CHAPTER-5

FIXED POINT THEOREMS IN $\mathcal{L}$-FUZZY METRIC SPACE.

The present research work is on $\mathcal{L}$-fuzzy metric space. It has two main objectives. The first objective is to extend the result of Singh et al. [77] proved on Fuzzy metric space. To ascertain the foremost result, biased maps of type ($R_M$) are introduced in $\mathcal{L}$-fuzzy metric space. Fang’s [78] property is also used in this fixed point theorem.

The second objective is the result for cyclic weak $\emptyset$-contraction in $\mathcal{L}$-fuzzy metric space. To prove this result, triangular condition cyclic and weak $\emptyset$-contraction mappings in $\mathcal{L}$-fuzzy metric space are defined.

A part of research work is published in journals entitled:

1. **A Common Fixed Point Theorem on $\mathcal{L}$-fuzzy metric space by Biased Mapping of Type ($R_M$),** (2013), *International Journal of Current Engineering and Technology*, 3(4).

CHAPTER-5

FIXED POINT THEOREMS IN $\mathcal{L}$-FUZZY METRIC SPACE

5.1 - INTRODUCTION

Fixed point theorems in fuzzy mathematics are emerging with vigorous hope and vital trust. The study of Fuzzy metric space and M-fuzzy metric space paves a soothing machinery to develop fixed point theorems for contractive type maps. The theory finds its applications in many fields of mathematics and beyond its domain.

On the other hand a generalisation of Fuzzy metric space and Intutionistic fuzzy metric space was developed naming $\mathcal{L}$-fuzzy metric space.

The notion of $\mathcal{L}$-fuzzy metric space is introduced by Saadati et al. [25] in the year 2006. $\mathcal{L}$-fuzzy metric space is a generalisation of Fuzzy metric spaces of George and Veeramani [17] and Intutionistic fuzzy metric spaces of Park [163,168], Park and Saadati [178] and Saadati [170]. The notion of Fuzzy metric space of George and Veeramani [17] was taken into consideration by Deng [116], Erceg [117], Kaleva and Seikkala [162], Kramosil and Michalek [118] and proved many results. These concepts are useful in modelling some phenomena where it is necessary to study the relationship between two probability functions. A spate of papers followed immediately showing results on fixed point theorems in $\mathcal{L}$-fuzzy metric space see ([171-179]).

The present chapter has two results on $\mathcal{L}$-fuzzy metric space. The first result is based on Singh et al. [77] notion of biased mappings of type $(R_M)$ introduced in Fuzzy metric space. They proved a common fixed point theorem for four self-mappings out of which one of the pair is continuous. The authors contribution to the chapter is to prove a common fixed point result for biased mapping of type $(R_M)$ in $\mathcal{L}$-fuzzy metric space for four self-mappings using the property $C$ introduced by Fang [78]. The notion of biased mapping of type $(R_M)$ in $\mathcal{L}$-fuzzy metric space is defined in preliminaries section that is in 5.2.10. The result based on the notion is 5.3.1 of main results.
The other result is a fixed point result for cyclic weak $\emptyset$-contraction mappings for two self-mappings in $\mathcal{L}$-fuzzy metric space. The author defines the triangular condition and cyclic weak $\emptyset$-contraction mappings in $\mathcal{L}$-fuzzy metric space. For this, a thorough study of Di Bari and Vetro’s [85] triangular condition, Pacurar and Rus’s [81] cyclic representation and Imdad et al.’s [84] cyclic weak $\emptyset$-contraction in Fuzzy metric spaces is done. Some of the related results of Karapinar [79], Mihet [80] and Vetro et al. [83] are also studied.

5.2 – PRELIMINARIES

**Definition 5.2.1**- A partially ordered set $(\mathcal{L}, \leq)$ in which all subsets have both supremum (join) and infimum (meet) is known as Complete Lattice.

**Definition 5.2.2**-[164] Let $\mathcal{L} = (\mathcal{L}, \leq_{\mathcal{L}})$ be a complete lattice. An $\mathcal{L}$-fuzzy set $A$ on $U$(universe) is a mapping $A: U \rightarrow \mathcal{L}$ such that for each $u$ in $U$, $A(u)$ represents the degree (in $\mathcal{L}$) to which $u$ satisfies $A$.

**Lemma 5.2.1**- [165] Consider the set $L^*$ and operation $\leq_{L^*}$ defined by

$$L^* = \{(x_1, x_2); (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } x_2 \geq y_2,$$

for every $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is complete Lattice.

**Definition 5.2.3**-[166]. An intuitionistic fuzzy set $A_{\zeta, \eta}$ on a universe $U$ is an object

$$A_{\zeta, \eta} = \{((\zeta_A(u), \eta_A(u)); u \in U\}$$

where for all $u \in U$, $\zeta_A(u) \in [0, 1]$ and $\eta_A(u) \in [0, 1]$ are called the membership degree and the non-membership degree respectively of $u$ in $A_{\zeta, \eta}$ and moreover

$$\zeta_A(u) + \eta_A(u) \leq 1.$$

A triangular norm $T$, on $([0, 1], \leq)$ is defined as an increasing, commutative, associative mapping $T: [0, 1]^2 \rightarrow [0, 1]$ satisfying $T(1, x) = x \forall x \in [0, 1]$.

Define $0_{\mathcal{L}} = \text{inf} \mathcal{L}$. 69
\[ 1_\mathcal{L} = \sup \mathcal{L} \]

**Definition 5.2.4-25** A triangular norm (t-norm) on \( \mathcal{L} \) is a mapping \( \mathcal{I} : \mathcal{L}^2 \rightarrow \mathcal{L} \) satisfying the respective conditions:

i. \( (\forall x \in \mathcal{L}) \ (\mathcal{I}(x, 1_\mathcal{L}) = x) \);
ii. \( (\forall (x, y) \in \mathcal{L}^2) \ (\mathcal{I}(x, y) = \mathcal{I}(y, x)) \);
iii. \( (\forall (x, y, z) \in \mathcal{L}^3) \ (\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(\mathcal{I}(x, y), z)) \);
iv. \( (\forall (x, x', y, y') \in \mathcal{L}^4) (x \leq_L x' \text{ and } y \leq_L y') \Rightarrow \mathcal{I}(x, y) \leq_L \mathcal{I}(x', y') \).

A t-norm can also be defined recursively as an \((n+1)\)-ary operation \((n \in \mathbb{N} \setminus \{0\})\) by \( \mathcal{I}^1 = \mathcal{I} \) and

\[ \mathcal{I}^n(x_{(1)}, x_{(2)}, \ldots, x_{(n+1)}) = \mathcal{I}(\mathcal{I}^{n-1}(x_{(1)}, x_{(2)}, \ldots, x_{(n)}), x_{(n+1)}) \]

for \( n \geq 2 \) and \( x_{(i)} \in \mathcal{L} \).

**Definition 5.2.5-167** A t-norm \( \mathcal{I} \) on \( \mathcal{L}^* \) is called t-representable iff there exists a t-norm \( T \) and t-conorm on \( S \) on \([0, 1]\) such that \( \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}^* \).

\[ \mathcal{I}(x, y) = (T(x_1, y_1), S(x_2, y_2)). \]

**Definition 5.2.6-25** A negation on \( \mathcal{L} \) is any decreasing mapping \( \mathcal{N} : \mathcal{L} \rightarrow \mathcal{L} \) satisfying \( \mathcal{N}(0_\mathcal{L}) = 1_\mathcal{L} \) and \( \mathcal{N}(1_\mathcal{L}) = 0_\mathcal{L} \). If \( \mathcal{N}(\mathcal{N}(x)) = x \ \forall x \in \mathcal{L} \), then \( \mathcal{N} \) is called an involutive negation.

If for all \( x \in [0, 1] \), \( \mathcal{N}_s(x) = 1 - x \), we say that \( \mathcal{N}_s \) is the standard negation on \(([0, 1], \leq)\).

**Definition 5.2.7-25** \( \mathcal{L} \)-fuzzy metric space is a 3-tuple space \((Z, \mathcal{M}, \mathcal{I})\) for an arbitrary (non-empty) set \( Z \), continuous t-norm \( \mathcal{I} \) on \( \mathcal{L} \) and an \( \mathcal{L} \)-fuzzy set \( \mathcal{M} \) on \( Z^2 \times (0, +\infty) \) such that

i. \( \mathcal{M}(z_1, z_2, t_1) \geq_L 0_\mathcal{L} \);
ii. \( \mathcal{M}(z_1, z_2, t_1) = 1_\mathcal{L} \) for all \( t_1 > 0 \) if and only if \( z_1 = z_2 \);
iii. \( \mathcal{M}(z_1, z_2, t_1) = \mathcal{M}(z_2, z_1, t_1) \);
iv. \( \mathcal{I}(\mathcal{M}(z_1, z_2, t_1), \mathcal{M}(z_2, z_3, t_2)) \leq_L \mathcal{M}(z_1, z_3, t_1 + t_2) \).
v. \( \mathcal{M}(z_1, z_2, \cdot) : ]0, \infty[ \to \mathbb{L} \) is continuous.

for every \( z_1, z_2, z_3 \) in \( \mathbb{Z} \) and \( t_1, t_2 \) in \( (0, +\infty) \)

In this case \( \mathcal{M} \) is an \( \mathbb{L} \)-fuzzy metric. The 3-tuple \((\mathbb{Z}, \mathcal{M}_{M,N}, \mathcal{J})\) is an Intuitionistic fuzzy metric space if \( \mathcal{M} = \mathcal{M}_{M,N} \) is an Intuitionistic fuzzy set.

Let \((\mathbb{Z}, \mathcal{M}, \mathcal{J})\) be an \( \mathbb{L} \)-fuzzy metric space. For \( t \in ]0, +\infty[ \), define the open ball \( B(z, s, t) \) with centre \( z \in \mathbb{Z} \) and a fixed radius \( s \in \mathbb{L} \setminus \{0, 1\} \) as

\[
B(z, s, t) = \{ y \in \mathbb{Z} : \mathcal{M}(z, y, t) >_L N(s) \}.
\]

A subset \( A \subseteq \mathbb{Z} \) is called open if for each \( z \in A \), there exist \( t > 0 \) and \( s \in \mathbb{L} \setminus \{0, 1\} \) such that \( B(z, s, t) \subseteq A \). Let \( \tau_M \) denote the family of all open subsets of \( \mathbb{Z} \). Then \( \tau_M \) is called the topology induced by the \( \mathbb{L} \)-fuzzy metric on \( \mathcal{M} \).

**Example 5.2.1-[178]** Let \((\mathbb{Z}, d)\) be a metric space. Set \( \mathcal{J}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1) ) \) for all \( a = (a_1, a_2) \) and \( b = (b_1, b_2) \) in \( \mathbb{L}^* \)and let \( M \) and \( N \) be fuzzy sets on \( \mathbb{Z}^2 \times (0, \infty) \) defined as follows:

\[
\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left( \frac{t}{d(x, y)} \frac{d(x, y)}{t+d(x, y)}, \frac{d(x, y)}{t+d(x, y)} \right) \text{ for all } t \in \mathbb{R}^+.
\]

Then \((\mathbb{Z}, \mathcal{M}_{M,N}, \mathcal{J})\) is an Intuitionistic fuzzy metric space.

**Example 5.2.2-[169]** Let \( \mathbb{Z} = \mathbb{N} \). Define \( \mathcal{J}(a, b) = (\max(0, a_1 + b_1 - 1, a_2 + b_2 - a_2 b_2) ) \) for all \( a = (a_1, a_2) \) and \( b = (b_1, b_2) \) in \( \mathbb{L}^* \)and let \( M \) and \( N \) be fuzzy sets on \( \mathbb{Z}^2 \times (0, \infty) \) defined as follows:

\[
\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \begin{cases} 
\left( \frac{x}{y}, \frac{y-x}{y} \right) & \text{if } x \leq y \\
\left( \frac{x}{y}, \frac{y-x}{y} \right) & \text{if } y \leq x 
\end{cases}
\]

for all \( x, y \in \mathbb{Z} \) and \( t > 0 \). Then \((\mathbb{Z}, \mathcal{M}, \mathcal{J})\) is an \( \mathbb{L} \)-fuzzy metric space.

**Lemma 5.2.2-[17]** Let \((\mathbb{Z}, \mathcal{M}, \mathcal{J})\) be an \( \mathbb{L} \)-fuzzy metric space. Then \( \mathcal{M}(x, y, t) \) is non-decreasing with respect to \( t \) for all \( x, y \in \mathbb{Z} \).
**Proof:** Let $t, s \in (0, +\infty)$ be such that $t < s$. Then $k = s - t > 0$ and
\[
\mathcal{M}(x, y, t) = J(M(x, y, t), 1_\mathcal{L}) = J(M(x, y, t), M(y, y, k)) \leq_t M(x, y, s).
\]

**Definition 5.2.8-[25]** A sequence $\{x_n\}_{n=1}^{\infty}$ in an $\mathcal{L}$-fuzzy metric space $(Z, \mathcal{M}, J)$ is called a Cauchy sequence, if for each $\varepsilon \in \mathcal{L}\setminus\{0, 1_\mathcal{L}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall m \geq n \geq n_0 (m \geq n \geq n_0)$,
\[
\mathcal{M}(x_m, x_n, t) >_t \mathcal{N}(\varepsilon).
\]

The sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x \in Z$ in the $\mathcal{L}$-fuzzy metric space $(Z, \mathcal{M}, J)$ (denoted by $x_n \to x$ in $\mathcal{M}$) if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \to 1_\mathcal{L}$ whenever $n \to +\infty$ for every $t > 0$.

A $\mathcal{L}$-fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent. Thus $J$ is a continuous $t$-norm on lattice $\mathcal{L}$ such that for every
\[
\mu \in \mathcal{L}\setminus\{0_\mathcal{L}, 1_\mathcal{L}\}, \text{ there is a } \lambda \in \mathcal{L}\setminus\{0_\mathcal{L}, 1_\mathcal{L}\} \text{ such that } J^{n-1}(N(\lambda), ..., N(\lambda)) >_t \mathcal{N}(\mu).
\]

**Definition 5.2.9-[174]** Let $(Z, \mathcal{M}, J)$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is said to be continuous on $Z \times Z \times ]0, \infty[$ that is
\[
\lim_{n \to \infty} \mathcal{M}(x_n, y_n, t_n) = \mathcal{M}(x, y, t)
\]
whenever a sequence $\{(x_n, y_n, t_n)\}$ in $Z \times Z \times ]0, \infty[$ converges to a point $(x, y, t) \in Z \times Z \times ]0, \infty[$ that is
\[
\lim_{n \to \infty} \mathcal{M}(x_n, x, t) = \lim_{n \to \infty} \mathcal{M}(y_n, y, t) = 1_\mathcal{L} \text{ and}
\lim_{n \to \infty} \mathcal{M}(x, y, t_n) = \mathcal{M}(x, y, t).
\]

**Lemma 5.2.3-[174]** Let $(Z, \mathcal{M}, J)$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is continuous function on $Z \times Z \times ]0, \infty[$.

**Lemma 5.2.4-[25, 174]** Let $(Z, \mathcal{M}, J)$ be an $\mathcal{L}$-fuzzy metric space. Define
\[
E_{\lambda, \mathcal{M}}: Z^2 \to \mathbb{R}^+ \cup \{0\} \text{ by } E_{\lambda, \mathcal{M}}(x, y) = \inf\{t > 0: \mathcal{M}(x, y, t) >_t \mathcal{N}(\lambda)\} \text{ for each } \lambda \in \mathcal{L}\setminus\{0_\mathcal{L}, 1_\mathcal{L}\} \text{ and } x, y \in Z. \text{ Then we have}
\]

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i. For any $\mu \in \mathcal{L}\backslash\{0_\mathcal{L}, 1_\mathcal{L}\}$ there exists $\lambda \in \mathcal{L}\backslash\{0_\mathcal{L}, 1_\mathcal{L}\}$ such that

$$E_{\mu,\mathcal{M}}(z_1, z_n) \leq E_{\lambda,\mathcal{M}}(z_1, z_2) + E_{\lambda,\mathcal{M}}(z_2, z_3) + \ldots + E_{\lambda,\mathcal{M}}(z_{n-1}, z_n)$$

for any $z_1, \ldots, z_n \in Z$.

ii. The sequence $\{z\}_{n \in \mathbb{N}}$ converges w.r.t $\mathcal{L}$-fuzzy metric $\mathcal{M}$ if and only if $E_{\lambda,\mathcal{M}}(z_n, z) \to 0$. Also the sequence $\{z_n\}_{n \in \mathbb{N}}$ is Cauchy w.r.t $\mathcal{L}$-fuzzy metric $\mathcal{M}$ if and only if it is Cauchy with $E_{\lambda,\mathcal{M}}$.

**Lemma 5.2.5** Let $(Z, \mathcal{M}, \mathcal{I})$ be an $\mathcal{L}$-fuzzy metric space. If

$$\mathcal{M}(z_n, z_{n+1}, t) \geq \mathcal{M}(z_0, z_1, k^nt)$$

for some $k > 1$ and $n \in \mathbb{N}$, then $\{z_n\}$ is a Cauchy sequence.

**Proof:** For every $\lambda \in \mathcal{L}\backslash\{0_\mathcal{L}, 1_\mathcal{L}\}$ and $z_n \in Z$, we have

$$E_{\lambda,\mathcal{M}}(z_{n+1}, z_n) = \inf \{t > 0 : \mathcal{M}(z_{n+1}, z_n, t) \geq \mathcal{N}(\lambda)\},$$

$$\leq \inf \{t > 0 : \mathcal{M}(z_0, z_1, k^nt) \geq \mathcal{N}(\lambda)\},$$

$$= \inf \{k^nt : \mathcal{M}(z_0, z_1, t) \geq \mathcal{N}(\lambda)\}$$

$$= \frac{1}{k^n} \inf \{t > 0 : \mathcal{M}(z_0, z_1, t) \geq \mathcal{N}(\lambda)\}$$

$$= \frac{1}{k^n} E_{\lambda,\mathcal{M}}(z_0, z_1).$$

From lemma 5.2.4, for every $\mu \in \mathcal{L}\backslash\{0_\mathcal{L}, 1_\mathcal{L}\}$ there exists $\lambda \in \mathcal{L}\backslash\{0_\mathcal{L}, 1_\mathcal{L}\}$, such that

$$E_{\mu,\mathcal{M}}(z_n, z_m) \leq E_{\lambda,\mathcal{M}}(z_n, z_{n+1}) + E_{\lambda,\mathcal{M}}(z_{n+1}, z_{n+2}) + \ldots + E_{\lambda,\mathcal{M}}(z_{m-1}, z_m),$$

$$\leq \frac{1}{k^n} E_{\lambda,\mathcal{M}}(z_0, z_1) + \frac{1}{k^{n+1}} E_{\lambda,\mathcal{M}}(z_0, z_1) + \ldots + \frac{1}{k^{m-1}} E_{\lambda,\mathcal{M}}(z_0, z_1),$$

$$= E_{\lambda,\mathcal{M}}(z_0, z_1) \sum_{l=n}^{m-1} \frac{1}{k^l} \to 0.$$

Hence the sequence $\{z_n\}$ is a Cauchy sequence.

The first main outcome is to justify the result for four self-mappings in $\mathcal{L}$-fuzzy metric space. To meet requirements of the first main result, biased mappings
of class \((R_M)\) and the property \(C\) are initiated in \(L\)-fuzzy metric space, which are as follows:

**Definition 5.2.10** - Two self-maps \(P\) and \(Q\) of \(L\)-fuzzy metric space \((Z, \mathcal{M}, J)\) are said to be \(Q\)-biased of type \((R_M)\) if

\[
\lim_{n \to \infty} \mathcal{M}(QPz_n, Qz_n, t) \geq L \lim_{n \to \infty} \mathcal{M}(PQz_n, Pz_n, t) \text{ for } t > 0,
\]

whenever \(\{z_n\}\) is a sequence in \(Z\) such that \(Pz_n, Qz_n \to p \in Z\).

Two self-maps \(P\) and \(Q\) of \(L\)-fuzzy metric space \((Z, \mathcal{M}, J)\) are said to be \(P\)-biased of type \((R_M)\) if

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\]

whenever \(\{z_n\}\) is a sequence in \(Z\) such that \(Pz_n, Qz_n \to p \in Z\).

**Definition 5.2.11** -\([78]\) \(L\)-fuzzy metric space \((Z, \mathcal{M}, J)\) is said to have a property \(C\), if \(\mathcal{M}(x, y, t) = C\) for all \(t > 0\) implying \(C = 1_L\).

Theorem 5.3.1 is the extension of Singh et al. \([77]\) result on Fuzzy metric space that is:

**Theorem 5.2.1** - Let \(A, B, S\) and \(T\) be self-maps of a Fuzzy metric space \((Z, M, *)\) with \(a*a \geq a, \forall a \in (0, 1]\) satisfying

i. \(A(Z) \subseteq T(Z)\) and \(B(Z) \subseteq S(Z)\),

ii. \(M(Ax, By, t) \geq \varphi(M(Sx, Ty, t)) \forall x, y \in Z\),

where \(\varphi: [0, 1] \to [0, 1]\) is a continuous function such that \(\varphi(s) > s\) and \(\varphi(1) = 1\).

iii. one of each pair \(\{A, S\}\) and \(\{B, T\}\) are continuous,

iv. \(\{A, S\}\) is \(S\)-biased of type \((R_M)\) and \(\{B, T\}\) is \(T\)-biased of type \((R_M)\).

Further, suppose that one of the range space of \(A, B, S\) and \(T\) is complete subspace of \(Z\) then \(A, B, S\) and \(T\) have a unique common fixed point in \(Z\).

Di Bari and Vetro \([85]\) defined triangular condition in Fuzzy metric space and at the same time Pacurar and Rus \([81]\) propounded cyclic representation of a non-void
set X with respect to a self-mapping f. Imdad et al. [84] put forward the notion of cyclic weak $\emptyset$-contraction mapping and proved fixed point results on Fuzzy metric spaces for which basic concepts are as follows:

**Definition 5.2.12-[85]** Let $(Z, M, *)$ be a Fuzzy metric space. The fuzzy metric $M$ is triangular if

\[
\frac{1}{M(x, y, t)} - 1 \leq \left( \frac{1}{M(x, z, t)} - 1 \right) + \left( \frac{1}{M(y, z, t)} - 1 \right)
\]

for every $x, y, z \in Z$ and every $t > 0$.

**Definition 5.2.13-[81]** Let $Z$ be a non-empty set, $m$ be a positive integer and $f : Z \to Z$ an operator. By definition, $Z = \bigcup_{i=1}^{m} Z_i$ is a cyclic representation of $Z$ with respect to $f$ if

i. $Z_i$, $i = 1, 2, \ldots, m$ are nonempty sets;

ii. $f(Z_1) \subset Z_2$, $\ldots$, $f(Z_{m-1}) \subset Z_m$, $f(Z_m) \subset Z_1$.

**Example 5.2.3-[84]** Let $Z = \mathbb{R}$. Assume $B_1 = [-2, 0]$ and $B_2 = [0, 2]$ so that $X = \bigcup_{i=1}^{4} B_i = [-2, 2]$. Define $g : X \to X$ such that $g(x) = -\frac{x}{2}$, for all $x \in X$. It is clear that $X = \bigcup_{i=1}^{4} B_i$ is a cyclic representation of $X$.

**Definition 5.2.14-[84]** Let $(Z, M, *)$ be a Fuzzy metric space, $B_1, B_2, B_3, \ldots, B_m$ be a closed subsets of $Z$ and $X = \bigcup_{i=1}^{m} B_i$. An operator $g : X \to X$ is called a cyclic weak $\emptyset$-contraction if the following conditions hold:

i. $X = \bigcup_{i=1}^{m} B_i$ is a cyclic representation of $X$ with respect to $g$,

ii. There exists a continuous, non-decreasing function $\emptyset : [0, +\infty) \to [0, +\infty)$ with $\emptyset(r) > 0$ for $r > 0$ and $\emptyset(0) = 0$, such that

\[
\frac{1}{M(gx, gy, t)} - 1 \leq \left( \frac{1}{M(x, y, t)} - 1 \right) - \emptyset \left( \frac{1}{M(x, y, t)} - 1 \right)
\]

for any $x \in B_i$, $y \in B_{i+1}$ ($i = 1, 2, \ldots, m$ where $B_{m+1} = B_1$).

**Theorem 5.2.2-** Let $(Z, M, *)$ be a Fuzzy metric space, $B_1, B_2, B_3, \ldots, B_m$ be a closed subsets of $Z$ and $X = \bigcup_{i=1}^{m} B_i$ be $G$-complete. Suppose that
\[ \emptyset : [0, +\infty) \to [0, +\infty) \text{ is a continuous, non-decreasing function with } \emptyset(r) > 0 \text{ for each } r \in (0, +\infty) \text{ and } \emptyset(0) = 0. \]

If \( g: X \to X \) is a cyclic weak \( \emptyset \)-contraction, then \( g \) has a unique fixed point \( z \in \bigcap_{i=1}^{m} B_i \).

Moreover Imdad et al. [84] proved the result by considering two Fuzzy metric spaces that is:

**Theorem 5.2.3-** Let \((Z, \mathcal{M}, \ast)\) and \((Z, \mathcal{M}', \ast)\) be two Fuzzy metric spaces, \( m \) be a positive integer, \( B_1, B_2, B_3, \ldots, B_m \) be closed non-empty subsets of \( Z \) and \( X = \bigcup_{i=1}^{m} B_i \). Assuming

i. \( X = \bigcup_{i=1}^{m} B_i \) is a cyclic representation of \( X \) with respect to \( g \);
ii. \( \mathcal{M}(x, y, t) \geq \mathcal{M}'(x, y, t), \forall x, y \in Z \);
iii. \((X, \mathcal{M}, \ast)\) is a \( G \)-complete fuzzy metric space;
iv. \( g: (X, \mathcal{M}, \ast) \to (X, \mathcal{M}, \ast) \) is continuous;
v. \( g: (X, \mathcal{M}', \ast) \to (X, \mathcal{M}', \ast) \) is a cyclic weak \( \emptyset \)-contraction;

where \( \emptyset : [0, +\infty) \to [0, +\infty) \) is a lower semi-continuous function with \( \emptyset(s) > 0 \) for each \( s \in (0, +\infty) \) and \( \emptyset(0) = 0. \) Then \( \{g^n(x_0)\} \to z \in (X, \mathcal{M}, \ast), \forall x_0 \in X \) and \( z \) is a unique fixed point of \( g \).

Inspired by Di Bari and Vetro [85], Pacurar and Rus [81] and Imdad et al. [84] we now elucidate the triangular condition and cyclic weak \( \emptyset \)-contraction mapping for two self-mappings in the \( L \)-fuzzy metric space.

**Definition 5.2.15-** Let \((Z, \mathcal{M}, J)\) be an \( L \)-fuzzy metric space. The \( L \)-fuzzy metric is triangular if

\[
\left( \frac{1}{\mathcal{M}(u, v, t)} - 1 \right) \leq \left( \frac{1}{\mathcal{M}(u, w, t)} - 1 \right) + \emptyset\left( \frac{1}{\mathcal{M}(v, w, t)} - 1 \right),
\]

for every \( u, v, w \in Z \) and every \( t > 0 \), where \( \emptyset \) is non-decreasing continuous function defined by \( \emptyset : [0, +\infty) \to [0, +\infty) \) with \( \emptyset(r) > 0 \) for \( r > 0 \) and \( \emptyset(0) = 0. \)
**Definition 5.2.16**- Let \((Z, \mathcal{M}, \mathcal{I})\) be an \(L\)-fuzzy metric space, \(B_1, B_2, \ldots, B_m\) be closed subsets of \(Z\) and \(X = \bigcup_{i=1}^{m} B_i\). An operator \(P_1, P_2 : X \to X\) are called cyclic weak \(\emptyset\)-contraction if the underneath conditions hold-

- iii. \(X = \bigcup_{i=1}^{m} B_i\) is a cyclic representation of \(X\) with respect to \(P_1\).
- iv. There exists a continuous, non-decreasing function \(\emptyset : [0, +\infty) \to [0, +\infty)\) with \(\emptyset(r) > 0\) for \(r > 0\) and \(\emptyset(0) = 0\), such that

\[
\frac{1}{\mathcal{M}(P_1 u, P_2 v, t)} - 1 \leq L \left( \frac{1}{\mathcal{M}(P_1 u, w, kt)} - 1 \right) - \emptyset \left( \frac{1}{\mathcal{M}(P_2 v, w, kt)} - 1 \right)
\]

For any \(u \in B_i, v \in B_{i+1}\) \((i = 1, 2\ldots m,\) where \(B_{m+1} = B_1\) and each \(t > 0\).

**5.3 - MAIN THEOREMS**

**THEOREM (I)**

**Theorem 5.3.1**- Let \(P_1, P_2, P_3\) and \(P_4\) be self-maps of a \(L\)-fuzzy metric space \((Z, \mathcal{M}, \mathcal{I})\) with a property \(C\) into itself and for \(\alpha \in (0, 1)\) satisfying

- i. \(P_1 (Z) \cup P_2 (Z) \subseteq P_3 (Z) \cap P_4 (Z)\);
- ii. \(\mathcal{M}(P_1 u, P_2 v, t) \geq L \mathcal{M}(P_3 u, P_4 v, t/\alpha)\);
- iii. One of each pair \(\{P_1, P_3\}\) and \(\{P_2, P_4\}\) is continuous;
- iv. The pair \(P_1, P_3\) is \(P_3\)-biased of type \((R_M)\) and \(\{P_2, P_4\}\) is \(P_4\)-biased of type \((R_M)\).

If one of the range space of \(P_1, P_2, P_3\) and \(P_4\) is complete subspace of \(Z\) then \(P_1, P_2, P_3\) and \(P_4\) have common fixed point in \(Z\).

**Proof:** Since from (i) \(P_1 (Z) \subseteq P_4 (Z)\) then choose randomly any point \(u_0 \in Z\) such that there exists a point \(u_1 \in Z\) so that \(P_1 u_0 = P_4 u_1\).

Similarly as \(P_2 (Z) \subseteq P_3 (Z)\), thus for any arbitrary point \(u_1 \in Z\) there exists a point \(u_2 \in Z\) so that \(P_2 u_1 = P_3 u_2\).

Therefore by induction we construct a sequence \(\{u_n\}\) such that
\[ v_{2n} = P_4 u_{2n+1} = P_1 u_{2n} \text{ and } v_{2n+1} = P_3 u_{2n+2} = P_2 u_{2n+1} \text{ for } n \in N \cup \{0\}. \]

We first prove \( \{v_n\} \) is a Cauchy sequence in \((Z, \mathcal{M}, J)\)

\[ \mathcal{M}(v_{2n}, v_{2n+1}, t) = \mathcal{M}(P_1 u_{2n}, P_2 u_{2n+1}, t) \]

\[ \geq \mathcal{L} \mathcal{M}(P_3 u_{2n}, P_4 u_{2n+1}, \frac{t}{\alpha}) = \mathcal{M}(v_{2n-1}, v_{2n}, \frac{t}{\alpha}). \]

\( \Rightarrow \mathcal{M} \) is non-decreasing in \((Z, \mathcal{M}, J)\).

\[ \Rightarrow \mathcal{M}(v_n, v_{n+1}, t) \geq \mathcal{L} \mathcal{M}(v_{n-1}, v_n, \frac{t}{\alpha}) \geq \mathcal{L} \mathcal{M}(v_{n-2}, v_{n-1}, \frac{t}{\alpha^2}) \cdots \]

\[ \geq \mathcal{L} \mathcal{M}(v_0, v_1, \frac{t}{\alpha^n}). \]

This entails \( E_{\lambda, \mathcal{M}}(v_n, v_{n+1}) \leq \alpha^n E_{\lambda, \mathcal{M}}(v_0, v_1). \)

Therefore for every \( \mu \in L \setminus \{0, 1\} \) there exists \( \gamma \in L \setminus \{0, 1\} \) such that

\[ E_{\mu, \mathcal{M}}(v_n, v_m) \leq E_{\gamma, \mathcal{M}}(v_n, v_{n+1}) + E_{\gamma, \mathcal{M}}(v_{n+1}, v_{n+2}) + \cdots + E_{\gamma, \mathcal{M}}(v_{m-1}, v_m) \]

\[ \leq \alpha^n E_{\gamma, \mathcal{M}}(v_0, v_1) + \alpha^{n+1} E_{\gamma, \mathcal{M}}(v_0, v_1) + \cdots + \alpha^{m-1} E_{\gamma, \mathcal{M}}(v_0, v_1) \]

\[ \leq E_{\gamma, \mathcal{M}}(v_0, v_1) \sum_{i=n}^{m-1} \alpha^i \to 0. \]

Thus by lemma (5.2.4) and (5.2.5), we get \( \{v_n\} \) as a Cauchy sequence. Now hypothesize that \( P_3 \) \((Z)\) is complete subspace of \( Z \) then the \( \{v_{2n+1}\} \subseteq P_3 \) \((Z)\) converges to a point \( x \in P_3(Z) \). Since \( \{v_{2n}\} \) is also a subsequence of \( \{v_n\} \) which converges to \( x \in P_3(Z) \). This shows that the sequence \( \{v_n\} \) converges to a point \( x \in P_3(Z) \).

Thus \( P_1 u_{2n}, P_2 u_{2n+1}, P_3 u_{2n+2} \) and \( P_4 u_{2n+1} \) converges to \( x \in P_3(Z) \).

As \( P_3 \) \((Z)\) is complete subspace of \( Z \), thus there exist a point \( y \in Z \) such that \( P_3 y = x \).

Put \( v = u_{2n+1} \) and \( u = y \) in (ii), we get

\[ \mathcal{M}(P_1 y, P_2 u_{2n+1}, t) \geq \mathcal{L} \mathcal{M}(P_3 y, P_4 u_{2n+1}, \frac{t}{\alpha}). \]

Take the limit as \( n \to \infty \), we get

\[ \mathcal{M}(P_1 y, x, t) \geq \mathcal{L} \mathcal{M}(P_3 y, x, \frac{t}{\alpha}) = \mathcal{M}(x, x, \frac{t}{\alpha}) = 1_{\mathcal{L}}. \]
This implies $P_1 y = x = P_3 y$.

Now assume that $P_3$ is continuous then $P_3 P_3 u_{2n} \to P_3 x$.

Also $\{P_1, P_3\}$ is $P_3$-biased of type $(R_M)$.

$\Rightarrow \lim_{n \to \infty} M(P_3 P_1 u_n, P_3 u_n, t) \geq L \lim_{n \to \infty} M(P_1 P_3 u_n, P_1 u_n, t)$ for $t > 0$.

Now to prove $P_3 x = x$, for this put $u = P_3 u_{2n}, v = u_{2n+1}$ in (ii).

$\mathcal{M} (P_1 P_3 u_{2n}, P_2 u_{2n+1}, t) \geq L \mathcal{M} (P_3 P_3 u_{2n}, P_4 u_{2n+1}, t/\alpha)$.

Now take the limit as $n \to \infty$

$\lim_{n \to \infty} \mathcal{M} (P_1 P_3 u_{2n}, P_1 u_{2n}, t) =
\lim_{n \to \infty} \mathcal{M} (P_1 P_3 u_{2n}, P_2 u_{2n+1}, t) \geq L \lim_{n \to \infty} \mathcal{M} (P_3 P_3 u_{2n}, P_4 u_{2n+1}, \frac{t}{\alpha})$.

This signifies

$\lim_{n \to \infty} \mathcal{M} (P_3 P_1 u_{2n}, P_3 u_{2n}, t) \geq L \lim_{n \to \infty} \mathcal{M} (P_1 P_3 u_{2n}, P_1 u_{2n}, t)$

$\geq L \lim_{n \to \infty} \mathcal{M} (P_3 P_3 u_{2n}, P_4 u_{2n+1}, \frac{t}{\alpha})$.

$\Rightarrow \mathcal{M} (P_3 x, x, t) \geq L \mathcal{M} (P_3 x, x, t/\alpha)$.

Therefore we have $\mathcal{M} (P_3 x, x, t/\alpha) \geq L \mathcal{M} (P_3 x, x, t/\alpha^2) \geq L \mathcal{M} (P_3 x, x, t/\alpha^3) \geq L \mathcal{M} (P_3 x, x, t/\alpha^n)$.

But by lemma (5.2.5) in $Z$.

$\Rightarrow \mathcal{M} (P_3 x, x, t/\alpha) \leq L \mathcal{M} (P_3 x, x, t/\alpha^n)$.

Hence $\mathcal{M} (P_3 x, x, t/\alpha) = C$ for all $t > 0$.

Since $\mathcal{M} (P_3 x, x, t/\alpha)$ has the property (C), it follows that $C = 1_L$.

That is $P_3 x = x$.

Currently to prove $P_1 x = P_3 x$.

Put $u = x$ and $v = u_{2n+1}$ in (ii), we get

$\mathcal{M} (P_1 x, P_3 x, t) = \mathcal{M} (P_1 x, x, t) = \mathcal{M} (P_1 x, P_2 u_{2n+1}, t) \geq L \mathcal{M} (P_3 x, P_4 u_{2n+1}, \frac{t}{\alpha})$. 79
Thus $P_1 x = P_3 x$.
So $P_1 x = x = P_3 x$.
Now to submit proof that $P_2 x = x = P_4 x$.

Suppose that $P_4$ is continuous, thus $P_4 P_4 u_{2n+1} 	o P_4 u$.

As $\{P_2, P_4\}$ is $P_4$-biased type of $(RM)$

$$\lim_{n \to \infty} M(P_4 P_2 u_n, P_4 u_n, t) \geq_L \lim_{n \to \infty} M(P_2 P_4 u_n, P_2 u_n, t) \text{ for } t > 0.$$ 

To produce $P_4 x = x$, for this take $u = u_{2n}, v = P_4 u_{2n+1}$ in (ii).

$M\left(P_1 u_{2n}, P_2 P_4 u_{2n+1}, t\right) \geq_L M\left(P_3 u_{2n}, P_4 P_4 u_{2n+1}, t/\alpha\right).$

$M\left(P_2 P_4 u_{2n+1}, P_1 u_{2n}, t\right) \geq_L M\left(P_4 P_4 u_{2n+1}, P_3 u_{2n}, t/\alpha\right).$

Now take the limit as $n \to \infty$

$$\lim_{n \to \infty} M(P_2 P_4 u_{2n+1}, P_2 u_{2n+1}, t) = $$

$$\lim_{n \to \infty} M(P_2 P_4 u_{2n+1}, P_1 u_{2n}, t) \geq_L \lim_{n \to \infty} M(P_4 P_4 u_{2n+1}, P_3 u_{2n}, \frac{t}{\alpha}).$$

$$\Rightarrow \lim_{n \to \infty} M(P_4 P_2 u_{2n+1}, P_4 u_{2n+1}, t) \geq_L \lim_{n \to \infty} M(P_2 P_4 u_{2n+1}, P_2 u_{2n+1}, t) \geq_L \lim_{n \to \infty} M(P_4 P_4 u_{2n+1}, P_3 u_{2n}, \frac{t}{\alpha}).$$

$$\Rightarrow M\left(P_4 x, x, t\right) \geq_L M\left(P_4 x, x, t/\alpha\right).$$

Then as proved above on using the lemma (5.2.5) in $Z$, we have

$$\Rightarrow M\left(P_4 x, x, t/\alpha\right) \leq_L M\left(P_4 x, x, \frac{t}{\alpha^n}\right).$$

This shows that $M\left(P_4 x, x, t/\alpha\right) = C$ for all $t > 0$.

Since $M\left(P_4 x, x, t/\alpha\right)$ has the property (C), it follows that $C = 1_L$, that is $P_4 x = x$.

Thus $P_4 x = x$.

To prove $P_2 x = P_4 x$, put $u = u_{2n}, v = x$ in (ii), we get

$M\left(P_1 u_{2n}, P_2 x, t\right) \geq_L M(P_3 u_{2n}, P_4 x, t/\alpha).$
Take limit as \( n \) tends to \( \infty \) we get

\[
\mathcal{M} (P_4x, P_2x, t) = \mathcal{M} (x, P_2x, t) \geq L \mathcal{M} (x, x, \frac{t}{\alpha}) = 1_L.
\]

Thus \( P_2x = P_4x \).

Therefore \( P_4x = x = P_2x \).

This implies \( P_1, P_2, P_3 \) and \( P_4 \) have a common fixed point \( x \in Z \).

**Corollary 5.3.2**-Let \( P_1, P_3 \) and \( P_4 \) be self-maps of a \( L \)-fuzzy metric space \( (Z, \mathcal{M}, I) \) with a property \( C \) into itself and for \( \alpha \in (0, 1) \) satisfying

i. \( P_1 (Z) \subseteq P_3 (Z) \cap P_4 (Z) \);

ii. \( \mathcal{M}(P_1u, P_1v, t) \geq L \mathcal{M}(P_3u, P_4v, \frac{t}{\alpha}) \);

iii. One of each pair \( \{P_1, P_3\} \) and \( \{P_1, P_4\} \) is continuous;

iv. The pair \( \{P_1, P_3\} \) is \( P_3 \)-biased of type \( (R_M) \) and \( \{P_1, P_4\} \) is \( P_4 \)-biased of type \( (R_M) \).

If one of the range space of \( P_1, P_3 \) and \( P_4 \) is complete subspace of \( Z \) then \( P_1, P_3 \) and \( P_4 \) have common fixed point in \( Z \).

**Proof:** To prove the result take \( P_2 = P_1 \) in the theorem 5.3.1.

**Corollary 5.3.3**-Let \( P_1, P_2 \) and \( P_4 \) be self-maps of a \( L \)-fuzzy metric space \( (Z, \mathcal{M}, I) \) with a property \( C \) into itself and for \( \alpha \in (0, 1) \) satisfying

i. \( P_1 (Z) \cup P_2 (Z) \subseteq P_4 (Z) \);

ii. \( \mathcal{M}(P_1u, P_2v, t) \geq L \mathcal{M}(P_4u, P_4v, \frac{t}{\alpha}) \);

iii. One of each pair \( \{P_1, P_4\} \) and \( \{P_2, P_4\} \) is continuous;

iv. The pair \( \{P_1, P_4\} \) is \( P_4 \)-biased of type \( (R_M) \) and \( \{P_2, P_4\} \) is \( P_4 \)-biased of type \( (R_M) \).

If one of the range space of \( P_1, P_2 \) and \( P_4 \) is complete subspace of \( Z \) then \( P_1, P_2 \) and \( P_4 \) have common fixed point in \( Z \).

**Proof:** To prove the result take \( P_2 = P_1 \) in the theorem 5.3.1.
THEOREM (II)

The next main out turn of the chapter 5 is on use of weakly $\emptyset$-contraction mappings in $\mathcal{L}$-fuzzy metric space. To prove this result the notions of the triangular condition and the weak $\emptyset$-contraction mappings are used in $\mathcal{L}$-fuzzy metric space. The fixed point theorem is proved as follows:

**Theorem 5.3.4**-Let $(Z, \mathcal{M}, \mathcal{I})$ be an $\mathcal{L}$-fuzzy metric space, $Q_1$, $Q_2$, $\ldots$, $Q_m$ be closed subsets of $Z$ and $X = \bigcup_{i=1}^{m} Q_i$ be complete. Suppose that

$\emptyset : [0, +\infty) \rightarrow [0, +\infty)$ is continuous, non-decreasing function with $\emptyset(r) > 0$ for $r \in (0, +\infty)$ and $\emptyset(0) = 0$.

If $P_1$, $P_2$, $P_3$ and $P_4 : X \rightarrow X$ be a cyclic weak $\emptyset$-contraction such that

1. $P_1(Z) \cup P_2(Z) \subseteq P_3(Z) \cap P_4(Z)$,
2. $(\frac{1}{\mathcal{M}(P_1u, P_2v)} - 1) \leq (\frac{1}{\mathcal{M}(P_3u, P_4v)} - 1) - \emptyset(\frac{1}{\mathcal{M}(P_1u, P_3u, kt)} - 1)$.

Then $P_1$, $P_2$, $P_3$ and $P_4$ has a unique fixed point $w \in \bigcap_{i=1}^{m} Q_i$.

**Proof:** Since $P_1(Z) \cup P_2(Z) \subseteq P_3(Z) \cap P_4(Z)$, thus we have $P_1(Z) \subseteq P_4(Z)$, therefore for any in consistent point $u_0 \in X = \bigcup_{i=1}^{m} Q_i$ there exists $u_1 \in X$ such that $P_1u_0 = P_4u_1$.

And $P_2(Z) \subseteq P_3(Z)$, for this point $u_1$, we select a point $u_2 \in X$ such that $P_2u_1 = P_3u_2$ and so on.

Thus by induction we can define a sequence $\{v_n\}$ such that $v_{2n} = P_4u_{2n+1} = P_1u_{2n}$. $v_{2n+1} = P_3u_{2n+2} = P_2u_{2n+1}$ for every $n \in \mathbb{N} \{0\}$.

Now to prove that $\{v_n\}$ is a cauchy sequence in $Z$.

Put $u = u_{2n}$, $v = u_{2n+1}$ in (ii)

$(\frac{1}{\mathcal{M}(P_1u_{2n}, P_2u_{2n+1})} - 1) \leq (\frac{1}{\mathcal{M}(P_3u_{2n}, P_4u_{2n+1})} - 1) - \emptyset(\frac{1}{\mathcal{M}(P_1u_{2n}, P_3u_{2n+1}, kt)} - 1)$.

$\Rightarrow (\frac{1}{\mathcal{M}(v_{2n}, v_{2n+1})} - 1) \leq (\frac{1}{\mathcal{M}(v_{2n-1}, v_{2n})} - 1) - \emptyset(\frac{1}{\mathcal{M}(v_{2n-1}, v_{2n-1}, kt)} - 1)$.
\[ M(v_{2n}, v_{2n+1}, t) \geq L M(v_{n-1}, v_{n}, kt) \geq L \cdots \geq L M(v_0, v_1, k^nt). \]

This implies \( M(v_{2n}, v_{2n+1}, t) \) is a non-decreasing sequence of positive real numbers in \([0, 1]\).

Thus we have

\[ M(v_{n}, v_{n+1}, t) \geq L M(v_{n-1}, v_{n}, kt) \geq L \cdots \geq L M(v_0, v_1, k^nt). \]

Therefore by lemma 5.5, we have \( \{v_n\} \) is a Cauchy sequence.

Since \( X \) is complete, there exists a point \( w \in X \) such that \( \lim_{n \to \infty} v_n = w. \)

Thus \( P_1 u_{2n}, P_2 u_{2n+1}, P_3 u_{2n+2} \) and \( P_4 u_{2n+1} \) converge to \( w \in X. \)

Since the mappings are \( \emptyset \)-contraction, thus the iterative sequence \( \{v_n\} \) has an infinite number of terms in \( Q_i \) for each \( i = 1, 2 \ldots m. \) As \( X \) is complete then for each \( Q_i, i=1, 2 \ldots m, \) one can extract a subsequence of \( \{v_n\} \) that converges to \( w. \) Also each \( Q_i, i=1, 2 \ldots m, \) is closed, we have that point \( v \in \bigcap_{i=1}^{m} Q_i \) and \( \bigcap_{i=1}^{m} Q_i \neq \emptyset. \)

Next we are to prove that \( w \) is fixed point of \( P_1, P_2, P_3 \) and \( P_4. \)

As \( P_2 (Z) \subseteq P_3 (Z) \) therefore there exists a point \( x \in Z \) such that \( P_3 x = w. \)

Now on putting \( u = x \) and \( v = u_{2n+1} \) in (ii), we get

\[ \left( \frac{1}{M(P_1 x, P_2 u_{2n+1}, t)} - 1 \right) \leq L \left( \frac{1}{M(P_3 x, P_4 u_{2n+1}, kt)} - 1 \right) - \emptyset \left( \frac{1}{M(P_1 x, P_3 x, kt)} - 1 \right). \]

Now on taking the limit as \( n \to \infty \) we get

\[ \left( \frac{1}{M(P_1 x, w, t)} - 1 \right) \leq L \left( \frac{1}{M(w, w, kt)} - 1 \right) - \emptyset \left( \frac{1}{M(P_1 x, w, kt)} - 1 \right). \]

\[ \Rightarrow M(P_1 x, w, t) \geq L M(w, w, kt) = 1_L \text{ as } M \text{ is continuous.} \]

Thus we have \( P_1 x = P_3 x = w. \)

Now as \( P_1 (Z) \subseteq P_4 (Z), \) therefore there exists a point \( y \in Z \) such that \( P_4 y = w. \)
Now on putting \( u = u_{2n} \) and \( v = y \) in (ii)

we get

\[
\left( \frac{1}{\mathcal{M}(P_{1}u_{2n}, P_{2}y, t)} - 1 \right) \leq \left( \frac{1}{\mathcal{M}(P_{3}u_{2n}, P_{4}y, kt)} - 1 \right) - \varnothing\left( \frac{1}{\mathcal{M}(P_{1}u_{2n}, P_{3}u_{2n}, kt)} - 1 \right).
\]

Now on taking the limit as \( n \to \infty \) we get,

\[
\Rightarrow \left( \frac{1}{\mathcal{M}(w, P_{2}y, t)} - 1 \right) \leq \left( \frac{1}{\mathcal{M}(w, w, kt)} - 1 \right) - \varnothing\left( \frac{1}{\mathcal{M}(w, w, kt)} - 1 \right).
\]

\[
\Rightarrow \mathcal{M}(w, P_{2}y, t) \geq \mathcal{M}(w, w, kt) = 1 \mathcal{L}, \text{ as } \mathcal{M} \text{ is continuous.}
\]

Thus we have \( P_{2}y = P_{4}y = w \).

\[
\Rightarrow P_{1}x = P_{3}x = P_{2}y = P_{4}y = w.
\]

Now we are to prove that \( P_{1}w = w \), for this put \( u = w \) and \( v = y \) in (ii) we get

\[
\left( \frac{1}{\mathcal{M}(P_{1}w, P_{2}y, t)} - 1 \right) \leq \left( \frac{1}{\mathcal{M}(P_{3}w, P_{4}y, kt)} - 1 \right) - \varnothing\left( \frac{1}{\mathcal{M}(P_{1}w, P_{3}w, kt)} - 1 \right).
\]

\[
\Rightarrow \left( \frac{1}{\mathcal{M}(P_{1}w, w, t)} - 1 \right) \leq \left( \frac{1}{\mathcal{M}(P_{1}w, w, kt)} - 1 \right) - \varnothing\left( \frac{1}{\mathcal{M}(P_{1}w, P_{1}w, kt)} - 1 \right)
\]

\[
\leq \left( \frac{1}{\mathcal{M}(P_{1}w, w, kt)} - 1 \right).
\]

\[
\Rightarrow \mathcal{M}(P_{1}w, w, t) \geq \mathcal{M}(P_{1}w, w, kt) \geq \mathcal{M}(P_{1}w, w, k^{n}t).
\]

But as \( \mathcal{M}(u, v, t) \) is non-decreasing with respect to \( t \) for all \( u, v \) in \( Z \).

Hence \( \mathcal{M}(P_{1}w, w, t) = C \) for all \( t > 0 \).

Since \((Z, \mathcal{M}, J)\) has property \((C)\), it follows that \( C = 1 \mathcal{L} \).

Thus we get \( P_{1}w = w \), therefore \( P_{1}w = P_{3}w = w \).

Similarly we can prove that \( P_{2}w = P_{4}w = w \).

Thus \( P_{1}, P_{2}, P_{3} \) and \( P_{4} \) have a fixed point \( w \in \bigcap_{i=1}^{m} Q_{i} \).

Now to prove their uniqueness let us choose another point \( z \in \bigcap_{i=1}^{m} Q_{i} \). Therefore \( z \) is another common fixed point of \( P_{1}, P_{2}, P_{3} \) and \( P_{4} \) that is \( P_{1}z = P_{2}z = P_{3}z = P_{4}z = z \).
Thus
\[
\frac{1}{M(w, z, t)} - 1 = \frac{1}{M(P_1w, P_2z, t)} - 1 \\
\geq L \left( \frac{1}{M(P_3w, P_4z, kt)} - 1 \right) - \varnothing \left( \frac{1}{M(P_1w, P_3w, kt)} - 1 \right).
\]
\[
\Rightarrow \left( \frac{1}{M(w, z, t)} - 1 \right) = \left( \frac{1}{M(w, z, t)} - 1 \right) \leq L \left( \frac{1}{M(w, z, kt)} - 1 \right) - \varnothing \left( \frac{1}{M(w, w, kt)} - 1 \right).
\]
\[
\leq L \left( \frac{1}{M(w, z, kt)} - 1 \right).
\]
\[
\Rightarrow M(w, z, t) \geq L M(w, z, kt) \ldots \geq L M(w, z, k^n t).
\]
Again as \(M(u, v, t)\) is non-decreasing with respect to \(t\) for all \(u, v\) in \(Z\).

Hence \(M(w, z, t) = C\) for all \(t > 0\).

Since \((Z, M, \mathcal{I})\) has property \((C)\), it follows that \(C = 1_L\).

Thus we get \(w = z\).

Therefore \(w\) is the unique common fixed point of maps \(P_1, P_2, P_3\) and \(P_4\).

5.4 - CONCLUSIONS

As a inference of 5th chapter of the thesis we say that a fixed point result is obtained for biased mapping of class \((R_M)\) in \(L\)-fuzzy metric space using the property \(C\) for four self-mappings. The same result could be secured for three self-mappings.

The second section of the main theorem leads to the conclusion that a successful fixed point result is proved for four self-mappings in \(L\)-fuzzy metric space. The triangular condition and notion of cyclic weak \(\varnothing\)-contraction mappings introduced in \(L\)-fuzzy metric space find its best use.