CHAPTER 3
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3.1 Introduction

Let \((X, d)\) be a \(f\)-orbitally complete metric space. Let \(\phi_1: \overline{F} \to [0, \infty)\) \([\overline{F} \text{ is the range of } d \text{ and } \overline{F} \text{ is closure of } F]\) be upper semicontinuous function from the right on \(\overline{F}\) and satisfy the condition

\[\phi_1(t) < t \text{ for } t > 0 \text{ and } \phi_1(0) = 0, \quad i = 1, 2. \quad (3.1.1)\]

Also, let \(f\) be an orbitally continuous mapping of \(X\) into itself such that

\[
\min\{d(fu_1, fu_2)d(fu_3, fu_4), d(u_1, u_2)d(fu_3, fu_4), [d(u_2, fu_4)]^2
- \min\{d(u_1, fu_3)d(u_2, fu_4), d(u_1, fu_4)d(u_2, fu_3)\}
\leq \phi_1[d(u_1, fu_3)] \phi_2[d(u_2, fu_4)] \tag{3.1.2}
\]

for \(u_1, u_2, u_3, u_4 \in X\).

In the following we establish a fixed point theorem which contains as a special case the result of Fakpatte [1979].

3.2 Fixed point theorem

THEOREM 1. Let \(f\) be an orbitally continuous mapping of \(X\) into itself satisfying (3.1.2), then \(f\) has a fixed point.

* Achari [1982]
PROOF. Let \( x, y \in X \) and put
\[
\begin{align*}
u_1 &= fy, \quad u_2 = fx, \quad u_3 = x, \quad u_4 = y
\end{align*}
\]
then the condition (3.1.2) takes the form
\[
\min\{d(f^2x, f^2y)d(fx, fy), \ d(fx, fy)d(fx, fy), \ [d(fx, fy)]^2 \}
\]
\[
- \min\{d(fy, fx)d(fx, fy), \ d(fy, fy)d(fx, fx) \}
\]
\[
\leq \phi_1[d(fx, fy)] \phi_2[d(fx, fy)]. \tag{3.2.1}
\]
Let \( x_0 \in X \) be an arbitrary and construct a sequence \( \{x_n\} \)
defined by
\[
\begin{align*}
fx_{n-2} &= x_{n-1}, \quad fx_{n-1} = x_n, \quad fx_n = x_{n+1}, \quad n = 1, 2, \ldots
\end{align*}
\]
Let us put \( x = x_{n-1}, \ y = x_{n-2} \) in (3.2.1), then we have
\[
\min\{d(f^2x_{n-1}, f^2x_{n-2})d(fx_{n-1}, fx_{n-2}),
\quad d(fx_{n-1}, fx_{n-2})d(fx_{n-1}, fx_{n-2}), \ [d(fx_{n-2}, fx_{n-1})]^2 \}
\]
\[
\leq \phi_1[d(fx_{n-1}, fx_{n-2})] \phi_2[d(fx_{n-1}, fx_{n-2})]
\]
or
\[
\min\{d(x_n, x_{n+1})d(x_{n-1}, x_n), \ d(x_n, x_{n-1})d(x_n, x_{n-1}), \ [d(x_n, x_{n-1})]^2 \}
\]
\[
\leq \phi_1[d(x_n, x_{n-1})] \phi_2[d(x_n, x_{n-1})].
\]
Since
\[ d(x_n, x_{n-1})d(x_n, x_{n-1}) \leq \phi_1[d(x_n, x_{n-1})] \phi_2[d(x_n, x_{n-1})] \]
\[ < d(x_n, x_{n-1})d(x_n, x_{n-1}) \]
and
\[ [d(x_n, x_{n-1})]^2 = d(x_n, x_{n-1})d(x_n, x_{n-1}) \]
\[ \leq \phi_1[d(x_n, x_{n-1})] \phi_2[d(x_n, x_{n-1})] \]
\[ < d(x_n, x_{n-1})d(x_n, x_{n-1}) \]
are impossible, we have
\[ d(x_{n+1}, x_n)d(x_n, x_{n-1}) \leq \phi_1[d(x_n, x_{n-1})] \phi_2[d(x_n, x_{n-1})] \]
\[ i.e. \quad d(x_{n+1}, x_n) \leq \phi_1[d(x_n, x_{n-1})]. \quad (3.2.2) \]
Let us take \( C_n = d(x_n, x_{n-1}) \), then
\[ C_{n+1} \leq \phi_1(C_n). \quad (3.2.3) \]
From this it is clear that \( C_n \) decreases with \( n \) and hence \( C_n \rightarrow C \) (say) as \( n \rightarrow \infty \). If possible, let \( C > 0 \). Then since \( \phi_1 \) is upper semicontinuous, we obtain in limit \( n \rightarrow \infty \)
\[ C \leq \phi_1(C) < C, \]
which is impossible unless \( C = 0 \).
Next we shall show that the sequence \( \{x_n\} \) is Cauchy. Suppose that it is not so, then there exists an \( \varepsilon > 0 \) and sequences of integers \( \{m(k)\}, \{n(k)\} \) with \( m(k) > n(k) \geq k \), such that

\[
d_k = d(x_{m(k)}, x_{n(k)}) \geq \varepsilon, \quad k = 1, 2, \ldots. \tag{3.2.4}\]

If \( m(k) \) is smallest integer exceeding \( n(k) \) for which (3.2.4) holds, then from well ordering principle, we have

\[
d(x_{m(k)-1}, x_{n(k)}) < \varepsilon. \tag{3.2.5}\]

Then, we get

\[
d_k = d(x_{m(k)}, x_{n(k)})
\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})
\leq C_{m(k)} + \varepsilon
< C_k + \varepsilon.
\]

Which implies that \( d_k \to \varepsilon \) as \( k \to \infty \).

Also we have

\[
d_k = d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1})
\leq C_{m+1} + C_{n+1} + d(x_{m+1}, x_{n+1}) \tag{3.2.6}\]
Putting $u_1 = x_n$, $u_2 = x_m$, $u_3 = x_{m-1}$, $u_4 = x_{n-1}$ in (3.1.2) we get

$$\min\{d(x_n, x_m) d(x_{n-1}, x_{m-1}), d(x_n, x_m) d(x_{n-1}, x_{m-1}),$$

$$[d(x_m, x_{n-1})]^2\}$$

$$- \min\{d(x_n, x_{m-1}) d(x_m, x_{n-1}), d(x_n, x_{m-1}) d(x_m, x_{n-1})\}$$

$$\leq \phi_1[d(x_n, x_{m-1})] \phi_2[d(x_m, x_{n-1})]$$

i.e.

$$\min\{d(x_n, x_{n-1}) d(x_n, x_m), d(x_n, x_m) d(x_n, x_m), [d(x_m, x_n)]^2\}$$

$$- \min\{d(x_n, x_m) d(x_m, x_n), d(x_n, x_m) d(x_m, x_n)\}$$

$$\leq \phi_1[d(x_n, x_m)] \phi_2[d(x_m, x_n)]$$

or

$$\min\{d(x_{m+1}, x_{m+1}) d(x_n, x_m), d(x_n, x_m) d(x_n, x_m), [d(x_n, x_m)]^2\}$$

$$\leq \phi_1[d(x_n, x_m)] \phi_2[d(x_n, x_m)].$$

Since

$$d(x_n, x_{n-1}) d(x_n, x_{n-1}) \leq \phi_1[d(x_n, x_{n-1})] \phi_2[d(x_n, x_{n-1})]$$

$$< d(x_n, x_{n-1}) d(x_n, x_{n-1})$$
and
\[ [d(x_n, x_{n-1})]^2 = d(x_n, x_{n-1})d(x_n, x_{n-1}) \]
\[ \leq \phi_1[d(x_n, x_{n-1})] \phi_2[d(x_n, x_{n-1})] \]
\[ < d(x_n, x_{n-1})d(x_n, x_{n-1}) \]
are impossible, we have
\[ d(x_n, x_{n+1})d(x_n, x_{n-1}) \leq \phi_1[d(x_n, x_{n-1})] \phi_2[d(x_n, x_{n-1})] \]
i.e. \[ d(x_n, x_{n+1}) \leq \phi_1[d(x_n, x_{n-1})]. \] (3.2.7)

Substituting (3.2.7) in (3.2.6), we get
\[ c_k \leq C_{n-1} + C_{n-1} + \phi_1[d(x_n, x_{n-1})] \]
\[ \leq C_{n-1} + C_{n-1} + \phi_1(d_k). \]
Letting \( k \to \infty \), we get \( \varepsilon \leq \phi_1(\varepsilon) < \varepsilon \), which is impossible if \( \varepsilon > 0 \). This leads us to conclude that the sequence \( \{x_n\} \) is Cauchy. \( X \) being \( f \)-orbitally complete, there is some \( z \in X \), such that \( z = \lim f^n x \). By the orbital continuity of \( f \) we have
\[ f z = \lim_{n \to \infty} f f^n x = z. \]
Thus \( z \) is a fixed point of \( f \).