CHAPTER-6

COMPARISION OF K-ITERATIVE SCHEME WITH VARIOUS ITERATIVE PROCEDURES

In this chapter we introduce a new iterative scheme named as K-iterative scheme. Using this scheme, the behaviour of strong convergence for quasi contractive operators in Banach spaces is studied. The present study proves that K-iterative scheme is equivalent to Picard, Mann, Ishikawa, Aggarwal et al., Noor, SP and CR iterative scheme for quasi contractive operators. Its rate of convergence is faster than above mentioned iterative schemes. Moreover, to compare the rate of convergence various examples are given. To elaborate the results computer programming and MAT LAB are used.

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CHAPTER 6

COMPARISION OF K-ITERATIVE SCHEME WITH VARIOUS ITERATIVE PROCEDURES

6.1-INTRODUCTION

There are non-identical techniques employed in order to solve the problems of nonlinear equations and of approaching specified points of corresponding contractive class operators. Assume a function $Q: Z \rightarrow Z$ on a complete metric space $(Z, d)$ and consider a set of stationary points of $Q$ say $S(Q) = \{z \in Z, Qz = z\}$. Over the years various researchers have been approximating the fixed point of operators for several iterative processes, which are mentioned below:

The Picard iterative scheme \( \{x_k\}_{k=0}^{\infty} \), for a complete metric space is defined by
\[
x_{k+1} = Qx_k, \quad k = 0, 1...
\] (1.1)
used in guessing the agreed points of mappings satisfying the Banach contraction condition
\[
d(Qx, Qy) \leq \alpha \, d(x, y), \text{ for all } x, y \in Z \text{ and } \alpha \in [0, 1).
\] (1.2)

The Mann iterative scheme (see [180]) is defined as
\[
x_{k+1} = (1 - \alpha_k) x_k + \alpha_k Qx_k,
\] (1.3)
where \( \{\alpha_k\} \) is a sequence of +ive numbers in \([0, 1]\).

The Ishikawa iterative scheme (see [181]) is defined as
\[
\begin{cases} 
  x_{k+1} = (1 - \alpha_k) x_k + \alpha_k Qy_k, \\
  y_k = (1 - \beta_k)x_k + \beta_k Qx_k,
\end{cases}
\] (1.4)
where \( \{\alpha_k\} \) and \( \{\beta_k\} \) are sequences of numbers in \([0, 1]\).

The Agarwal et al. Iterative scheme (see [182]) is defined as
\[
\begin{cases} 
  x_{k+1} = (1 - \alpha_k) Qx_k + \alpha_k Qy_k, \\
  y_k = (1 - \beta_k)x_k + \beta_k Qx_k,
\end{cases}
\] (1.5)
where \( \{\alpha_k\} \) and \( \{\beta_k\} \) are sequences defined in \([0, 1]\).
The new two-step iteration (see [183]) of S. Thianwan is defined as

\[
\begin{align*}
x_{k+1} &= (1 - \alpha_k)y_k + \alpha_k Qy_k, \\
y_k &= (1 - \beta_k)x_k + \beta_k Qx_k,
\end{align*}
\]  
(1.6)

where \(\{\alpha_k\}\) and \(\{\beta_k\}\) are sequences taken in [0, 1].

The Noor iterative scheme (see [184]) is as follows:

\[
\begin{align*}
x_{k+1} &= (1 - \alpha_k)x_k + \alpha_k y_k, \\
y_k &= (1 - \beta_k)x_k + \beta_k z_k, \\
z_k &= (1 - \gamma_k)x_k + \gamma_k Qx_k,
\end{align*}
\]  
(1.7)

where \(\{\alpha_k\}\), \(\{\beta_k\}\) and \(\{\gamma_k\}\) are sequences notified in [0, 1].

The SP iterative scheme (see [185]) of Phuengrattana and Suantai is

\[
\begin{align*}
x_{k+1} &= (1 - \alpha_k)y_k + \alpha_k Qy_k, \\
y_k &= (1 - \beta_k)z_k + \beta_k Qz_k, \\
z_k &= (1 - \gamma_k)x_k + \gamma_k Qx_k,
\end{align*}
\]  
(1.8)

where \(\{\alpha_k\}\), \(\{\beta_k\}\) and \(\{\gamma_k\}\) are sequences of positive numbers in [0, 1].

The CR iterative scheme (see [186]) of Renu Chugh et al. is defined as

\[
\begin{align*}
x_{k+1} &= (1 - \alpha_k)y_k + \alpha_k Qy_k, \\
y_k &= (1 - \beta_k)Qx_k + \beta_k Qz_k, \\
z_k &= (1 - \gamma_k)x_k + \gamma_k Qx_k,
\end{align*}
\]  
(1.9)

where \(\{\alpha_k\}\), \(\{\beta_k\}\) and \(\{\gamma_k\}\) are sequences of positive numbers in [0, 1] with \(\{\alpha_n\}\) satisfying \(\sum_{n=0}^{\infty} \alpha_k = \infty\).

One can remember the following arguments:

1.1. If \(\gamma_k = 0\), then (1.8) reduces to the new two step iteration of S. Thianwan (1.6).

1.2. If \(\gamma_k = \beta_k = 0\), then (1.8), (1.7), (1.6) and (1.4) lessens to the Mann iteration (1.3).

1.3. If \(\gamma_k = 0\), then (1.7) diminishes to the Ishikawa iteration (1.4).

Zamfirescu [187] in the year 1972 stated that:
Theorem 6.1.1.-[187] Let $Q$ be a self-map on a complete metric space $Z$ for which there prevails real numbers $k$, $l$ and $m$ fulfilling $k \in (0,1)$ and $l, m \in (0, \frac{1}{2})$ such that for each pair $x, y \in Z$, at least one of the following conditions hold

1. $d(Qx, Qy) \leq k d(x, y)$
2. $d(Qx, Qy) \leq l [d(x, Qx) + d(y, Qy)]$
3. $d(Qx, Qy) \leq m [d(x, Qy) + d(y, Qx)]$. (1.10)

Then $Q$ has a fixed point $z$ and the Picard iteration $\{x_k\}_{k=0}^{\infty}$ defined by

$$x_{k+1} = Qx_k, \quad k = 0, 1, \ldots$$

coversges to $z$ for any arbitrary but fixed $z_0 \in Z$.

The operators pleasing (1.10) are Zamfirescu operators.

Berinde [188] on inconsistent Banach space declared a new class of operators as:

$$d(Qx, Qy) \leq 2\delta d(x, Qx) + \delta d(x, y), \text{ for all } x, y \in Z \text{ and some } \delta \in [0,1). \quad (1.11)$$

clarifying the fact that Berinde type operators are vast as compared to Zamfirescu operators. He [188] stated the result of approximation in an arbitrary Banach space as:

Theorem 6.1.2.-[188] Let $T \neq \emptyset$ be a closed convex subset of an arbitrary Banach space $Z$ and $P$ be a self-mapping on $T$ gratifying Berinde operators (1.11). Let $\{x_k\}$ be defined by the Ishikawa iteration (1.4) and $x_0 \in Z$ where $\{\alpha_k\}, \{\beta_k\}$ are sequences of positive numbers in $[0, 1]$ with $\{\alpha_k\}$ satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$. Then $\{x_k\}$ converges strongly to the fixed point of $P$.

In the last four decades many researchers [189-194] put their efforts in the introduction and convergence of various iteration procedures for estimating fixed points of quasi–contrastive operators. Researchers like Rhoades and Solutz [191], Berinde [188], Solutz [193], Olaleru [196] made their valuable contribution to the study of equivalence among various iterative schemes. Renu Chugh and Vivek Kumar [197] explained that earlier defined repetition process are equivalent for quasi-contractive operators satisfying (1.11). Recently Renu Chugh et al. [198] proved that CR scheme is identical to all two step and three step defined procedures for quasi-contractive operators. Phuengratan and Suantai [185] propounded SP iterative scheme showing that SP iterative scheme is equal to and quicker as compared to Mann, Ishikawa and Noor iterative schemes for increasing functions.

These iterative schemes have wide use in Physics. Iterations are used in physics engine for doing collisions pair-wise in which one is to determine the number
of iterations to be used for successful box stacking. Chipmunk 2D a 2D rigid body physics library is used in hundreds of games uses iterative solver to figure out the forces between objects in the space. Iterations are used for rigid body dynamics with contact which requires linear time and space to open up the possibility for exploiting temporal coherence.

Wazwaz [199] used iterations to solve linear and non-linear Schrodinger equations. Montri [200] used iterations to get exact solution to linear and non-linear Fokker-Plank equations, which have wide applications in plasma physics, surface physics, biophysics, laser physics. Yucheng Liu [201] used iteration method to solve free vibration problems for an Euler-Bernoulli beam under various supporting conditions. The iterations have wide applications of Statistical Physics to coding theory.

To authentify our result we need the following lemma and definitions-

6.2.-PRELIMINARIES

**Lemma6.2.1-[202]** If $\delta$ is a real number such that $0 \leq \delta < 1$ and $\{\varepsilon_k\}_{k=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{k \to \infty} \varepsilon_k = 0$, then for any sequence of positive numbers $\{\varepsilon_k\}_{k=0}^{\infty}$ satisfying

$$u_{k+1} \leq \delta u_k + \varepsilon_k, \quad k = 0, 1, 2...$$

we have $\lim_{k \to \infty} u_k = 0$.

**Definition6.2.1**-Assume two real convergent sequence $\{x_k\}$ and $\{y_k\}$ converging to $x$ and $y$ respectively. Then $\{x_k\}$ is converges faster than $\{y_k\}$ if $\lim_{k \to \infty} \left| \frac{x_k - x}{y_k - y} \right| = 0$.

**Definition6.2.2-[203]** Consider two stationary point iterative procedures $\{a_k\}$ and $\{b_k\}$ converging to the same fixed point $z$ with the error estimates

$$\|a_k - z\| \leq x_k, \quad k = 0, 1, 2...$$
$$\|b_k - p\| \leq y_k, \quad k = 0, 1, 2...$$

With two real convergent sequences $\{x_k\}$ and $\{y_k\}$ converging to 0. If $\{x_k\}$ convergences faster than $\{y_k\}$, then we say $\{a_k\}$ converges faster than $\{b_k\}$ to $z$.

The personal contribution to the topic on iteration is to introduce a new iterative scheme with the name K-iterative process:
Let $Q$ be a self-map on a Banach space $Z$, and $z_0 \in Z$. Define the sequence $\{x_k\}_{k=0}^{\infty}$ by

$$
\begin{align*}
    x_{k+1} &= (1 - \alpha_k) y_k + \alpha_k Qy_k, \\
    y_k &= (1 - \beta_k) Qw_k + \beta_k Qz_k, \\
    z_k &= (1 - \gamma_k) Qx_k + \gamma_k Qw_k, \\
    w_k &= (1 - \lambda_k) x_k + \lambda_k Qx_k,
\end{align*}
$$

(1.12)

where $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ and $\{\lambda_k\}$ are sequences of positive numbers in $[0, 1]$ with $\{\alpha_k\}$ satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$.

Here we explore that rate of convergence of the K-iterative scheme for quasi-contractive operators with the condition of Berinde class of operators and compare it with all two step and three step iteration process. We would provide examples to authentify the result by using C programme and MAT LAB.

6.3. CONVERGENCE THEOREMS

Theorem 6.3.1- Let $P$ be a non-empty closed convex subset of an arbitrary Banach space $Z$ and $Q$ be a self-mapping on $P$ satisfying (1.11). Let $\{x_k\}_{k=0}^{\infty}$ be defined through the K-iteration (1.12) and $x_0 \in Z$, where $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ and $\{\lambda_k\}$ are sequences of positive numbers in $[0, 1]$ with $\{\alpha_k\}$ satisfying $\sum_{k=0}^{\infty} \alpha_k = \infty$. Then $\{x_k\}_{k=0}^{\infty}$ converges strongly to the fixed point of $Q$.

Proof. Theorem 1.1. shows that $Q$ has a unique fixed point in $P$, say $u$. From (1.12), we have

$$
\|x_{k+1} - u\| \leq (1 - \alpha_k) \|y_k - u\| + \alpha_k \|Qy_k - u\|. 
$$

(2.1)

Using (1.11), (2.1) becomes

$$
\|x_{k+1} - u\| \leq [1 - \alpha_k (1 - \mu)] \|y_k - u\|. 
$$

(2.2)

Using (1.12) and (1.11), we get

$$
\|x_{k+1} - u\| \leq [1 - \alpha_k (1 - \mu)] (1 - \beta_k) \|Qw_k - u\| + [1 - \alpha_k (1 - \mu)] \beta_k \|Qz_k - u\|. 
$$

$$
\leq [1 - \alpha_k (1 - \mu)] (1 - \beta_k) \mu \|w_k - u\| + [1 - \alpha_k (1 - \mu)] \beta_k \mu \|z_k - u\|. 
$$

$$
\leq [1 - \alpha_k (1 - \mu)] (1 - \beta_k) \mu \lambda_k \|Qx_k - u\| + [1 - \alpha_k (1 - \mu)] \beta_k \mu \|Qx_k - u\| + \\
[1 - \alpha_k (1 - \mu)] \beta_k \mu \gamma_k \|Qw_k - u\|. 
$$
\leq [1- \alpha_k(1-\mu)] \mu (1-\lambda_k) \|x_k - u\| + [1-\alpha_k(1-\mu)] (1-\beta_k) \mu^2 \\
\lambda_k \|x_k - u\| + [1-\alpha_k(1-\mu)] \beta_k \mu^2 (1-\gamma_k) \|x_k - u\| + \\
[1-\alpha_k(1-\mu)] \beta_k \mu^2 \gamma_k \|w_k - u\|.
\leq [1-\alpha_k(1-\mu)] \mu ((1-\beta_k) (1-\lambda_k) \|x_k - u\| + (1-\beta_k) \mu \lambda_k \|x_k - u\| + \\
\beta_k \mu (1-\gamma_k) \|x_k - u\| + \beta_k \mu \gamma_k (1-\lambda_k) \|x_k - u\| + \beta_k \mu \gamma_k \lambda_k \|Qx_k - u\|).
\quad \leq [1-\alpha_k(1-\mu)] \mu ((1-\beta_k) (1-\lambda_k) + (1-\beta_k) \mu \lambda_k + \beta_k \mu (1-\gamma_k) + \\
\beta_k \gamma_k (1-\lambda_k) + \beta_k \mu^2 \gamma_k \lambda_k) \|x_k - u\|
\quad = [1-\alpha_k(1-\mu)] \mu (1-\lambda_k (1-\mu) - \beta_k (1-\mu) + \lambda_k \beta_k (1-\mu) - \mu \beta_k \\
\gamma_k \lambda_k (1-\mu)) \|x_k - u\|.
\|x_{k+1} - u\| \leq \mu^{k+1} \prod_{i=0}^{k} [1 - \alpha_i (1-\mu)] \|x_0 - u\|, k = 0, 1, 2...
\tag{2.3}
\]Since, 0 \leq \mu < 1, \alpha_i \in [0,1] and \sum_{k=0}^{\infty} \alpha_k = \infty, we get
\lim_{k \to \infty} \mu^k \prod_{i=0}^{k} [1 - \alpha_i (1-\mu)] = 0.
This implies \lim_{k \to \infty} \|x_{k+1} - u\| = 0. Therefore \{x_k\}^{\infty}_{k=0} converges strongly to u.

**Theorem 6.3.2** Let P be a non-empty closed convex subset of a Banach space Z and Q be self-mapping on P satisfying (1.11). If the initial point is same, \alpha_k \geq A > 0, \forall k \in \mathbb{N}, then the Mann iteration (1.3) converges to u implies the K-iteration (1.12) converges to u and vice-versa.

**Proof:** First we assume that the Mann iteration (1.3) converges to u. We would now show that the K-iteration (1.12) also converges to u.

On using Mann iteration (1.3) and K-iteration (1.12), we have
\|x_{k+1} - u\| \leq (1-\alpha_k) \|y_k - u\| + \alpha_k \|Qy_k - Qu_k\|
\tag{2.2.1}

Using (1.11), we get
\|x_{k+1} - u\| \leq (1-\alpha_k) \|y_k - u\| + \alpha_k \mu \|y_k - u\| + 2 \alpha_k \mu \|Qy_k - u\k|
\quad = (1-\alpha_k (1-\mu)) \|y_k - u\| + 2 \alpha_k \mu \|Qy_k - u\|
\tag{2.2.2}

Now
\|y_k - u\| \leq (1 - \beta_k) \|Qw_k - u\| + \beta_k \|Qz_k - u\|
\tag{2.2.3}

On using (1.11), (2.2.3) becomes

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\[ \| y_k - u_k \| \leq (2\mu + 1) \| Qu_k - u_k \| + (1 - \beta_k)\mu \| w_k - u_k \| + \beta_k\mu \| z_k - u_k \|. \]  
\text{(2.2.4)}

Now \[ \| z_k - u_k \| \leq (1 - \gamma_k) \| Qx_k - u_k \| + \gamma_k \| Qw_k - u_k \|. \]
\[ \leq (2\mu + 1) \| Qu_k - u_k \| + (1 - \gamma_k)\mu \| x_k - u_k \| + \gamma_k\mu \| w_k - u_k \|. \]  
\text{(2.2.5)}

And \[ \| w_k - u_k \| \leq (1 - \lambda_k) \| x_k - u_k \| + \lambda_k \| Qx_k - u_k \|. \]
\[ \leq [1 - \lambda_k(1 - \mu)] \| x_k - u_k \| + (2\mu + 1)\lambda_k \| Qu_k - u_k \|. \]  
\text{(2.2.6)}

Substituting (2.2.5) in (2.2.4), we have
\[ \| y_k - u_k \| \leq (2\mu + 1)(1 + \beta_k\mu) \| Qu_k - u_k \| + [(1 - \beta_k)\mu + \beta_k\gamma k\mu^2] \] \[ \| w_k - u_k \| + (1 - \gamma_k)\beta_k\mu^2 \| x_k - u_k \|. \]  
\text{(2.2.7)}

Substituting (2.2.6) in (2.2.7), we get
\[ \| y_k - u_k \| \leq (2\mu + 1)[1 + \lambda_k\mu(1 - \beta_k) + \beta_k\mu(1 + \gamma k\lambda k\mu)] \| Qu_k - u_k \| + \mu[1 - \lambda_k(1 - \mu) - \beta_k(1 - \mu) + \beta_k\lambda k(1 - \mu) - \beta_k\gamma k\lambda k\mu(1 - \mu)] \] \[ \| x_k - u_k \|. \]  
\text{(2.2.8)}

Now on substituting (2.2.8) in (2.2.1), we have
\[ \| x_{k+1} - u_{k+1} \| \]
\[ \leq [1 - \alpha_k(1 - \mu)](2\mu + 1)[1 + \lambda_k\mu(1 - \beta_k) + \beta_k\mu(1 + \gamma k\lambda k\mu)] \| Qu_k - u_k \| + \] \[ [1 - \alpha_k(1 - \mu)]\mu[1 - \lambda_k(1 - \mu) - \beta_k(1 - \mu) + \beta_k\lambda k(1 - \mu) - \beta_k\gamma k\lambda k\mu(1 - \mu)] \] \[ \| x_k - u_k \| + 2\alpha_k\mu \| Qu_k - u_k \|. \]  
\text{(2.2.9)}

Also
\[ \| Qu_k - u_k \| \leq \| Qu_k - u \| + \| u_k - u \|. \]
\[ \leq (1 + \mu) \| u_k - u \|. \]  
\text{(2.2.10)}

On substituting (2.2.10) in (2.2.9), we have
\[ \| x_{k+1} - u_{k+1} \| \]
\[ \leq [1 - \alpha_k(1 - \mu)](2\mu + 1)[1 + \lambda_k\mu(1 - \beta_k) + \beta_k\mu(1 + \gamma k\lambda k\mu)(1 + \mu)] \| u_k - u \| + \] \[ [1 - \alpha_k(1 - \mu)]\mu[1 - \lambda_k(1 - \mu) - \beta_k(1 - \mu) + \beta_k\lambda k(1 - \mu) - \beta_k\gamma k\lambda k\mu(1 - \mu)] \]
\[ \| x_k - u_k \| + 2\alpha_k \mu (1 + \mu) \| u_k - u \|. \]
\[ \leq h \| x_k - u_k \| + p \| u_k - u \|. \quad (2.2.11) \]

where
\[ h = [1 - \alpha_k (1 - \mu)] \mu [1 - \lambda_k (1 - \mu) - \beta_k (1 - \mu) + \beta_k \lambda_k (1 - \mu) - \beta_k \gamma_k \lambda_k \mu (1 - \mu)] < 1. \]

(Using \( \alpha_k \geq A > 0 \), \( \forall k \in \mathbb{N} \) and
\[ p = [1 - \alpha_k (1 - \mu)] (2 \mu + 1)[1 + \lambda_k (1 - \beta_k \mu) + \beta_k \mu (1 + \gamma_k \lambda_k \mu)] (1 + \mu) + 2\alpha_k \mu (1 + \mu). \]

As \( u_k \to u \) as \( k \to \infty \) and lemma 6.2.1, (2.2.11) yields
\[ \| x_k - u_k \| \to 0 \text{ as } k \to \infty. \]

Also
\[ \| x_k - u \| \leq \| x_k - u_k \| + \| u_k - u \|. \]

This implies \( x_k \to u \) as \( k \to \infty \).

Conversely, we prove that \( x_k \to u \) implies \( u_k \to u \).

On using (1.3), (1.11) and (1.12), we have
\[ \| x_{k+1} - u_{k+1} \| \leq (1 - \alpha_k) \| y_k - u_k \| + \alpha_k \mu \| Q y_k - Qu_k \|. \]
\[ \leq [1 - \alpha_k (1 - \mu)] \| y_k - u_k \| + 2\mu \alpha_k \| Q y_k - y_k \|. \quad (2.2.12) \]

From (1.12), we get
\[ \| y_k - u_k \| \leq (1 - \beta_k) \| Q w_k - u_k \| + \beta_k \mu \| Q z_k - u_k \|. \]
\[ \leq (1 - \beta_k) \| Q w_k - x_k \| + (1 - \beta_k) \| x_k - u_k \| + \beta_k \| Q z_k - x_k \| + \beta_k \| x_k - u_k \|. \]
\[ \leq (1 - \beta_k) \| Q w_k - u \| + \| x_k - u \| + \| x_k - u_k \| + \beta_k \| Q z_k - u \|. \]
\[ \leq (1 - \beta_k) \| Q w_k - u \| + \| x_k - u \| + \| x_k - u_k \| + \beta_k \| z_k - u \|. \quad (2.2.13) \]

From (1.12) and on using (1.11), we have
\[ \| w_k - u \| \leq (1 - \lambda_k) \| x_k - u \| + \lambda_k \| Q x_k - u \|. \]
\[ \leq [1 - \lambda_k (1 - \mu)] \| x_k - u \|. \quad (2.2.14) \]
\[
\| z_k - u \| \leq (1 - \gamma_k) \| Q x_k - u \| + \gamma_k \| Q w_k - u \|.
\]
\[
\leq (1 - \gamma_k) \mu \| x_k - u \| + \gamma_k \mu \| w_k - u \|.
\]
(2.2.15)

Substituting (2.2.14) in (2.2.15), we have
\[
\| z_k - u \| \leq (1 - \gamma_k) \mu \| x_k - u \| + \gamma_k \mu [1 - \lambda_k (1 - \mu)] \| x_k - u \|.
\]
\[
= \mu [1 - \gamma_k \lambda_k (1 - \mu)] \| x_k - u \|.
\]
(2.2.16)

Substituting (2.2.14), (2.2.16) in (2.2.13), we have
\[
\| y_k - u_k \| \leq (1 - \beta_k) \mu [1 - \lambda_k (1 - \mu)] \| x_k - u \| + \| x_k - u \| + \| x_k - u_k \|
\]
\[
+ \beta_k \mu^2 [1 - \gamma_k \lambda_k (1 - \mu)] \| x_k - u \|.
\]
\[
\leq \| x_k - u_k \| + \{1 + \mu - \mu \lambda_k (1 - \mu) - \mu \beta_k (1 - \mu) + \mu \lambda_k \beta_k (1 - \mu) -
\]
\[
\mu^2 \lambda_k \gamma_k \beta_k (1 - \mu) \} \| x_k - u \|.
\]
(2.2.17)

Also
\[
\| Q y_k - y_k \| \leq \| Q y_k - u \| + \| y_k - u \|.
\]
\[
\leq (1 + \mu) \| y_k - u \|.
\]
\[
\leq (1 + \mu) (1 - \beta_k) \| Q w_k - u \| + (1 + \mu) \beta_k \| Q z_k - u \|.
\]
\[
\leq \mu (1 + \mu) (1 - \beta_k) \| w_k - u \| + \mu (1 + \mu) \beta_k \| z_k - u \|.
\]
(2.2.18)

On substituting (2.2.14) and (2.2.16) in (2.2.18), we get
\[
\| Q y_k - y_k \| \leq \mu (1 + \mu) (1 - \beta_k) [1 - \lambda_k (1 - \mu)] \| x_k - u \| + \mu (1 + \mu) \beta_k \mu
\]
\[
[1 - \gamma_k \lambda_k (1 - \mu)] \| x_k - u \|.
\]
\[
= \mu (1 + \mu) [1 - \lambda_k (1 - \mu) - \beta_k (1 - \mu) + \lambda_k \beta_k (1 - \mu) - \mu \lambda_k \gamma_k \beta_k (1 - \mu)] \| x_k - u \|.
\]
(2.2.19)

Substituting (2.2.17) and (2.2.19) in (2.2.12), we have
\[
\| x_{k+1} - u_{k+1} \| \leq [1 - \alpha_k (1 - \mu)] \| x_k - u_k \| + [1 - \alpha_k (1 - \mu)]
\]
\[
\{1 + \mu - \mu \lambda_k (1 - \mu) - \mu \beta_k (1 - \mu) + \mu \lambda_k \beta_k (1 - \mu)
\]
\[
- \mu^2 \lambda_k \gamma_k \beta_k (1 - \mu) \} \| x_k - u \| +
\]
\[
2 \mu \alpha_k \mu (1 + \mu) \{1 - \lambda_k (1 - \mu) - \beta_k (1 - \mu) + \lambda_k \beta_k (1 - \mu) -
\]
\[ \mu \lambda_k \beta_k (1 - \mu) \| x_k - u \| . \]  

(2.2.20)

As \( \alpha_k \geq A > 0, \forall \ k \in \mathbb{N} \).
So \( 0 \leq 1 - \alpha_k (1 - \mu) < 1, \forall \ k \in \mathbb{N} \).
Also \( x_k \rightarrow u \) as \( k \rightarrow \infty \).

Hence on using Lemma 6.2.1, (2.2.20) gives \( \| x_k - u_k \| \rightarrow 0 \) as \( k \rightarrow \infty \).
Also \( \| u_k - u \| \leq \| x_k - u_k \| + \| x_k - u \| \)
This implies that \( u_k \rightarrow u \) as \( k \rightarrow \infty \).
Thus the result.

Results of Soltuz’s [192, 193], Renu Chugh and Vivek Kumar [197] and Renu Chugh et al. [198] results lead to the following corollary to Theorem 6.3.1:

**Corollary 6.3.3**-Let \( P \) be a non-empty closed convex subset of a Banach space \( Z \) and \( Q: P \rightarrow P \) a mapping satisfying (1.11). If the initial point is the same, \( \alpha_k \geq A > 0, \forall \ k \in \mathbb{N} \), then the Picard iteration (1.1), the Mann iteration (1.3), the Ishikawa iteration (1.4), the Noor iteration (1.7), the Aggarwal e al. iteration (1.5), the SP iteration (1.8), the CR iteration (1.9), the K- iteration (1.12) converges to the same fixed point \( u \).

### 6.4 - INFEERENCE ON SPEED

As already interpreted, analysed and observed that in the last four decades researchers have worked a lot in proving the fastness of different iterative processes by giving different examples. They have their results comparable with the results calculated from C-programming. The recent comparison was done by Renu Chugh et al. [198] for CR iterations proving that CR iteration converges faster than other iterative schemes. Now we are giving illustrations to prove that the K-iteration process is quicker than the earlier defined iterative procedures for quasi-contractive operators satisfying (1.11).

**Example 6.1**- Let \( Q: [0, 1] \rightarrow [0, 1] \) be defined by \( Q(x) = \frac{x}{2} \),
\[ \alpha_k = \beta_k = \gamma_k = \lambda_k = 0, \]
k = 1, 2... 15, \( \alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{4}{\sqrt{k}}, \ k \geq 16 \). Then \( Q \) is a quasi-contractive operator satisfying (1.11) with a unique fixed point 0. Also \( Q, \ alpha_k, beta_k, gamma_k and lambda_k \)
satisfy all the conditions of strong convergence of Theorem 6.1. Now we would prove that K-iterative scheme is quicker than CR and Picard iterative schemes.

**Proof:** Here we show that K-iterative scheme is quicker than CR iterative scheme.

Let \( k \geq 16 \) and \( p_0 = x_0 \). Then from (1.12), we have K-iteration as

\[
w_k = (1 - \frac{4}{\sqrt{k}}) x_k + \frac{2}{\sqrt{k}} x_k = (1 - \frac{2}{\sqrt{k}}) x_k.
\]

\[
z_k = (1 - \frac{4}{\sqrt{k}}) \frac{x_k}{2} + \frac{4}{\sqrt{k}} (1 - \frac{2}{\sqrt{k}}) \frac{x_k}{2} = (1 - \frac{8}{k}) \frac{x_k}{2}.
\]

\[
y_k = (1 - \frac{4}{\sqrt{k}} + \frac{8}{k} - \frac{16}{k^2}) \frac{x_k}{2}.
\]

\[
x_{k+1} = (1 - \frac{4}{\sqrt{k}})(1 - \frac{4}{\sqrt{k}} + \frac{8}{k} - \frac{16}{k^2}) \frac{x_k}{2} + \frac{4}{\sqrt{k}} (1 - \frac{4}{\sqrt{k}} + \frac{8}{k} - \frac{16}{k^2}) \frac{x_k}{4}.
\]

\[
= \frac{1}{2} - \frac{3}{\sqrt{k}} + \frac{8}{k} - \frac{16}{k^2} \frac{x_k}{2}.
\]

\[
= \prod_{i=16}^{k} \left( \frac{1}{2} - \frac{3}{\sqrt{i}} + \frac{8}{i} - \frac{16}{i^2} \right) x_0.
\]

Also, from [184], for CR iteration (1.9), we have

\[
x_{k+1} = \prod_{i=16}^{k} \left( \frac{1}{2} - \frac{1}{\sqrt{i}} - \frac{4}{i} + \frac{8}{i^2} \right) p_0.
\]

So,

\[
\frac{x_{k+1}(K)}{x_{k+1}(CR)} = \frac{\prod_{i=16}^{k} \left( \frac{1}{2} - \frac{3}{\sqrt{i}} + \frac{8}{i} + \frac{16}{i^2} \right) x_0}{\prod_{i=16}^{k} \left( \frac{1}{2} - \frac{1}{\sqrt{i}} - \frac{4}{i} + \frac{8}{i^2} \right) p_0}.
\]

\[
\leq \prod_{i=16}^{k} \left( 1 - \frac{2(2\sqrt{i}-12i+24\sqrt{i}-16)}{(i^2-i\sqrt{i}-2i+4\sqrt{i})} \right).
\]

Thus

\[
0 \leq \lim_{k \to \infty} \prod_{i=16}^{k} \left( 1 - \frac{2(2\sqrt{i}-12i+24\sqrt{i}-16)}{(i^2-i\sqrt{i}-2i+4\sqrt{i})} \right).
\]

\[
\leq \lim_{k \to \infty} \prod_{i=16}^{k} \left( 1 - \frac{1}{i} \right) = \lim_{k \to \infty} \frac{15}{k} = 0.
\]

Hence, we have \( \lim_{k \to \infty} \left| \frac{x_{k+1}(K)}{x_{k+1}(CR)} \right| = 0. \)

Therefore, K-iterative scheme converges quicker than CR iterative scheme to the fixed point 0.
From the following equations we would show that K-iterative scheme is quicker than Picard iteration.

Let \( k \geq 16 \), then

\[
\left| \frac{x_{k+1}(K)}{x_{k+1}(\text{Picard})} \right| \leq \left| \frac{\prod_{i=16}^{k} \left( \frac{1}{2} - \frac{3}{16} + \frac{8}{17} + \frac{16}{17^2} \right)x_0}{(\frac{1}{2})^k p_0} \right|. 
\]

\[
\leq \left| \frac{\prod_{i=16}^{k} \left( \frac{1}{2} - \frac{3}{16} + \frac{8}{17} + \frac{16}{17^2} \right)x_0}{(\frac{1}{2})^{k-15} \prod_{i=16}^{k} \left( 1 - \frac{6}{17} + \frac{32}{17^2} + \frac{32}{17^3} \right)} \right|. 
\]

\[< 1.\]

Therefore \( \lim_{n \to \infty} \left| \frac{x_{k+1}(K)}{x_{k+1}(\text{Picard})} \right| = 0. \)

Thus, K-iterative scheme converges quicker than Picard iteration.

Considering the aforementioned results of example 6.4.1 as well as results of Ciric et al. [204], Renu Chugh et al. [197], we conclude that K-iterative scheme is quicker than other iterative schemes for a certain class of quasi-contractive operators.

### 6.5-APPLICATIONS

In this section and its sub-sections, we will compare the rate of convergence of Picard, Mann, Ishikawa, Noor, Aggarwal et al., SP, CR and K iteration procedures by giving different examples. The rate of convergence is calculated by making use of C-language programming whose outcomes are listed in tabular form shown in the tables from 1-3. The graphs of the aforementioned iterative procedures for different cited examples are traced with the help of MAT LAB.

#### 6.5.1.-Decreasing function

Let \( g: [0, 1] \to [0, 1] \) be defined by \( g(x) = (1-x)^m \), \( m = 7, 8, \ldots \). Then \( g \) is a decreasing function. By taking \( m = 11 \), initial approximation \( x_0 = 0.4 \) and defining four sequences \( \{ \alpha_k \}, \{ \beta_k \}, \{ \gamma_k \}, \{ \lambda_k \} \) of positive numbers on \( [0, 1] \) with \( \{ \alpha_k \} \) satisfying \( \sum_{k=0}^{\infty} \alpha_k = \infty \) where \( \alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^{\frac{7}{3}}} \). Then the comparison of
convergence of the above-mentioned iterative processes shows that the exact fixed point is \( p = 0.155602 \) which is listed in Table 1.

6.5.2.- Increasing function
Let \( g: [0, 8] \rightarrow [0, 8] \) be defined by \( g(x) = \frac{x^2+9}{10} \). Then \( f \) is an increasing function. By defining four sequences \( \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}, \{\lambda_k\} \) of positive numbers on \([0, 1]\) with \( \{\alpha_k\} \) satisfying \( \sum_{k=0}^{\infty} \alpha_k = \infty \) and on taking \( \alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^{7/3}} \). The convergence of aforementioned iterations by C-language programming shows that the increasing function converges to the fixed point \( p = 1 \) listed in Table 2.

6.5.3.- Super linear functions with multiple roots
The function defined by \( g(x) = 2x^3 - 7x^2 + 8x - 2 \) is a super linear function with multiple real roots. Let \( \{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}, \{\lambda_k\} \) be four sequences of positive numbers on \([0, 1]\) with \( \{\alpha_k\} \) satisfying \( \sum_{k=0}^{\infty} \alpha_k = \infty \) where \( \alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^{7/3}} \).

Again on comparing the rate of convergence we observe that the different iterative schemes converges to the same fixed point \( p = 1 \), listed in Table 3.
Table 1: Decreasing Function.

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Table: 3-Super Linear function with Multiple Roots

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6.6. - GRAPHS OF DIFFERENT ITERATIVE SCHEMES FOR DECREASING, INCREASING AND SUPER LINEAR FUNCTION WITH MULTIPLE ROOTS-

K-iteration

[Graph]

CR-iteration

[Graph]
SP-iteration

Noor-iteration
Ishikawa-iteration

Mann-iteration

Here in all the above plots blue line is the plot for decreasing function, green line for increasing function and the red line shows the result for super linear function with multiple roots.
6.7. - OBSERVATIONS

6.7.1. - Decreasing Function \( (1-x)^m \).

1) For \( m = 11 \) and on taking the initial approximation as \( x_0 = 0.4 \), Picard iteration oscillate between 0 and 1, the Mann iterative process converges in 14\(^{th}\) iterations, the Ishikawa iterative process in 45\(^{th}\) iteration, the Aggarwal scheme does not converge, the Noor iterative scheme in 33\(^{rd}\) iteration, the SP iterative scheme in 15\(^{th}\) iteration, the CR iterative scheme in 12\(^{th}\) iteration and the K-iterative scheme in 8\(^{th}\) iteration for 

\[
\alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^{\frac{k}{3}}} \quad \text{where } \{ \alpha_k \}, \{ \beta_k \}, \{ \gamma_k \}, \{ \lambda_k \} \text{ are sequences of positive numbers on } [0, 1] \text{ with } \{ \alpha_k \} \text{ satisfying } \sum_{n=0}^{\infty} \alpha_k = \infty.
\]

2) If \( m = 11 \) and the starting point is \( x_0 = 0.6 \), Picard iteration oscillate between 0 and 1, the Mann iterative process converges in 166\(^{th}\) iterations, the number of iterations required for the Ishikawa iterative process to converge are 2358, the Aggarwal scheme requires 84 iterations, the Noor iterative scheme in 424 iterations, the SP iterative scheme in 118\(^{th}\) iteration, the CR iterative scheme needs 76 iterations and the K-iterative scheme converges in 55\(^{th}\) iteration for four sequences \( \{ \alpha_k \}, \{ \beta_k \}, \{ \gamma_k \}, \{ \lambda_k \} \) of positive numbers on \([0, 1]\) with \( \{ \alpha_k \} \) satisfying \( \sum_{k=0}^{\infty} \alpha_k = \infty \).

6.7.2. - Increasing Function \( \frac{(x^2+9)}{10} \).

1) By taking the initial value \( x_0 = 0.4 \) and \( \alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^{\frac{k}{3}}} \), the Picard iteration converges in 8\(^{th}\) iteration, the Mann scheme in 27\(^{th}\) iteration, the Ishikawa iterative process in 17\(^{th}\) iteration, the Aggarwal iterative scheme in 6\(^{th}\) iteration, the Noor iterative scheme in 13\(^{th}\) iteration, the SP scheme in 4\(^{th}\) iteration, the CR schemes converges in 4\(^{th}\) iteration and the K-iterative process in 3\(^{rd}\) iteration with four sequences \( \{ \alpha_k \}, \{ \beta_k \}, \{ \gamma_k \}, \{ \lambda_k \} \) of positive numbers on \([0, 1]\) with \( \{ \alpha_k \} \) satisfying \( \sum_{k=0}^{\infty} \alpha_k = \infty \)

2) If we take the initial approximation \( x_0 = 0.8 \) and \( \alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^{0.1}} \), where \( \{ \alpha_k \}, \{ \beta_k \}, \{ \gamma_k \}, \{ \lambda_k \} \) are sequences of positive numbers on \([0, 1]\) with \( \{ \alpha_k \} \)
satisfying $\sum_{n=0}^{\infty} \alpha_k = \infty$, the Picard iteration converges in $7^{th}$ iteration, the Mann scheme requires 10 iterations, the Ishikawa iterative process needs 6 iterations, the Aggarwal iterative scheme converges in $4^{th}$ iteration, the Noor iterative scheme will converge in $5^{th}$ iteration, the SP scheme needs 3 iterations, the CR schemes also converges in $3^{rd}$ iteration and the K-iterative process requires the minimum that is 2 iterations only.

6.7.3. - Super linear Equation with Multiple Roots $2x^3 - 7x^2 + 8x - 2$.

1) If the initial value is $x_0 = 0.6$ and $\alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^{1/3}}$, the Picard iteration converges in $4^{th}$ iteration, the Mann scheme in $23^{rd}$ iteration, the Ishikawa iterative process in $17^{th}$ iteration, the Aggarwal iterative scheme in $3^{rd}$ iteration, the Noor iterative scheme in $14^{th}$ iteration, the SP scheme in $4^{th}$ iteration, the CR schemes converges in 2 iteration and the K-iterative process in $1^{st}$ iteration. Here $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}, \{\lambda_k\}$ are sequences of positive numbers on $[0, 1]$ with $\sum_{k=0}^{\infty} \alpha_k = \infty$.

2) If we take the initial approximation as $x_0 = 0.7$ and $\alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^{0.6}}$, we need 3 iterations for the Picard iterative scheme to converge, 78 iterations for the Mann iterative scheme, 62 iterations for the Ishikawa process, Aggarwal et al. needs 2 iteration to converge, the Noor iterative scheme also converges in 22 iteration, the SP iterative scheme in 4 iterations, the CR iterative scheme in 1 iteration and the K-iteration also in the first iteration for four sequences $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}, \{\lambda_k\}$ of positive numbers on $[0, 1]$ with $\sum_{k=0}^{\infty} \alpha_k = \infty$.

6.8. - CONCLUSIONS

The endmost section of the thesis based on K-iterative scheme leads to the subsequent conclusions:

6.8.1. K-iterative scheme is more superior to earlier defined iterative techniques.

6.8.2. K-iterative series of steps converges quickly as compared with the rest.

6.8.3. The author concludes with calculation of decreasing function as follows:

1) The calculation oscillates between 0 and 1 as the value of initial approximation increases from 0.1 to 1, the solution for Picard plan.
2) The Aggarwal et al. scheme does not converge as the value of \( p \) varies from 0.3 to 0.6 in \( \alpha_k = \beta_k = \frac{1}{(1+k)^p} \) for the initial approximations from 0.1 to 1.

3) The scale of convergence of the Mann iterative scheme, the Ishikawa, the SP, the CR, and the K iterative scheme decreases as the value of \( p \) varies from 0.1 to 0.6 for \( \alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^p} \), whatever the initial approximation may be from 0.1 to 1.

In this case all results show that K-iterative scheme is fleet-footed than the remaining plans.

6.8.4. The increasing function winds up with the following facts:

1) The convergence rate becomes richer in sequence from single step to two step and more efficient as one moves the answers to three step iterative procedures leading with four step K-iterative scheme.

2) The number of iterations required to get fixed point decreases as the value of initial approximation becomes closer to the fixed point but it almost remain same for each approximation from 0.1 to the fixed point in the K-iterative scheme.

6.8.5. For the cubic equation with multiple roots:

1) The iteration numeral declines when the point of initial approximation becomes closer to the fixed point.

2) The iterative digit grows as value of \( p \) from 0.1 to 0.6 in \( \alpha_k = \beta_k = \gamma_k = \lambda_k = \frac{1}{(1+k)^p} \) but almost remains same for the K-iteration.