CHAPTER 7

FIXED POINT THEOREM
FOR QUASI-CONTRACTION TYPE MAPPING
ON NONARCHIMEDEAN PROBABLISTIC
METRIC SPACE
CHAPTER 7

FIXED POINT THEOREM FOR QUASI-CONTRACTION TYPE MAPPING ON NONARCHIMEDEAN PROBABILISTIC METRIC SPACE.

7.1 Introduction

This chapter is devoted to study the fixed points for Quasi-contraction type mappings in nonarchimedean probabilistic metric spaces. The result obtained here generalises the result of Istratescu [1978].

7.2 Definitions

DEFINITION 1. A t-norm is a function $T:[0,1] \times [0,1] \rightarrow [0,1]$, which is associative, commutative, non decreasing in each place and satisfies $T(a,1) = a$ for each $a \in [0,1]$

DEFINITION 2. A t-norm is T archimedean if in addition to satisfying definition 1, it is continuous on $[0,1] \times [0,1]$ and $T(x,x) < x$ for all $x \in (0,1)$. A characterization of archimedean t-norm is due to Ling [1965]. He proved that a t-norm T is archimedean iff it admits the representation

$$T(x,y) = g^{-1}(g(x) + g(y)).$$

Where $g$ is continuous and decreasing function from $[0,1]$ into $[0,\infty)$ with $g(1) = 0$ and $g^{-1}$ is the Pseudo-inverse of $g$. The continuous, decreasing function $g$ appearing in
this characterization is called an additive generator
of the archimedean t-norm T.

**DEFINITION 3.** A nonarchimedean Menger space is
an ordered triple \((S, f^-, T)\), where \((S, f^-)\) is a
nonarchimedean probabilistic metric space, T is a t-norm
and the nonarchimedean Menger inequality.

\[
F_{pq}(\max\{x,y\}) \geq T(F_{pr}(x), F_{rq}(y)) \quad (7.2.1)
\]

holds for all \(p, q \in S\) and all \(x, y \geq 0\), where
\(f^-(p,q) = F_{pq}\) is a mapping from \(S \times S\) into the set of
nondecreasing, left continuous function \(F\) such that \(F(0) = 0\)
and
\[
\lim_{x \to \infty} F(x) = 1.
\]

**DEFINITION 4.** Let \((S, f^-)\) be a nonarchimedean
probabilistic metric space, let \(g\) be an additive generator
and let \(\alpha\) be a number such that \(0 < \alpha < 1\). A mapping
\(f : S \to S\) is a quasi-contraction type mapping on \(S\) with
respect to \(g\) and \(\alpha\) if for every \(p, q \in S\)

\[
g \circ F_{f(p)f(q)}(x) \leq \alpha \ g \ \max\{F_{pq}(x/\alpha), F_{pf(p)}(x/\alpha), F_{qf(q)}(x/\alpha)\} \quad (7.2.2)
\]
DEFINITION 5. Let \((S, f)\) be a nonarchimedean probabilistic metric space. A mapping \(f: S \rightarrow S\) is a quasi-contraction type map on \((S, f)\) iff there exists a constant \(\alpha \in (0,1)\) such that

\[
F_f(p)f(q)(x) \geq \max \{F_{pq}(x/\alpha), F_{pf}(x/\alpha), F_{qf}(x/\alpha)\}.
\]

(7.2.3)

for \(p, q \in S\). This may be interpreted as the probability that the distance between the image points \(f(p), f(q)\) is less than \(x\) is at least equal to the probability that the maximum of the distances between \(p, q, p, f(p)\) and \(q, f(q)\) is less than \((x/\alpha)\).

DEFINITION 6. A sequence of point \(\{p_n\}\) in \(S\) is a Cauchy sequence if \(F_{p_n p_m} \rightarrow H(\text{pointwise})\) as \(n, m \rightarrow \infty\), where \(H(x) = 0\) if \(x \leq 0\) and \(H(x) = 1\) if \(x > 0\).

7.3 \textbf{Fixed point theorem for quasi-contraction type mappings on nonarchimedean probabilistic metric spaces.}

THEOREM 1. Let \((S, f, T)\) be a complete nonarchimedean Menger space under the archimedean \(t\)-norm \(T\) with the additive generator \(g\). Let \(f\) be a quasi-contraction type mapping on \(S\) with respect to \(g\) and \(\alpha\) where \(0 < \alpha < 1\).
Then there exists a point \( p \) in \( S \) such that \( f(p) = p \).

**Proof.** Let \( q \) be an arbitrary element of \( S \). Define a sequence \( \{p_n\} \) inductively by \( p_1 = f(q) \) and \( p_{n+1} = f(p_n) \) for every positive integer \( n \). Now we have from (7.2.2)

\[
g(F_{p_1p_2}(x)) = g(F_{f(p_1)f(q)}(x))
\]

\[
\leq a \cdot g \max \{F_{p_1q}(x/a), F_{p_1f(p_1)}(x/a), F_{f(q)}(x/a)\}
\]

\[
\leq a \cdot g \max \{F_{qf(q)}(x/a), F_{qf(q)}(x/a), F_{p_1p_1q}(x/a)\}
\]

\[
\leq a \cdot g \max \{F_{qf(q)}(x/a)\}.
\]

Similarly we have

\[
g(F_{p_2p_3}(x)) \leq a^2 \cdot g[F_{qf(q)}(x/a^2)].
\]

Hence it follows by induction that for every positive integer \( n \),

\[
g(F_{p_np_{n+1}}(x)) \leq a^n \cdot g[F_{qf(q)}(x/a^n)].
\]

Then for \( m > n \) and \( x > 0 \) we have

\[
F_{p_np_{n+m}}(x) \geq T(F_{p_{n+p_{n+1}}}(x), F_{p_{n+1}p_{n+m}}(x))
\]
\[ g^{-1}[g(T(p_{n+1}, p_{n+2}, p_{n+2}^2)) + g(p_{n+2}^2)] \]
\[ = g^{-1}[g(g^{-1}(g(p_{n+1}^3, p_{n+2}^3)) + g(p_{n+2}^2))] \]
\[ = g^{-1}[g(g^{-1}(g(p_{n+1}^3, p_{n+2}^3)) + g(p_{n+2}^2))] e^g(p_{n+2}^2) \]
\[ + a^n g(f_{p_0}^n(x/a^n)) \]

We conclude that \( \{p_n\} \) is a Cauchy sequence, since \( g^{-1} \) and \( g \) are continuous, \( a^n \to 0 \) as \( n \to \infty \), \( F_{pq}(x) \to 1 \) as \( x \to \infty \) and \( g^{-1}(0) = 1 \). Since \( (S, f, T) \) is complete, there is a point \( p \in S \) such that \( p_n \to p \). We shall now show that \( p \) is a fixed point of \( f \). For every \( x \) and for every positive integer \( n \) we have

\[ F_{pf(p)}(x) \geq T(p_{n+1}^3, p_{n+2}^3, p_{n+2}^2) \]
\[ = g^{-1}(g(p_{n+1}^3) + g(p_{n+2}^2)) \]
\[ \geq g^{-1}[g(F_{ppn}(x)) + \alpha g(F_{p_{n-1}p}(x/\alpha))] \]

\[ \geq \lim_{n \to \infty} g^{-1}[g(F_{ppn}(x)) + \alpha g(F_{p_{n-1}p}(x/\alpha))] \]

= 1

which implies \( f(p) = p \). This completes the proof.