CHAPTER 4

FIXED POINT FOR A CLASS OF MAPPINGS
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4.1 Introduction

Let \( M \) be a closed convex subset of a Banach space \( X \) and \( f \) be a self mapping on \( M \) satisfying the condition

\[
||fx-fy|| \leq \alpha \max\{||x-y||, \frac{1}{2}[||x-fx|| + ||y-fy||]\}.
\]

\[
\frac{1}{2}[||y-fx|| + ||x-fy||]
\]

for \( 0 < \alpha < 1 \) and \( x, y \in M \).

In this chapter we have first established a result on the existence of fixed points for the operator \( f \), satisfying (4.1.1). In proving the theorem, we have not assumed the space \( X \) to be either uniformly convex or reflexive. We have then given some results using the inequality (4.1.1). Finally, we have proved some fixed point theorems when \( f \) satisfies the following condition.

\[
||fx-fy|| \leq \max\{||x-y||, \frac{1}{2}[||x-fx|| + ||y-fy||]\}.
\]

\[
\frac{1}{2}[||x-fy|| + ||y-fx||]
\]

(4.1.2)

The results obtained by using inequality (4.1.2) generalize the results of Edelstein (1964), Browder and Petryshyn (1967) and Kirk (1969).
4.2 **Definition**

**DEFINITION 1.** A set $M$ in a linear space is said to be star-shaped about a point $y \in M$ if for every $z \in M$ there is a non-negative number $t_z : 0 \leq t_z \leq \infty$ such that the set \{ $y + t_z | 0 \leq t \leq t_z$ \} is in $M$ and the set \{ $y + t_z | t_z < t$ \} is outside of $M$.

4.3 **Fixed point theorems**

**THEOREM 1.** Let $M$ be a closed convex subset of a Banach space $X$ and $f$ be a self mapping on $M$ such that it satisfies (4.1.1) for all $x, y \in M$ and for some $0 \leq \alpha < 1$.

Let $x_0 \in M$, $t \in (0, 1)$ and $x_{n+1} = (1-t)x_n + tfx_n$ for each integer $n \geq 0$. If the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a point $u$ in $M$, then $u$ is a unique fixed point of $f$.

**PROOF.** We have

$$||x_{n+1} - fu|| = ||(1-t)x_n + tfx_n - fu||$$

$$= ||(1-t)x_n + tfx_n - tfu + tfu - fu||$$

$$\leq ||(1-t)(x_n - fu)|| + ||t(fx_n - fu)||$$

$$\leq ||(1-t)||x_n - fu|| + t||fx_n - fu||.$$

By using (4.1.1) for $||fx_n - fu||$, we get
\[ \|x_{n+1} - fu\| \leq (1-t)\|x_n - fu\| + t \max(\|x_n - u\|), \]
\[ \frac{1}{2}[\|x_n - fx_n\| + \|u - fu\|], \]
\[ \frac{1}{2}[\|x_n - fu\| + \|u - fx_n\|] \]
\[ \leq (1-t)\|x_n - fu\| + t \alpha(\|x_n - u\| + \|x_n - fx_n\|) \]
\[ + \|u - fu\| \]. \quad (4.3.2) \]

Since \( \{x_n\} \) converges to \( u \) and \( x_{n+1} - x_n = t(fx_n - x_n) \), the sequence \( \{\|fx_n - x_n\|\}_{n=1}^{\infty} \) tends to zero. Therefore, letting \( n \to \infty \) in (4.3.2), we have

\[ \|u - fu\| \leq (1-t)\|u - fu\| + \alpha t\|u - fu\| \]
\[ \leq [1-(1-\alpha)t]\|u - fu\| \]
\[ = \beta\|u - fu\|, \]

where \( \beta = [1-(1-\alpha)t] \in (0,1) \). Hence \( u = fu \). We now show the uniqueness of the fixed point. Suppose \( v \neq u \) be another fixed point of \( f \). Then we have from (4.1.1)

\[ \|u - v\| = \|fu - fv\| \]
\[ \leq \max(\|u - v\|, \frac{1}{2}[\|u - fu\| + \|v - fv\|]) \]
\[ \frac{1}{2}[\|u - fv\| + \|v - fu\|]). \]
\[ \leq \max( ||u-v||, \frac{1}{2}[||u-v|| + ||u-v||]) \]

\[ \leq \alpha ||u-v||, \]

which is impossible since \( \alpha < 1 \). Hence \( u = v \). This completes the proof.

**THEOREM 2.** Let \( X \) be a Banach space and let \( f \) be a mapping of \( X \) into itself. If there exists a mapping \( f_1 \) of \( X \) into itself which has a right inverse \( f_1^{-1} \) (i.e. \( f_1 f_1^{-1} = I \), the identity mapping) such that \( f_1^{-1} ff_1 \) satisfies the condition

\[ ||f_1^{-1} ff_1 x - f_1^{-1} ff_1 y|| \leq \max( ||x-y||, \]

\[ \frac{1}{2}[||x-f_1^{-1} ff_1 x|| + ||y-f_1^{-1} ff_1 y||], \]

\[ \frac{1}{2}[||x-f_1^{-1} ff_1 y|| + ||y-f_1^{-1} ff_1 x||]) \]

(4.3.3)

for all \( x, y \in X \) and \( \alpha \in (0,1) \), then \( f \) has a unique fixed point.

**PROOF.** Since \( f_1^{-1} ff_1 \) satisfies (4.3.3) and \( X \) is a Banach space, \( f_1^{-1} ff_1 \) has a unique fixed point \( u \in X \).

Hence
\[ f_1^u = f_1 f_1^{-1} f_1^u = f_1^u. \]

If \( v \in X \) is a fixed point of \( f \), then since

\[ f_1^{-1} f_1 f_1^{-1} v = f_1^{-1} v, \]

we have \( f_1^{-1} v = u \). Therefore \( f_1 v \) is a unique fixed point of \( f \).

**Theorem 3.** Let \( f \) be a mapping of a Banach space \( X \) into itself satisfying the condition (4.1.1) for all \( x, y \in X \) and \( \alpha \in (0,1) \). For each positive integer \( n \), let \( a_n \in X \) be a solution of the equation \( u - fu = A_n \) (\( A_n \in X \)). If \( A_n \to 0 \) as \( n \to \infty \), then the sequence \( \{a_n\}_{n=1}^{\infty} \) converges to the solution of the equation \( u = fu \).

**Proof.** We have

\[ \|a_n - a_m\| = \|a_n - fa_n + fa_n - fa_m + fa_m - a_m\| \]

\[ \leq \|a_n - fa_n\| + \|fa_n - fa_m\| + \|fa_m - a_m\| \]

\[ \leq \|A_n\| + \|A_m\| + \alpha \max\{\|a_n - a_m\|, \}

\[ \frac{1}{2}[\|a_n - fa_n\| + \|a_m - fa_m\|], \]

\[ \frac{1}{2}[\|a_n - fa_m\| + \|a_m - fa_n\|] \]
\[ \leq ||A_n|| + ||A_m|| + \alpha \left( ||a_n - a_m|| + ||a_n - fa_n|| + ||a_m - fa_m|| \right) \]

\[ \leq ||A_n|| + ||A_m|| + \alpha ||a_n - a_m|| + \alpha ||A_n|| + \alpha ||A_m|| \]

implying

\[ ||a_n - a_m|| \leq \frac{1 + \alpha}{1 - \alpha} \left( ||A_n|| + \frac{1 + \alpha}{1 - \alpha} ||A_m|| \right). \]

It follows therefore that \{a_n\} is a Cauchy sequence.

Hence it converges to a (say) of \( X \). Also

\[ ||a - fa|| = ||a - a_n + a_n - fa_n + fa_n - fa|| \]

\[ \leq ||a - a_n|| + ||a_n - fa_n|| + ||fa_n - fa|| \]

\[ \leq ||a - a_n|| + ||A_n|| + \alpha \max\{||a_n - a||, \frac{1}{2}(||a_n - fa_n|| + ||a - fa||)\} \]

\[ \leq ||a - a_n|| + ||A_n|| + \alpha \left( ||a - a_n|| + \frac{1}{2}(||a_n - fa_n|| + ||a - fa||) \right) \]

\[ \leq ||a - a_n|| + ||A_n|| + \alpha \left( ||a - a_n|| + \frac{1}{2}(||a_n - fa_n|| + ||a - fa||) \right) \]
\[ \leq ||e_a|| + ||A_n|| + \alpha ||e_a|| + \alpha ||A_n|| + \alpha ||e_{fa}|| \]

implying

\[ ||e_{fa}|| \leq \frac{1+\alpha}{1-\alpha} ||e_a|| + \frac{1+\alpha}{1-\alpha} ||A_n|| \]

for any positive integer \( n \). Hence it follows that \( a = fa \) and this completes the proof.

**Theorem 4.** Let \( M \) be a closed bounded convex subset of a Banach space \( X \) and let \( K \) be a compact subset of \( M \). Suppose \( f : M \rightarrow M \) is a continuous mapping which satisfies (4.1.2) and that for each \( x \in M \)

We \( \{f^R_n\} \cap K \neq \emptyset \) (where \( \{f^R_n\} \) stands for weak closure of \( \{f_n^R\} \))

\[ ||y-fy|| \leq ||x-fx|| \] if \( y \in \text{Co} \{f^R_n\} \) (\( \text{Co} \{f^R_n\} \) stands for convex hull of \( \{f^R_n\} \))

then there exists a \( x \in K \) such that \( Tx = x \).

**Proof.** Let \( \varepsilon > 0 \) be arbitrary and let \( x_0 \) be a fixed element of \( M \). Define \( f_\alpha : M \rightarrow M \) as follows.

\[ f_\alpha(x) = (1-\alpha)x_0 + \alpha fx, \quad 0 < \alpha < 1. \]
This definition makes the sense due to convexity of $M$.

Now

$$|| f_{a}x - f_{a}y || = || afx - afy ||$$

$$= |a| ||fx - fy||$$

$$\leq \alpha \max(||x-y||, \frac{1}{2}[||x-fx|| + ||y-fy||]).$$

$$\frac{1}{2}([||x-fy|| + ||y-fx||]).$$

Hence by Theorem 1, there is an element $x_{a}$ such that $f_{a}x_{a} = x_{a}$.

Also

$$||x_{a} - fx_{a}|| = ||f_{a}x_{a} - fx_{a}||$$

$$= ||(1-a)x_{0} + afx_{a} - fx_{a}||$$

$$= ||(1-a)x_{0} - (1-a)fx_{a}||$$

$$= (1-a) ||x_{0} - fx_{a}|| .$$

Since $M$ is bounded, we can choose $a$ so close to 1 such that $||x_{a} - fx_{a}|| < \epsilon/6$. By condition (4.3.5) there is an element $y \in Co \{f^{n}x_{a}\}$ such that $||y-z|| < \epsilon/8$ for some $z \in K$. Now
\[ ||z-fz|| = ||z-y + y - fy + fy - fz|| \]
\[ \leq ||z-y|| + ||y-fy|| + ||fy-fz|| \]
\[ \leq ||z-y|| + ||y-fy|| + \max\{ ||z-y||, \frac{1}{2}[||y-fy|| + ||z-fz||], \frac{1}{2}[||y-fz|| + ||z-fy||]\} \]
\[ \leq ||z-y|| + ||y-fy|| + ||z-y|| + ||y-fy|| / 2 + \]
\[ ||z-fz|| / 2 \]

implying

\[ ||z-fz|| \leq 4||z-y|| + 3||y-fy|| \]
\[ \leq 4 \frac{\varepsilon}{6} + 3 \frac{\varepsilon}{6} \]
\[ = \varepsilon. \]

Hence \( \inf_{z \in K} ||z-fz|| \) = 0. Since \( K \) is compact and \( \varphi \) is continuous, this infimum is attained at some point \( x \in K \), which is the fixedpoint of \( f \).

**THEOREM 5.** Let \( X \) be a Banach space. Let \( M \) be a bounded closed starshaped subset of \( X \). Let \( f : M \to M \) be a continuous mapping of \( M \) into itself satisfying (4.1.2). Suppose there exists a compact set \( K \) in \( X \) such that for
every \( x \in X \), the closure of the sequence of iterates \( \{f^n x\} \) contains a point of \( K \). Then \( f \) has a fixed point.

**Proof.** Without loss of generality, we assume that the origin \( 0 \) is in \( X \) and \( M \) is starshaped about \( 0 \). We define the mapping \( f_\alpha \) as follows

\[
f_\alpha x = \alpha f x, \quad 0 < \alpha < 1 \text{ and } x \in M.
\]

Now,

\[
||f_\alpha x - f_\alpha y|| = ||\alpha f x - \alpha f y||
\]

\[
= \alpha ||f x - f y||
\]

\[
\leq \alpha \max(||x - y||, \tfrac{1}{2}[||x - f x|| + ||y - f y||])
\]

\[
\leq \tfrac{1}{2}[||x - f y|| + ||y - f x||].
\]

(since \( f \) satisfies (4.1.2))

Therefore \( f_\alpha \) has a unique fixed point \( x_\alpha \in M \) i.e. \( f_\alpha x_\alpha = x_\alpha \).

Also

\[
||x_\alpha - f x_\alpha|| = ||f_\alpha x_\alpha - f x_\alpha||
\]

\[
= ||\alpha f x_\alpha - f x_\alpha||
\]

\[
\leq (1 - \alpha) ||f x_\alpha||
\]

\[
< \delta (1 - \alpha)
\]

(4.3.6)

where \( \delta \) is the diameter of \( M \).
On the other hand by assumption, there exists an integer \( n(a) \) and a point \( y_\alpha \in K \) such that

\[
||y_\alpha - x_n(a)x_\alpha|| < (1 - \alpha) \tag{4.3.7}
\]

Then

\[
||y_\alpha - fy_\alpha|| = ||y_\alpha - x_n(a)x_\alpha + x_n(a)x_\alpha - x_n(a)x_\alpha + x_n(a)x_\alpha - fy_\alpha||
\]

\[
\leq ||y_\alpha - x_n(a)x_\alpha|| + ||x_n(a)x_\alpha - x_n(a)x_\alpha|| + ||x_n(a)x_\alpha - fy_\alpha|| \tag{4.3.8}
\]

Now

\[
||x_n(a)x_\alpha - x_n(a)x_\alpha|| \leq \max\{||x_n(a)x_\alpha - x_n(a)x_\alpha||,
\]

\[
\frac{1}{2}||x_n(a)x_\alpha - x_n(a)x_\alpha|| + \frac{1}{2}||x_n(a)x_\alpha - x_n(a)x_\alpha||
\]

\[
||x_n(a)x_\alpha - x_n(a)x_\alpha|| \leq \left\{ \frac{1}{2}||x_n(a)x_\alpha - x_n(a)x_\alpha|| + \frac{1}{2}||x_n(a)x_\alpha - x_n(a)x_\alpha|| \right\}.
\]
If we take the maximum as $||f^{n(a)} - 1_x - f^{n(a)} x_u||$, then we have

$$||f^{n(a)} x_u - f^{n(a)} x_u|| \leq ||f^{n(a)} - 1_x - f^{n(a)} x_u||.$$  \hspace{1cm} (4.3.9)

If we take \( \frac{1}{2}[||f^{n(a)} - 1_x - f^{n(a)} x_u|| + ||f^{n(a)} x_u - f^{n(a)} + 1_x||] \) as maximum, then

$$||f^{n(a)} x_u - f^{n(a)} x_u|| \leq \frac{1}{2}[||f^{n(a)} - 1_x - f^{n(a)} x_u|| + ||f^{n(a)} x_u - f^{n(a)} + 1_x||].$$

implying

$$||f^{n(a)} x_u - f^{n(a)} + 1_x|| \leq ||f^{n(a)} - 1_x - f^{n(a)} x_u||.$$ \hspace{1cm} (4.3.10)

Finally if \( \frac{1}{2}||f^{n(a)} - 1_x - f^{n(a)} x_u|| \) is maximum then

$$||f^{n(a)} x_u - f^{n(a)} + 1_x|| \leq \frac{1}{2}||f^{n(a)} - 1_x - f^{n(a)} x_u||$$
\[ \leq \frac{1}{2} \| f(n(a)-1)^{\frac{1}{2}} x_a - f(n(a) x_a + f(n(a)^{\frac{1}{2}} x_a \| + f(n(a)^{\frac{1}{2}} x_a \| ) \]

\[ \leq \frac{1}{2} \| f(n(a)-1)^{\frac{1}{2}} x_a - f(n(a) x_a \| + \frac{1}{2} \| f(n(a) x_a - f(n(a)^{\frac{1}{2}} x_a \| \]

implying

\[ \| f(n(a) x_a - f(n(a)^{\frac{1}{2}} x_a \| \leq \| f(n(a)-1)^{\frac{1}{2}} x_a - f(n(a) x_a \| \].

(4.3.11)

From inequalities (4.3.9), (4.3.10) and (4.3.11), it follows that

\[ \| f(n(a) x_a - f(n(a)^{\frac{1}{2}} x_a \| \leq \| f(n(a)-1)^{\frac{1}{2}} x_a - f(n(a) x_a \| .

Continuing this process, we get

\[ \| f(n(a) x_a - f(n(a)^{\frac{1}{2}} x_a \| \leq \| x_a - f x_a \|. \quad (4.3.12)

Also

\[ \| f(n(a)^{\frac{1}{2}} x_a - f y_a \| \leq \max \{ \| f(n(a) x_a - y_a \| ,

\[ \frac{1}{2} \| f(n(a) x_a - f(n(a)^{\frac{1}{2}} x_a \| + \| y_a - f y_a \| ,

\[ \frac{1}{2} \| f(n(a) x_a - f y_a \| + \| y_a - f(n(a)^{\frac{1}{2}} x_a \| \} \]
\[ \begin{align*} 
\| y_a - x_a \| & \leq \| y_a - x^{(a)}_a \| + \frac{1}{2} \| x_a - f x_a \| + \frac{1}{2} \| y_a - f y_a \| \\
& \quad + \frac{1}{2} \| x_a - x^{(a)}_a \| \\
& \quad + \frac{1}{2} \| x_a - x^{(a)}_a \| \\
& \quad + \frac{1}{2} \| x_a - x^{(a)}_a \| \\
& \quad + \frac{1}{2} \| y_a - f y_a \| \\
& \leq 2 \| y_a - x^{(a)}_a \| + \| x_a - f x_a \| + \frac{1}{2} \| x_a - f x_a \| + \frac{1}{2} \| y_a - f y_a \| \\
& \quad + \frac{1}{2} \| y_a - f y_a \| \\
& \leq 2 \| y_a - x^{(a)}_a \| + \frac{3}{2} \| x_a - f x_a \| + \frac{1}{2} \| y_a - f y_a \| \\
\end{align*} \]

i.e.

\[ \| y_a - f y_a \| \leq 4 \| y_a - x^{(a)}_a \| + 3 \| x_a - f x_a \| . \]  

(4.3.14)

Substituting (4.3.6) and (4.3.7) in (4.3.14), we get

\[ \| y_a - f y_a \| \leq 4(1-a) + 3 \delta (1-a) \]

\[ \leq (1-a) (4 + 3 \delta) . \]  

(4.3.15)
Now let \( \{a\} \) be a sequence converging to 1, using compactness of \( K \) it follows that there exists a subsequence of \( y_a \) which we denote by \( y_a \) that converges to \( y \in K \). From (4.3.15) we immediately conclude that \( y \) is a fixed point of \( f \).

**THEOREM 6.** Let \( H \) be a Hilbert space. Let \( M \) be a closed convex bounded subset of \( H \). Let \( f : M \rightarrow M \) be a continuous mapping satisfying (4.1.2). Then \( f \) has a fixed point in \( M \).

**PROOF.** Let \( u_0 \) be a fixed element of \( M \). For each \( a \) with \( 0 < a < 1 \) we define \( f_a : M \rightarrow M \) by

\[
f_a(x) = (1 - a)u_0 + af(x).
\]

Now \( f_a \) satisfies the condition

\[
||f_a(x) - f_a(y)|| = ||af_x - af_y||
\]

\[
\leq a \max\{|||x-y|||, \frac{1}{2}[||x-fx|| + ||y-fy||] \}.
\]

Therefore, by Theorem 1 there exists a unique fixed point \( x_a \) in \( M \) such that \( f_a(x_a) = x_a \). Since \( M \) is closed bounded and convex in the Hilbert space \( H \), it is weakly compact. Hence we confind a sequence \( a_i \rightarrow 1 \) as \( i \rightarrow \infty \) such that \( x_i = x_{a_i} \) converges weakly to an element \( x_0 \) of \( H \). Since
$M$ is weakly closed $x_0$ lies in $M$. We shall prove that $x_0$ is a fixed point of $f$. If $x$ is any point in $M$, we note that

$$
||x_1 - x||^2 = ||x_1 - x_0 + x_0 - x||^2

= ||x_1 - x_0||^2 + ||x_0 - x||^2 + 2(x_1 - x_0, x_0 - x)
$$

where $2(x_1 - x_0, x_0 - x) \to 0$ as $i \to \infty$ because $x_1 - x_0$ converges weakly to zero in $M$. However since $a_i \to 1$, we have

$$
f_{x_1 - x_1} = (a_i f(x_1) + (1-a_i)u_0) - x_1 + (1-a_i)(f(x_1) - u_0)

= (a_i f(x_1) = (x_1) + (1-a_i)(f(x_1) - u_0)

= (1-a_i)(f(x_1) - u_0)

\to 0 \text{ as } i \to \infty.
$$

Setting $x = f_{x_0}$, we have

$$
\lim_{i \to \infty} \{||x_1 - f_{x_0}||^2 - ||x_1 - x_0||^2\} = ||x_0 - f_{x_0}||^2.
$$

On the other hand, since $f$ satisfies (4.1.2), we get

$$
||fx_1 - fx_0|| \leq \max(||x_1 - x_0||, \frac{1}{2}[||x_1 - fx_1|| + ||x_0 - fx_0||]),
$$

$$
\frac{1}{2}[||x_1 - fx_0|| + ||x_0 - fx_1||]. \quad (4.3.16)
$$
If the maximum is $||x_1 - x_0||$, then

$$||x_1 - x_0|| \leq ||x_1 - x_0||.$$  \hfill (4.3.17)

If the maximum is $\frac{1}{2}(||x_1 - x_1|| + ||x_0 - x_0||)$, then

$$||x_1 - x_0|| \leq \frac{1}{2}||x_1 - x_1|| + \frac{1}{2}||x_0 - x_0||$$

$$\leq \frac{1}{2}||x_1 - x_1|| + \frac{1}{2}||x_0 - x_1|| +$$

$$\frac{1}{2}||x_1 - x_1|| + \frac{1}{2}||x_1 - x_0||$$

i.e.

$$||x_1 - x_0|| \leq ||x_0 - x_1|| + 2||x_1 - x_1||.$$  \hfill (4.3.18)

If the maximum is $\frac{1}{2}(||x_0 - x_1|| + ||x_1 - x_0||)$, then

$$||x_1 - x_0|| \leq \frac{1}{2}||x_0 - x_1|| + \frac{1}{2}||x_1 - x_0||$$

$$\leq \frac{1}{2}||x_0 - x_1|| + \frac{1}{2}||x_1 - x_1|| +$$

$$\frac{1}{2}||x_1 - x_1|| + \frac{1}{2}||x_0 - x_1||$$

i.e.

$$||x_1 - x_0|| \leq ||x_0 - x_1|| + 2||x_1 - x_1||.$$  \hfill (4.3.19)
Also we have

\[ ||x_1 - f x_0|| = ||x_1 - f x_1 + f x_1 - f x_0|| \]

\[ \leq ||x_1 - f x_1|| + ||f x_1 - f x_0|| \quad (4.3.20) \]

substituting the value of \( ||f x_1 - f x_0|| \) from (4.3.17), (4.3.18), (4.3.19) in (4.3.20) we get

\[ ||x_1 - f x_0|| \leq ||x_1 - f x_1|| + ||x_1 - x_0|| \quad (4.3.21) \]

and

\[ ||x_1 - f x_0|| \leq ||f x_1 - x_1|| + ||x_1 - x_0|| + 2 ||x_1 - f x_1|| \]

\[ (4.3.22) \]

respectively.

But from (4.3.21) and (4.3.22) it is clear that

\[ \text{Im} (||x_1 - f x_0|| - ||x_1 - x_0||) \leq 0 \]

and hence

\[ \text{Im} (||x_1 - f x_0||^2 - ||x_1 - x_0||^2) \leq 0 \]

Finally, we have \( ||x_0 - f x_0||^2 = 0 \) and hence \( x_0 \) is a fixed point of \( f \). This completes the proof.