L)^{c_2} \)  

(8.21)

\( \pi_3 = \rho (F)^{a_3}(t)^{b_3}(L)^{c_3} \)  

(8.22)

\( \pi_4 = V (F)^{a_4}(t)^{b_4}(L)^{c_4} \)  

(8.23)

Now, putting the dimensions (refer Table 8.3) of all the parameters in Eqns. (8.20-8.23) and further equating these equations to zero, we can achieve the final exponent for each basic dimension as the “\( \pi \)’s” are dimensional groups. So, \( x_i, y_i \) and \( z_i \) (where \( i = 1, 2, 3... \)) can be solved. On solving \( \pi_1 \), we get:

\[ \pi_1 = L (ML^{-1}T^{-2})^{a_1}(T)^{b_1}(L)^{c_1} \]  

(8.24)

Here,

\( M: x_1 = 0 \)

\( L: 1 - y_1 + z_1 = 0 \)

\( T: -2x_1 + y_1 = 0 \)

So,

\( x_1 = 0, y_1 = 0 \) and \( z_1 = -1 \)
Thus
\[ \pi_1 = \Delta d/l \] (8.25)

Similarly, we get;
\[ \pi_2 = M (ML^{-1}T^2)^{a2} (T)^{b2} (L)^{c2} \] (8.26)

Here,

M: 1+x = 0
L: -x + z = 0
T: -2x + y = 0

So,
\[ x = -1, y = -2 \text{ and } z = -1 \]

Thus
\[ \pi_2 = W/Ft^2l \] (8.27)

Similarly, we get;
\[ \pi_3 = ML^{-3} (ML^{-1}T^2)^{a3} (T)^{b3} (L)^{c3} \] (8.28)

Here,

M: 1 + x = 0
L: -3 - x + z = 0
T: -2x + y = 0

So,
\[ x = -1, y = -2 \text{ and } z = 2 \]

Thus
\[ \pi_3 = \rho l^2/Ft^2 \] (8.29)
Similarly, we get;

\[ \pi_4 = L^3 \left( ML^{-1} T^2 \right)^{a_4} (T)^{b_4} (L)^{c_4} \]  \hspace{1cm} (8.30)

Here,

M: \( x_4 = 0 \)

L: \( 3 - x_4 + z_4 = 0 \)

T: \( -2x_4 + y_4 = 0 \)

So,

\( x_4 = 0, \ y_4 = 0 \) and \( z_4 = -3 \)

Thus

\[ \pi_4 = V/l^3 \]  \hspace{1cm} (8.31)

The final relationship can be presented in the form:

\[ \pi_1 = f(\pi_2, \pi_3 \text{ and } \pi_4) \]

Or

\[ \Delta d/l = f \left( \frac{W}{F t^2 l}, \frac{\rho l^2}{t^2} \text{ and } \frac{V}{l^3} \right) \]

\[ \Delta d = K \cdot p.W.V/F^2.l.t^4 \]  \hspace{1cm} (8.32)

“K” is constant of proportionality.

It has been experimentally found that dimensional accuracy is significantly affected by parameter ‘E’ (refer Table 8.2). Therefore “barrel finishing media weight” can be assumed to representative for generating the mathematical model. For doing the same, the data from the final experimentation has been plotted in the form of graph as shown in Figure 8.2. Excel software was used for generating best suitable curve from available data. A 2\(^{nd}\) order polynomial equation has been determined with regression equals to 1.
Figure 8.2. Dimensional accuracy vs. barrel finishing media weight plot.

So the final mathematical model for dimensional accuracy is:

\[ \Delta d = \left[ -4 \times 10^{-6} W^2 + 0.0012W - 0.048 \right] \rho \cdot \frac{V}{F^2} \cdot l^4 \]  

(8.33)

8.2.3 Modeling for Predicting Hardness of Castings

In the present study, hardness as third output parameter is kept as a function of all input process parameters as given in Eqn. 8.34.

\[ H = f(P, V, \rho, t, W, l) \]  

(8.34)

So, \( H = f(P, V, \rho, t, W, l) \)

Based on the Table 8.2, parameter ‘A’, ‘D’ and ‘E’ are the least significant in present case so they will directly go in “\( \pi \)” groups.

\[ \pi_1 = H (F)^{a_1} (t)^{b_1} (W)^{c_1} \]  

(8.35)

\[ \pi_2 = \rho (F)^{a_2} (t)^{b_2} (W)^{c_2} \]  

(8.36)

\[ \pi_3 = l (F)^{a_3} (t)^{b_3} (W)^{c_3} \]  

(8.37)

\[ \pi_4 = V (F)^{a_4} (t)^{b_4} (W)^{c_4} \]  

(8.38)
Now, putting the dimensions (refer Table 8.3) of all the parameters in Eqns. (8.35-8.38) and further equating these equations to zero, we can achieve the final exponent for each basic dimension as the “π’s” are dimensional groups. So, xi, yi and zi (where i = 1, 2, 3...) can be solved. On solving π1, we get:

\[ \pi_1 = ML^{-1}T^{-2} (ML^{-1}T^{-2})^{a_1} (T)^{b_1} (M)^{c_1} \]  

(8.39)

Here,

M: \(1+x_1+z_1 = 0\)

L: \(-1 - x_1 = 0\)

T: \(-2 -2x_1 + y_1 = 0\)

So,

\(x_1 = -1, \ y_1 = 0\) and \(z_1 = 0\)

Thus

\[ \pi_1 = H/F \]  

(8.40)

Similarly, we get;

\[ \pi_2 = ML^{-3} (ML^{-1}T^{-2})^{a_2} (T)^{b_2} (M)^{c_2} \]  

(8.41)

Here,

M: \(1+x_2+z_2 = 0\)

L: \(-3 - x_2 = 0\)

T: \(-2x_2 + y_2 = 0\)

So,

\(x_2 = -3, \ y_2 = -6\) and \(z_2 = 2\)

Thus
\[ \pi_2 = \rho F^3 t^6 \]  
(8.42)

Similarly, we get;

\[ \pi_3 = L (ML^{-1}T^{-2})^{a_3} (T)^{b_3} (M)^{c_3} \]  
(8.43)

Here,

\begin{align*}
M: & \quad x_3 + y_3 = 0 \\
L: & \quad 1 - x_3 = 0 \\
T: & \quad -2x_3 + y_3 = 0
\end{align*}

\[ x_3 = 1, \quad y_3 = 2 \text{ and } z_3 = -1 \]

Thus

\[ \pi_3 = lFT^2/W \]  
(8.44)

Similarly, we get;

\[ \pi_4 = L^3 (ML^{-1}T^{-2})^{a_4} (T)^{b_4} (M)^{c_4} \]  
(8.45)

Here,

\begin{align*}
M: & \quad x_4 + z_4 = 0 \\
L: & \quad 3 - x_4 = 0 \\
T: & \quad 0 - 2x_4 + y_4 = 0
\end{align*}

\[ x_4 = 3, \quad y_4 = 6 \text{ and } z_4 = -3 \]

Thus

\[ \pi_4 = VF^3 t^6 / W^3 \]  
(8.46)
The final relationship can be presented in the form:

$$\pi_1 = f(\pi_2, \pi_3 \text{ and } \pi_4)$$

Or

$$H/F = f \left( \frac{\rho}{F^3 t^6}, \frac{lF^2}{W} \text{ and } VF^3 t^6 / W^3 \right)$$

Or

$$H = K \cdot \rho \cdot F^2 \cdot l^2 \cdot V / W^4 \quad (8.47)$$

“K” is constant of proportionality.

It has been experimentally found that hardness is significantly affected by parameter ‘C’ (refer Table 8.2). Therefore “density of FDM pattern” can be assumed to representative for generating the mathematical model. For doing the same, the data from the final experimentation has been plotted in the form of graph as shown in Figure 8.3. Microsoft Excel software was used for generating best suitable curve from available data. A 2nd order polynomial equation has been determined with regression equals to 1.

![Graph showing hardness vs. density of FDM pattern plot.](image)

**Figure 8.3. Hardness vs. density of FDM pattern plot.**

So the final mathematical model for hardness is:

$$H = [(3E+13(\rho^3) - 4E+08(\rho) + 2190.7]F^2 \cdot l^2 \cdot V / W^4 \quad (8.48)$$
8.2.4 Modeling for Predicting Wear of Castings

In the present study, wear as fourth output parameter is kept as a function of all input process parameters as given in Eqn. 8.49.

Now, \( W = f(P, V, \rho, t, m, l) \) \hspace{1cm} (8.49)

Based on the Table 3; parameter ‘D’, ‘E’ and ‘F’ are least significant in present case so they will directly go in “\( \pi \)” groups.

\[
\begin{align*}
\pi_1 &= W (t)^{a_1} (m)^{b_1} (l)^{c_1} \hspace{1cm} (8.50) \\
\pi_2 &= V (t)^{a_2} (m)^{b_2} (l)^{c_2} \hspace{1cm} (8.51) \\
\pi_3 &= \rho (t)^{a_3} (m)^{b_3} (l)^{c_3} \hspace{1cm} (8.52) \\
\pi_4 &= P (t)^{a_4} (m)^{b_4} (l)^{c_4} \hspace{1cm} (8.53)
\end{align*}
\]

Now, putting the dimensions (refer Table 8.3) of all the parameters in Eqns. (8.50-8.53) and further equating these equations to zero, we can achieve the final exponent for each basic dimension as the “\( \pi \)”s” are dimensional groups. So, \( x_i, y_i \) and \( z_i \) (where \( i = 1, 2, 3... \)) can be solved. On solving \( \pi_1 \), we get:

\[
\pi_1 = L (T)^{a_1} (M)^{b_1} (L)^{c_1} \hspace{1cm} (8.54)
\]

Here,

\[
\begin{align*}
M: y_1 &= 0 \\
L: 1 + z_1 &= 0 \\
T: x_1 &= 0
\end{align*}
\]

So,

\[
x_1 = 0, \hspace{0.1cm} y_1 = 0 \hspace{0.1cm} \text{and} \hspace{0.1cm} z_1 = -1
\]

Thus

\[
\pi_1 = m/l \hspace{1cm} (8.55)
\]
Similarly, we get;

\[ \pi_2 = L^3 (T)^{a_2} (M)^{b_2} (L)^{c_2} \]  (8.54)

Here,

M: \( y_2 = 0 \)

L: \( 3 + z_2 = 0 \)

T: \( x_2 = 0 \)

So,

\( x_2 = 0, y_2 = 0 \) and \( 3 + z_2 = 0 \), so \( z_2 = -3 \)

Thus

\[ \pi_2 = v/l^3 \]  (8.55)

Similarly, we get;

\[ \pi_3 = ML^{-3} (T)^{a_3} (M)^{b_3} (L)^{c_3} \]  (8.56)

Here,

M: \( 1 + y_3 = 0 \)

L: \( -3 + z_3 = 0 \)

T: \( x_3 = 0 \)

So,

\( x_3 = 0, y_3 = -1 \) and \( -3 + z_3 = 0 \), so \( z_3 = 3 \)

Thus

\[ \pi_3 = \rho l^3/m \]  (8.57)

Similarly, we get;
\( \pi_4 = ML^{-1}T^{-2} (T)^a (M)^b (L)^c \)  \hspace{1cm} (8.58)

Here,

M: \( 1 + y = 0 \)

L: \( -1 + z = 0 \)

T: \( -2 + x = 0 \)

So,

\( x = 2, y = -1 \) and \( z = 1 \)

Thus

\( \pi_4 = Pt^2l/m \)  \hspace{1cm} (8.59)

The final relationship can be presented in the form:

\( \pi_1 = f(\pi_2, \pi_3 \text{ and } \pi_4) \)

Or

\( W/L = f\left(\frac{V}{l^2}, \frac{\rho l^3}{m} \text{ and } Pt^2l/m\right) \)

Or

\( W = K.V . \rho . P.t^2l/m^2 \)  \hspace{1cm} (8.60)

“K” is constant of proportionality.

It has been experimentally found that wear is significantly affected by parameter ‘B’ (refer Table 8.2). Therefore “volume of FDM pattern” can be assumed to representative for generating the mathematical model. For doing the same, the data from the final experimentation has been plotted in the form of graph as shown in Figure 8.1. Microsoft Excel software was used for generating best suitable curve from available data. A 2\(^{nd}\) order polynomial equation has been determined with regression equals to 1.
Figure 8.4. Wear vs. volume of FDM pattern.

So, \( W = (-2 \times 10^{-10} V^2 + 1 \times 10^{-5} V - 0.03) \text{p. P.t}^2 \text{l/m}^2 \)  

(8.60)

8.2.5 Corollary

By using Eqn. 8.60, (with data of optimum values selected from Table 8.1) the experiment no. 3 has been used for verification of mathematical equation. The experimental value for “W” is 0.087mm. By putting the input parameter values of experiment no. 3 (as per Table 8.1) in Eqn. 15:

Result of wear comes out as, \( W = 0.084 \text{mm} \) (predicted value)

Comparison of result of wear obtained using mathematical equation agrees well with the experimental value (see 3rd row of Table 5.14 of Chapter 5). Further the model error (i.e. 3.44%) is below the permissible limit of 5%, justifying towards the validity of the model.

8.3 Summary

Mathematical modeling on the basis of Buckingham’ \( \pi \) based approach is very efficient technique for data interpretation obtained after conducting experiments as per Taguchi L18 OA. In the present research work, Buckingham’s \( \pi \)-theorem was employed for the development of mathematical models of surface roughness, dimensional accuracy, surface hardness and castings obtained using FDMAIC process using reinforced pattern.
By using developed mathematical modelling equations; surface roughness, dimensional accuracy, surface hardness and wear of the castings may be predicted with minimum experimentations cost/expensive.