PART-B

A study of ideal
Theory in Rings
Without Involution

And

★-Rings
In Part-B of the Thesis, we shall investigate ideal theory in an associative ring with unity which may or may not have involution. The ring which we shall consider may be noncommutative as was the case in Part-A. By an ideal, we shall always mean a two-sided ideal. As usual, an ideal $P$ of a ring $A$ is called prime if, for ideals $I, J$ of $A$, $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$. In Chapter-III we carry out investigations involving prime ideals of a ring $A$ without nonzero nilpotent elements; such rings are called semiprime rings. We take up the study of prime strict ideals in a Rickart $^*$-ring in Chapter-IV. We give the necessary definitions. Recall

**Definition B.1**: An ideal $I$ of a $^*$-ring $A$ is called a $^*$-ideal if $xI$ implies $x^*\in I$.

**Definition B.2**: An ideal $I$ of a $^*$-ring $A$ is called a strict ideal if $xI$ implies $RP(x)\in I$. 
These two concepts were discussed in the last section of Chapter-I. Here is a concept that is almost a folk notion:

**DEFINITION B.3:** A strict ideal $P$ of a *-ring $A$ is called prime strict, if for strict ideals $I, J$ of $A$, $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$.

It can be easily seen that if a strict ideal is prime in the usual sense then it is prime strict. Also, if $e, f$ are projections in a strict ideal $I$ then $eVfI$.

We denote by $\Sigma(A)$, the set of all prime ideals (prime strict ideals) of a ring $A$ with unity (of a Rickart *-ring $A$). A nonzero element of $\Sigma(A)$, will be called a minimal prime ideal (minimal prime strict ideal) of $A$, if it does not contain, as subset any other nonzero element of $\Sigma(A)$. The set of all minimal prime ideals (minimal prime strict ideals) of $A$ will be denoted by $\pi(A)$. Interestingly, however, the minimal prime strict ideals and prime strict ideals coincide when $A$ is a semiprime Rickart *-ring. More explicitly, for such rings we have $\Sigma(A) = \pi(A)$. 
The concepts of hulls, kernels and hull-kernel topology in rings, lattices, semilattices, semigroups have been studied by several authors. See for example, Atiyah and Macdonald [1], Lambek [4], Speed [8], Pawar and Thakare [5,6], Kist [3], DeMarco and Orsatti [2], Simmons [7] et al. Such a study for commutative rings is abundant as compared to such a study in noncommutative rings. We could not lay our hands on such type of study in *-rings carried out by others.

For a nonempty subset \( \Theta \) of \( \Sigma(A) \), we define the kernel of \( \Theta \), denoted by \( K(\Theta) \), as the set

\[
Ker\Theta = K(\Theta) = \bigcap \{ P : P \in \Theta \} \tag{B.1}
\]

For any ideal (strict ideal) \( I \) of \( A \) (of a Rickart *-ring \( A \)), we define its hull, denoted by \( H(I) \), as the set

\[
H(I) = \{ P : P \in \Sigma(A), \quad I \subseteq P \} \tag{B.2}
\]

For any \( x \in A \), \( \{ P \}_x \) denotes the set

\[
\{ P \}_x = \{ P : P \in \Sigma(A), \quad x \in P \} \tag{B.3}
\]
We shall also adopt the notation \((x)\) to denote ideal generated by the element \(x\in A\). In the case of semiprime rings with unity, we shall see that \(x\in P\) if and only if \((x)\subseteq P\) for \(P\in \Sigma(A)\). This permits us not to make distinction between \(\{ P \}_x\) and \(\{ P \}_{(x)}\) and similarly between \(H(x)\) and \(H((x))\) in such rings.

Let us digress a little more on these points when the underlying ring \(A\) is a Rickart \(*\)-ring and \(\Sigma(A)\) is the set of prime strict ideals of \(A\). It is easy to see that \(H(x) = H(RP(x))\) where \(RP(x)\), is the right projection of \(x\), has the meaning that was assigned in the introduction to Part-A. On account of this, we have \(\{ P \}_x \subseteq \{ P \}_{RP(x)}\). Moreover, if \(A\) is a semiprime ring then \(\{ P \}_{RP(x)} = \{ P \}_{(RP(x))}\) where \((RP(x))\) is the strict ideal generated by \(RP(x)\).

For both types of rings, we can show that the sets \(\{ P \}_x\) have the following properties.

\[
\bigcup_{i\in I} \{ P \}_{x_i} = \{ P \}_{\{ x_i : i \in I \}} \quad \ldots \quad (B.4)
\]

\[
\{ P \}_{I \cap J} = \{ P \}_I \cap \{ P \}_J \quad \ldots \quad (B.5)
\]
for any ideals (strict ideals) $I, J$ of $A$, where $\{ P \}_I$
denotes the set $\{ P : P \in \Sigma(A), I \notin P \}$:

$$\{ P \}_o = \emptyset \quad \ldots \quad (B.6)$$

$$\{ P \}_{\Sigma} = \{ P \}_I = \Sigma(A) \quad \ldots \quad (B.7)$$

It can be shown that for a set $\{ X_i : i I \}$ of
ideals (strict ideals) of $A$,

$$\bigcup_{i I} \{ P \}_{X_i} = \{ P \}_{\Sigma X_i} \quad \ldots \quad (B.3)$$

where $\Sigma X_i$ denotes the ideal sum of the ideals
$i I$ (strict ideals) $X_i$.

These properties are so much well entrenched in the
literature that we resist the temptation of giving proofs of
these results. In view of the above properties it is
straightforward to note that the sets of the type
$\{ \{ P \}_x : x \in A \}$ form a basis for open sets and these
sets define a topology on $\Sigma(A)$. We denote this topology on
$\Sigma(A)$ by $\mathcal{F}^h$ and call it the hull-kernel topology. There are
several equivalent terms for this topology. As per one's own inclination, this topology is equivalently called Zariski topology or Stone topology or Jacobson topology.

In the same way, it can be shown that the sets of the type \( \{ H(x): x \in A \} \) have the following properties.

\[
\begin{align*}
H(0) &= \Sigma(A) \quad \ldots \quad (B.9) \\
H(A) &= H(1) = \emptyset \quad \ldots \quad (B.10) \\
H(\bigcup_{i \in I} E_i) &= \bigcap_{i \in I} H(E_i) \quad \ldots \quad (B.11)
\end{align*}
\]

where \( (E_i)_{i \in I} \) is a family of nonempty subsets of \( A \).

\[
H(X \cap Y) = H(X) \cup H(Y) \quad \ldots \quad (B.12)
\]

for any ideals (strict ideals) \( X \) and \( Y \) of \( A \).

\[
H(\sum_{i \in I} X_i) = \bigcap_{i \in I} H(X_i) \quad \ldots \quad (B.13)
\]

where \( X_i, i \in I \) are ideals (strict ideals) of \( A \) and \( \sum_{i \in I} X_i \) denotes their ideal sum.
Thus, it can be seen that the sets \( \{ H(x) : x \in A \} \) form a basis for closed sets and define a topology on \( \Sigma(A) \). This topology is called the dual hull-kernel topology and is denoted by \( J^d \).

\( \Sigma(A) \) with the hull-kernel topology is called the prime-spectrum of \( A \). \( \pi(A) \) with the restriction of the hull-kernel topology is called the minimal prime spectrum or the minimal spectrum of \( A \).

We shall need all the properties involving \( \{ P \} \) and hulls and kernels in Chapter-III and Chapter-IV in their appropriate context.

In Chapter-III we consider the restriction of the hull-kernel topology to the set \( \pi(A) \) of minimal prime ideals of a semiprime ring \( A \) with unity. We characterize the compactness of \( \pi(A) \) in several ways. A functional correspondence between the category of rings having the property that every prime ideal contains a unique minimal prime ideal and their minimal spectra is established.

In Chapter-IV, a novel separation theorem for Rickart *-rings is proved. It is shown that for a semiprime Rickart *-ring \( A \), the prime spectrum \( \Sigma(A) \) is Hausdorff and
compact. Several equivalent properties of compactness of the prime spectrum $\Sigma(A)$ are obtained. In fact, we also correlate and investigate the concept of general comparability (GC) for projections in Rickart $*$-ring in the light of hull-kernel topology on $\Sigma(A)$. A functorial representation as well as a general sheaf representation is given."

The details are, however, postponed to the respective chapters.

We conclude this introduction with an observation; The following authors have used our paper based on the text of Chapter-III:

1) Thakare, N.K., Manjarekar, C.S., and Maeda, S.:

   Abstract spectral theory-II; Minimal characters and minimal spectrums of multiplicative lattices.


This indicates the fruitfulness of our approach; In fact, Marconi's work is completely based on our work. He proves that the semiprime Baer rings are complementedly normal rings whose minimal spectrum is compact.
REFERENCES


Alg. Universalis 7 (1977), 259-263.


CHAPTER III

Space of Minimal Prime Ideals of a Semiprime Ring

A Paper based on the text of this chapter has appeared in
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[27 (1983) 75-85]
The study of ideal theory in commutative rings with unity has been carried out in abundance. The ideal theory in noncommutative ring theory is as intrinsically beautiful as the one for commutative case. The study of the prime spectrum, maximal spectrum, minimal spectrum is so extensive for commutative rings that one need not give any specific reference at all. That is, however, not the case for non-commutative rings. There is almost a scarce literature about the study of prime spectrum or maximal spectrum or minimal spectrum of non-commutative rings when they carry with them the hull-kernel topology.

In both the chapters of this part, we shall discuss these aspects for noncommutative rings. In Chapter-III we take up the noncommutative rings without involution.
§ 1: Introduction:

In Section-2 of this chapter, we shall obtain a separation theorem involving an ideal I of a ring A with unity and an m-system M not intersecting the ideal I. Further, a minimal prime ideal, belonging to an ideal is characterized. Moreover, for a semiprime ring A with unity, we characterize a maximal m-system.

In Section-3 we investigate the prime spectrum \( \Sigma(A) \) of a semiprime ring A. Topological properties such as compactness and Hausdorffness of \( \Sigma(A) \) are adequately investigated. An elegant characterization of a minimal prime ideal M of A in terms of complement of M, as a maximal m-system is given.

In Section-4, our emphasis is on the minimal spectrum \( \pi(A) \) with the usual hull-kernel topology. Two useful characterizations of minimal prime ideals are given. The relationship between annihilators of ideals and their hulls and kernels are looked into.

We introduce and investigate the concept of a normal ideal that turns out to be the intersection of all minimal prime ideals containing it. As an offshoot, we are
readily led to a nice characterization of Rickart rings."
We then obtain six characterizations of the compactness of
minimal prime spectrum \( \pi(A) \). Finally we show that
there exists a functorial correspondence between the category
of normal rings and their minimal spectra.

\[ \text{Section 2: Preliminaries:} \]

In this section, \( A \) will be a ring with unity.

We recall the concept of an \( m \)-system in the following

**DEFINITION 2.1:** A nonempty subset \( M \) of a ring \( A \) is
called an \( m \)-system, if for \( x, y \in M \), there exists \( r \in A \)
such that \( xry \in M \).

In our analysis, that will follow, we shall need a
Separation Theorem that plays a crucial role.

**THEOREM 2.1:** If \( A \) is a ring with unity, \( I \) is an ideal
of \( A \) and \( M \) is an \( m \)-system of \( A \) such that \( M \cap I = \emptyset \)
then there exists a maximal ideal \( Q \) of \( A \) such that \( I \subseteq Q \)
and \( M \cap Q = \emptyset \). This ideal \( Q \) is prime.

**Proof:** Let \( C \) denote the set of all ideals \( J \) of \( A \) such
that \( I \subseteq J \) and \( J \cap M = \emptyset \). Clearly, If \( C \) and so \( C \) is
nonempty. Let \( C \) be a chain in \( C \). Let \( K = \bigcup_{J \in C} J \).
Clearly, \( K \) is an ideal of \( A \) and \( I \subseteq K \). If \( x \in K \setminus M \) then \( x = \sum x_i \) where \( x_i \in J \cap C \) and the sum contains only finitely many nonzero terms. Since \( C \) is a chain, there exists an ideal \( X \subseteq C \) such that all the nonzero \( x_i \) in the above representation of \( x \) belong to \( X \). Then \( x \in X \) and so \( X \cap M \neq \emptyset \), a contradiction which shows that \( K \cap M = \emptyset \). Thus, \( K \subseteq C \). By using Zorn's lemma, we conclude the existence of a maximal ideal \( Q \subseteq C \). i.e. \( Q \) is an ideal of \( A \), maximal with the properties \( I \subseteq Q \) and \( M \cap Q = \emptyset \).

We show that this ideal \( Q \) is prime. Let \( J, D \) be ideals of \( A \) such that \( JD \subseteq Q \) but \( J \not\subseteq Q \), \( D \not\subseteq Q \). By maximality of \( Q \), there exist \( x, y \in A \) such that \( x \in (J + Q) \cap M \) and \( y \in (D + Q) \cap M \). Then \( x, y \in M \) and for some \( r \in A \), \( xry \in M \). Also \( x = a + b \), \( y = c + d \) for some \( a \in J \), \( c \in D \), \( b, d \in Q \). Then \( xry = ar + ard + brc + brc \). Clearly \( brc, ard, brc \in Q \) and \( ar \in JD \). Hence \( xry \in Q \cap M \) a contradiction. Thus \( Q \) is a prime ideal of \( A \).

\[ \text{Q.E.D.} \]

The above theorem is a variant of the following interesting result of Lambek [3, p.16].
**Lemma 2.1**: Let $T$ be any subset of a ring $A$ with unity. Then any ideal $B$ of $A$ which has no element in common with $T$ except possibly 0 is contained in an ideal $M$ which is maximal with respect to this property.

Next we state a sort of folk theorem. Its proof essentially involves usual techniques. See McCoy [4].

**Proposition 2.1**: Let $X$ be an ideal of a ring $A$ and $M$ be an $m$-system of $A$ with $X \cap M = \emptyset$. Then $M$ is contained in a maximal $m$-system $N$ with the property $N \cap X = \emptyset$.

Suppose $X$ is a nonempty subset of $A$, then $C(X)$ denotes the set theoretic complement of $X$ in $A$. We also recall the following highly known result to be found in McCoy [5].

**Lemma 2.2**: An ideal $P$ of $A$ is prime if and only if $C(P)$ is an $m$-system.

We shall use this proposition repeatedly to obtain several interesting results. Here is one more concept that we shall allude to.
DEFINITION 2.2.: For a ring \( A \), a prime ideal \( P \) is called a minimal prime ideal belonging to an ideal \( I \) if and only if \( I \subseteq P \) and there is no prime ideal \( Q \) such that \( I \subseteq Q \subseteq P \).

We give below a characterization of a minimal prime ideal belonging to an ideal \( I \). This characterization essentially uses the Separation Theorem 2.1, and Proposition 2.1, along with the earlier mentioned characterization of prime ideals, namely Lemma 2.2.

THEOREM 2.2.: A nonempty subset \( P \) of \( A \) is a minimal prime ideal belonging to the ideal \( I \) if and only if \( C(P) \) is an \( m \)-system, maximal with respect to the property of not meeting \( I \).

Proof: Suppose that \( P \) is a minimal prime ideal belonging to the ideal \( I \).

By Lemma 2.2, \( C(P) \) is an \( m \)-system. As \( I \subseteq P \), clearly \( I \cap C(P) = \emptyset \). By Proposition 2.1, \( C(P) \) is contained in an \( m \)-system say \( K \), maximal with the property \( K \cap I = \emptyset \). Then by Lemma 2.2, \( C(K) \) is a prime ideal. We have \( I \subseteq C(K) \). Clearly \( C(P) \subseteq K \) implies \( C(K) \subseteq P \). This contradicts the assumption that \( P \) is a minimal prime ideal belonging to the ideal \( I \).
Conversely, suppose that $P$ is a nonempty subset of $A$ and $C(P)$ is an $m$-system maximal with respect to the property of not meeting $I$. Clearly by Lemma 2.2, $P$ is a prime ideal containing $I$. If $P$ is not minimal prime ideal belonging to $I$, then there exists a prime ideal $J$ of $A$ such that $I \subseteq J \subseteq P$. Then $C(J)$ is an $m$-system not meeting $I$ and $C(P) \subseteq C(J)$. This contradicts the maximal property of $C(P)$ of not meeting $I$. Hence $P$ must be a minimal prime ideal belonging to $I$.

Q.E.D.

As every maximal ideal in a ring with unity is prime, the next two results are immediate consequences of Proposition 2.1 and Theorem 2.2.

**COROLLARY 2.1**: Any prime ideal containing the ideal $I$ contains a minimal prime ideal belonging to $I$.

**COROLLARY 2.2**: Every minimal prime ideal belonging to an ideal $I$ is contained in a maximal ideal belonging to $I$.
Finally, in this section, we give a characterization of a maximal $m$-system in a semiprime ring $A$ with unity. It needs to be pointed out that we use in the proof of the said characterization a minor but interesting observation of Herstein [1, p. 4].

**PROPOSITION 2.2**: For a semiprime ring $A$ with unity, the following statements are equivalent.

1) $M$ is a maximal $m$-system of $A$.

2) For any nonzero $a \notin M$, there exists an element $b \in M$ with $ab = 0$.

**Proof**: (i) implies (ii): Let $M$ be a maximal $m$-system for which (ii) does not hold. Let $a \notin M$ be such that $ab \neq 0$ for all $b \in M$. Let $K$ be the multiplicative system (i.e. for any $x, y \in K$, $xy \in K$) generated by $M \cup \{a\}$. We claim that $0 \notin K$. For, if $0 \in K$, let $m_1 m_2 \ldots m_k = 0$ where $m_i \in K (1 \leq i \leq k)$. Since $C(M)$ is a prime ideal, for some $x_1, x_2, \ldots, x_{k-1}$ in $A$, $m_1 x_1 m_2 x_2 \ldots x_{k-1} m_k \neq 0$. However, $m_1 m_2 \ldots m_k x_1 \ldots x_{k-1} = 0$. As $A$ does not have nonzero nilpotent elements we conclude that $m_1 x_1 \ldots x_{k-1} m_k = 0$ a contradiction. Hence $0 \notin K$. Since $K$ is an $m$-system containing $M$, this contradicts the maximality of $M$. 
(ii) implies (i): Let M be an m-system satisfying (ii) but which is not maximal. By Proposition 2.1, M is contained in a maximal m-system, say N. Let \( x \in N - M \); by (ii) there exists \( y \in M \) with \( xy = 0 \). Since \( A \) has no nonzero nilpotent elements, we have \( (x)(y) = 0 \). As \( x, y \notin C(N) \), this contradicts the primeness of \( C(N) \) and we are through.

Q.E.D.

This proposition, together with Lemma 2.2 implies that any element of a minimal prime ideal is a zero-divisor.

\[ \text{\S} 3. \textbf{Topological Considerations}: \]

In this section \( A \) is a semiprime ring with unity.

In the introduction to Part-B, we referred to the hull-kernel topology on \( \Sigma(A) \). As mentioned there, it is called the prime spectrum of \( A \). Our next theorem explicitly shows that the prime spectrum of \( A \) is compact.

**Theorem 3.1:** \( \Sigma(A) \) is a compact space.

**Proof:** Let \( \Sigma(A) = \bigcup_{i \in I} \left\{ \mathcal{P} \right\}_{i} X_{i} \), where \( X_{i}, i \in I \), are ideals of \( A \). Clearly, by (B.S), we have \( \Sigma(A) = \bigcup_{i \in I} \left\{ \mathcal{P} \right\}_{i} X_{i} \).
This shows that $\sum_{i \in I} x_i$ is an ideal of $A$ not contained in any prime ideal of $A$ and so it must contain 1. Thus
$1 = \sum_{i \in I} a_i$ where $a_i \in x_i$ and only finitely many $a_i$ are nonzero. Thus, we may assume that $a_i \neq 0$ for $i \in \mathcal{F}$, where $\mathcal{F}$ is a finite subset of $I$. Thus $1 \in \sum_{i \in \mathcal{F}} x_i$ and

$$\Sigma(A) = \left\{ \mathcal{P} \right\} \sum_{i \in \mathcal{F}} x_i = \bigcup_{i \in \mathcal{F}} \left\{ \mathcal{P} \right\} x_i.$$

Hence $\Sigma(A)$ is compact.

Q.E.D.

We need to introduce one more notation.

**DEFINITION 3.1**: For a prime ideal $\mathcal{P}$ of a ring $A$, $O(\mathcal{P})$ denotes the set, $O(\mathcal{P}) = \left\{ ra : ra=0 \text{ for some } a \notin \mathcal{P} \right\}$.

As $A$ has no nonzero nilpotent elements, it is clear that $O(\mathcal{P})$ is an ideal of $A$ which is contained in $\mathcal{P}$ i.e. $O(\mathcal{P}) \subseteq \mathcal{P}$. Now it becomes possible for us to characterize the Hausdorffness of the prime spectrum $\Sigma(A)$ of $A$, in terms of $O(\mathcal{P})$. We have
THEOREM 3.2: The prime spectrum $\Sigma(A)$ of a semiprime ring $A$ is Hausdorff if and only if $P$ is the unique prime ideal containing $O(P)$.

Proof: Suppose that $\Sigma(A)$ is Hausdorff. Let $Q, S$ be distinct prime ideals of $A$ such that $O(Q) \subseteq S$. Let $\{ P \}_x, \{ P \}_y$ be disjoint neighbourhoods of $Q$ and $S$ respectively, where $x \in S - Q$, $y \in Q - S$. Hence there is no prime ideal $P$ with $x \notin P$ and $y \notin P$. Thus every prime ideal contains either $x$ or $y$. Hence $x \in \bigcap \{ P : P \subseteq \Sigma(A) \}$. It is known that $\bigcap \{ P : P \subseteq \Sigma(A) \} = \{ O \}$. Hence $xy = 0$. Thus, $y \in O(Q)$; a contradiction to the choice of $y$.

Conversely, let $Q$ be the unique prime ideal containing $O(Q)$. If $S 
subseteq Q$ is a prime ideal, then there exists $x \notin O(Q) - S$. But $xy = 0$ for some $y \notin Q$. Since $A$ has no nonzero nilpotent elements, $xy = 0$ if and only if $(x)(y) = 0$. Hence it follows that $y \notin S$. Consider the open neighbourhoods $\{ P \}_x, \{ P \}_y$ in $\Sigma(A)$ of $S$ and $Q$ respectively. Further, by properties (E3.5) and (8.6) we have,

$$\{ P \}_x \cap \{ P \}_y = \{ P \}_x \cap \{ P \}_y = \{ P \}_0 = \emptyset$$

and so we are through. Q.E.D.
The next result gives a sufficient condition for a particular subset of \( \Sigma(A) \) to be totally ordered (i.e. linearly ordered).

**Theorem 3.3**: If any two incomparable elements of \( \Sigma(A) \) have disjoint neighbourhoods, then for any \( Q \in \Sigma(A) \), the set \( \{ P : P \subseteq A, Q \subseteq P \} \) is a chain.

**Proof**: Let \( X, Y \) be two incomparable elements of \( \{ P : P \subseteq A, Q \subseteq P \} \). By assumption, there exist disjoint neighbourhoods \( \{ P \}_x \), \( \{ P \}_y \) of \( X \) and \( Y \) respectively, where \( x \notin X, y \notin Y \). Now \( \{ P \}_y \cap \{ P \}_x \neq \emptyset \) implies \( (x)(y) = \omega \) and so either \( x \notin Q \) or \( y \notin Q \). In either case, this contradicts the choice of \( x \) and \( y \) and this completes the proof.

\[ Q.E.D. \]

Next, we state a known result; see Koh [2, Theorem 2.4], which shall be used to characterize minimal prime ideals.

**Lemma 3.1**: Let \( A \) be a ring without nonzero nilpotent elements. Then \( P \) is a minimal prime ideal if and only if \( P = \text{C}(P) \).
We now glue together our considerations of Section-2, and Section-3, to have the following characterization.

**THEOREM 3.4:** Let $M$ be a nonempty subset of a semiprime ring $A$. Then the following are equivalent:

i) $C(M)$ is a maximal $m$-system,

ii) $M$ is a minimal prime ideal,

iii) For any $a \in M$, there exists $b \in C(M)$ such that $ab = 0^+$.

**Proof:** (i) if and only if (ii) follows from Theorem 2.2 and Lemma 2.2.

(ii) if and only if (iii) follows from Lemma 3.1 and the definition of $0(F)$.

§ 4. The Minimal Spectrum:

We restrict the hull-kernel topology $\Sigma(A)$ to $\pi(A)$, where $\pi(A)$ is the set of minimal prime ideals of $A$. As stated earlier, $\pi(A)$, with induced topology from $\Sigma(A)$, is frequently referred to as the minimal spectrum of $A$. In this section, we concentrate our attention on the minimal spectrum. However, in the beginning we list two characterizations of minimal prime ideals of $A$. In this section also $A$ is a semiprime ring with unity.
For a nonempty subset $S$ of $A$, we have defined in the introduction to Part-A, the right and the left annihilator of $S$ in $A$. We note that if $A$ has no nonzero nilpotent elements, then the right and left annihilators coincide and we shall denote the annihilator of $S$ by $S^\ast$. It can be easily verified that $S^\ast$ is an ideal of $A$. If $S = \{x\}$ then we shall denote by $(x)^{\ast}$ the annihilator of $S$. In view of the fact that $A$ has no nonzero nilpotent elements, it is clear that $(x)^{\ast} = ((x))^{\ast}$ where $(x)$ is the ideal generated by $x$. Hence we use $(x)^{\ast}$, $(\{x\})^{\ast}$ and $((x))^{\ast}$ interchangeably without any explanation. It can be shown that $SS^\ast(o)$ for any nonempty subset $S$ of $A$, where $SS^\ast = \{xy : x \in S, y \in S^\ast\}$. We also note that $S \subseteq S^{**}$.

Here is the first characterization of minimal prime ideals that involves the concept of annihilation.

**Theorem 4.1**: A prime ideal $M$ is minimal prime if and only if $(x)^{\ast} - M \neq \emptyset$ for any $x \in M$.

**Proof**: Let $x$ be an element of a minimal prime ideal $M$. As $G(M)$ is a maximal $m$-system, by Proposition 2.2, there exists $y \in G(M)$ such that $xy = 0$. Thus $y(x)^{\ast} - M$. 
Conversely, suppose that for a prime ideal $M$, 
$(x)^* \not\in M$ for any $x \in M$. To show $M$ is minimal prime.
Suppose that $N$ is a prime ideal of $A$ such that $N \subseteq M$.
Let $y \in M - N$. By assumption $(y)^* \not\in M$. Let $z \in (y)^* - M$.
Then $yz = 0$ implies $(y)(z)(z) = (0)$. Hence $z \in N$ but $z \notin M$, a
contradiction, which proves the theorem.

Q.E.D.

Next we have second characterization of minimal
prime ideals.

**Theorem 4.2:** A prime ideal $M$ is minimal prime if and
only if it contains precisely one of $(x), (x)^*$ for any $x \in A$.

**Proof:** Let $M$ be a minimal prime ideal. If $x \in M$, then
by Theorem 4.1, $(x)^* \not\subseteq M$. On the other hand, if $(x)^* \subseteq M$
and $(x) \subseteq M$ for some $x \in A$. By Theorem 3.4, $x \in M$ implies
there exists $y \in C(M)$ with $xy = 0$. But then $y \in (x)^* \subseteq M$ leads
to a contradiction as $M \cap C(M) = \emptyset$. Hence $M$ contains
precisely one of $(x), (x)^*$ for any $x \in A$.

Conversely, let a prime ideal $M$ satisfy the given
condition. Let $N$ be a prime ideal of $A$ such that $N \subseteq M$.
Let $x \in M - N$. Then, by assumption, $(x)^* \not\in M$. But $(x)(x)^* = (0)$
implies, by primeness of $N$, $(x)^* \subseteq N \subseteq M$, a contradiction.
Hence $M$ is minimal prime.

Q.E.D.
In the next result we establish a relationship between annihilators and hulls and kernels. We have

**Theorem 4.3**: For any ideal $I$ of $A$.

$$I^* = K(\pi(A) - H(I)).$$

**Proof**: Since $II^* = (0)$, it follows that whenever, $I \not\subseteq M \pi(A)$, then $I^* \subseteq M$. Thus

$$I^* \subseteq \cap \{M : M \pi(A), I \not\subseteq M\}.$$

Let $x \in \cap \{M : M \pi(A), I \not\subseteq M\}$ and $x \not\subseteq I^*$. Then for some $y \in I$, $xy \not\in 0$. Consider the multiplicative system $T = \{(xy)^i : i = 1, 2, \ldots\}$. Clearly $0 \not\in T$ and so by Proposition 2.1, $T$ is contained in a maximal $m$-system say $F$ of $A$. Clearly $xy \not\in C(F)$ and hence $x \not\in C(F)$, $y \in C(F)$ [as $C(F)$ is a (prime) ideal].

Thus $I \not\subseteq C(F)$ and so

$$\cap \{M : M \pi(A), I \not\subseteq M\} \subseteq C(F),$$

but then $x \not\in C(F)$, a contradiction. Hence $I^* = \cap \{M : M \pi(A), I \not\subseteq M\} = K(\pi(A) - H(I)).$

Q.E.D.
This theorem readily yields the following sequence of results.

**Corollary 4.1:** For any \( x \notin A \), \( (x)^* = k(\{ M \}_{x}) \).

A more useful consequence of our considerations is proved in the next result.

**Corollary 4.2:** For any \( x \notin A \),

\[
H(k(\{ M \}_{x})) = H((x)^*) = \{ M \}_{x}.
\]

In particular, \( H(x) \) and \( H((x)^*) \) are both open and closed sets in \( \pi(A) \) that are disjoint.

**Proof:** The proof follows from Corollary 4.1 and Theorem 4.2.

Q.E.D.

The next Corollary is immediate from Theorem 4.3 and Corollary 4.2.

**Corollary 4.3:** For any \( x \notin A \), \( H(x) = H((x)^{**}) \).

After the above sequence of observations we take up a consideration involving an element and a prime ideal of \( A \). In fact, we have the following nice characterization.
THEOREM 4.4: For each element \( r \notin A \) and for a prime ideal \( P \) of a semiprime ring \( A \) the following statements are equivalent.

i) \( (r)^* \subseteq P \).

ii) There is some \( Q \in \pi(A) \) such that \( Q \subseteq P \) and \( r \not\in Q \).

Proof: (i) implies (ii): Let \( (r)^* \subseteq P \) where \( r \notin A \), \( P \in \pi(A) \). Then \( (r)^* \cap C(P) = \emptyset \). Hence by Proposition 2.1, \( C(P) \) is contained in a maximal \( m \)-system say \( T \) with \( T \cap (r)^* = \emptyset \). But then \( C(T) \in \pi(A) \) is a prime ideal containing \( P \) and \( (r)^* \subseteq C(T) \). By Theorem 4.2, \( (r)^* \not\subseteq C(T) \) i.e. \( r \not\in C(T) \). Thus we have \( (r)^* \subseteq C(T) \subseteq P \).

(ii) implies (i): By Theorem 4.2, \( r \not\in Q \) implies \( (r)^* \subseteq Q \). Hence \( (r)^* \subseteq P \).

Q.E.D.

If \( \{ M \}_x \subseteq \{ M \}_y \) then \( K(\{ M \}_x) \subseteq K(\{ M \}_y) \) and by Corollary 4.1, we have \( (y)^* \subseteq (x)^* \). Hence \( (x)^* \subseteq (y)^* \). Conversely, from Corollaries 4.1 and 4.2, we can get \( (x)^* \subseteq (y)^* \) implies \( \{ M \}_x \subseteq \{ M \}_y \). This proves the following
**Corollary 4.4:** \( \{ M \}_{x} \subseteq \{ M \}_{y} \) if and only if \((x)^{**} \subseteq (y)^{**}\).

One readily notes that \( \{ M \}_{x} \) are clopen sets of \( \pi(A) \).

In fact, we have

**Theorem 4.5:** The hull-kernel topology on \( \pi(A) \) is Hausdorff.

The base sets \( \{ M \}_{x} \) are open as well as closed.

**Proof:** Let \( X, Y \) be distinct minimal prime ideals in \( A \).

Let \( x \notin X-Y \). As \( x \notin C(X) \) and \( C(X) \) is a maximal \( m \)-system by Theorem 3.4, there exists \( y \in X \) with \( xy=0 \). Clearly \( Y \subseteq \{ M \}_{x} \) and \( X \subseteq \{ M \}_{y} \). From \( xy=0 \) it follows that \( \{ M \}_{x} \) and \( \{ M \}_{y} \) are disjoint i.e. \( \pi(A) \) is Hausdorff. The Corollary 4.2, implies that \( \{ M \}_{x} \) are clopen.

Q.E.D.

Now we give two definitions.

**Definition 4.1:** An ideal \( I \) of a ring \( A \) is called dense if \( I^{*} = (0) \).
DEFINITION 4.2: An ideal $I$ of a ring $A$ is called normal if $I = I^{**}$.

In the next theorem we show that any non-dense ideal is contained in a minimal prime ideal.

THEOREM 4.6: Any non-dense ideal $I$ of a ring $A$ is contained in a minimal prime ideal.

Proof: Clearly $I$ is non-dense if and only if $I^* 
eq (0)$. For $x \in I^*$, $x \neq 0$, $T = \{ x^i ; i=1,2,... \}$ is a multiplicative system of $A$ not meeting $I$ (as $I \cap I^* = (0)$). By Proposition 2.1, $T$ is contained in a maximal $m$-system say $F$ not meeting $I$. Then $C(F)$ is a minimal prime ideal and $I \subseteq C(F)$ implies the result.

Q.E.D.

Let us discuss normal ideals

THEOREM 4.7: Any normal ideal $I$ of $A$ is the intersection of all minimal prime ideals containing it.
Proof: From Theorem 4.3, it follows that

\[ I^{**} = \cap \{ M : M \in \pi(A), \ I^* \nsubseteq M \} . \]

By normality of \( I \) we have

\[ I = \cap \{ M : M \in \pi(A), \ I^* \nsubseteq M \} . \] Then Theorem 4.2, implies \( I \subseteq M \) if and only if \( I^* \nsubseteq M \). Thus

\[ I = \cap \{ M : M \in \pi(A), \ I \subseteq M \} . \]

Q.E.D.

Here is an immediate consequence:

**Corollary 4.5**: An ideal \( I \) is normal if and only if \( I \) is the intersection of all minimal prime ideals containing it.

A ring \( A \) is called a Rickart ring if the right annihilator of any element is a right ideal generated by an idempotent.

For a semiprime ring it was mentioned that the right annihilator and the left annihilator of an element coincide. For such rings we may say that, \( A \) is a Rickart ring if and only if \( (r)^* = eA \) for any \( r \in A \) and for some idempotent \( e \in A \).
We give a characterization of Rickart rings in the next theorem.

**THEOREM 4.8:** For a semiprime ring $A$ the following statements are equivalent.

1) $A$ is a Rickart ring.

2) For each $r \in A$, there is some idempotent $e$ such that for any $P \in \pi(A)$, $rP$ if and only if $eP$.

**Proof.** (i) implies (ii): Let $rP \in \pi(A)$. By Theorem 4.2, $(r)^* \subseteq P$. But $(r)^* = eA$ for some idempotent $e$ and so $eP$. But then $(e)^* = (1-e)A \subseteq P$. Thus $1-eP$ where $1-e$ is an idempotent. Thus $rP$ implies there is an idempotent $f = 1-e$ in $P$.

On the other hand, if $f = 1-e$ is in a minimal prime ideal $Q$ then, as $(r) \subseteq (r)^*$, it follows that $r(1-e)A$ and so $rQ$.

(ii) implies (i). Let $r \in A$. If $rM$ for any $M \in \pi(A)$, then trivially, $(r)^* = 0A$.

If $rM \in \pi(A)$ then there exists an idempotent $eA$ such that $rM$ if and only if $eM$. Then $\{ M \}_{r} = \{ M \}_{e}$ and by Corollary 4.4, $(r)^* = (e)^*$. But $r(e)^*$ if and only if $r(1-e)A$. Thus $(r)^* = fA$ where $f = 1-e$ is an idempotent.

Thus $A$ is a Rickart ring. Q.E.D.
Now we concentrate on $\pi(A)$ as a space. We have already defined on $\pi(A)$ two topologies namely, the hull-kernel topology $\mathcal{F}^h$ and the dual hull-kernel topology $\mathcal{F}^d$.

In the next theorem we show the relationship between the hull-kernel and dual hull-kernel topology.

**Theorem 4.9** : The hull-kernel topology $\mathcal{F}^h$ is finer than the dual hull-kernel topology $\mathcal{F}^d$.

**Proof.** The sets $\{ H(x) : x \in A \}$ form a basis for $\mathcal{F}^d$ and $H(x) = \pi(A) - \{ M \}_{x}$ for any $x \in A$. By Theorem 4.5, $\{ M \}_{x}$ is closed in $(\pi(A), \mathcal{F}^h)$. Hence $H(x)$ is open in $(\pi(A), \mathcal{F}^h)$. Thus $\mathcal{F}^h$ is finer than $\mathcal{F}^d$.

Q.E.D.

In fact, the reverse inclusion is valid under some restriction.

**Theorem 4.10** : If for any $x \in A$, there exists $y \in A$ such that $(x)^* = (y)^*$ then $\mathcal{F}^h = \mathcal{F}^d$. 
Proof: From Theorem 4.9, we know $\mathcal{J}^d \subseteq \mathcal{J}^h$. Let 
\[ \{ M \}_x \in \mathcal{J}^h. \] We have, from Corollary 4.2, $H((x)^* ) = \{ M \}_x$.

By Corollary 4.3, we have $H(x) = H(x)^{**}$. Hence if for any $x \in A$, there is $y \in A$ such that $(x)^{**} = (y)^*$ we have 
\[ \{ M \}_x = H((x)^*) = H((y)^{**}) = H(y), \]

i.e.: 
\[ \{ M \}_x \text{ is closed in } \mathcal{J}^d. \]

Hence $\mathcal{J}^h \subseteq \mathcal{J}^d$. Thus $\mathcal{J}^h = \mathcal{J}^d$.

Q.E.D.

Now we are in a position to obtain the main result:

**Theorem 4.11**: The statements given below are equivalent in a semiprime ring $A$.

1) $(\pi(A), \mathcal{J}^h)$ is compact.

2) Finite unions of $\{ \{ M \}_x : x \in A \}$ form a Boolean lattice.

3) For any $x \in A$, there exist $t_i \in A$, $1 \leq i \leq n$ such that $t_i f(x)^*$, $1 \leq i \leq n$ and $(x)^* \cap \bigcap_{i=1}^{n} (t_i)^* = (0)$. 


4) For any $x \in A$ there exist $t_i, 1 \leq i \leq n$, in $A$ such that $(x)^* = \bigcap_{i=1}^{n} (t_i)^*$.

5) $\mathcal{T}^h = \mathcal{T}^d$.

6) $\{ H(x) : x \in A \}$ is a subbasis for the open sets of $(\pi(A), \mathcal{T}^d)$.

7) $\{ \{ M \}_{x} : x \in A \}$ is a subbasis for the open sets of $(\pi(A), \mathcal{T}^h)$.

Proof: The equivalence of (5), (6) and (7) is trivial because topologies are completely determined by any of their subbases. The theorem would be proved if we show

$$ (1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (1) $$

(1) implies (2): By Theorem 4.5, $\pi(A)$ is Hausdorff and as $H(x)$ is a closed subset of $\pi(A)$ we conclude that $H(x)$ is compact in its relative topology. By Theorem 4.2, it follows that $H(x) \cap H((x)^*) = \emptyset$ and so $H(x) \cap \{ H(t) : t \in (x)^* \} = \emptyset$. By compactness, there exist $t_i, 1 \leq i \leq n$ in $(x)^*$ such that

$$ H(x) \cap \{ H(t_i) : t_i \in (x)^*, 1 \leq i \leq n \} = \emptyset. $$
On taking complements in \( \pi(A) \) we get

\[
\pi(A) = \left\{ M \right\}_x \cup \left\{ M \right\}_{t_1} \cup \ldots \cup \left\{ M \right\}_{t_n}
\]

As \( t_i \in (x)^* \) for \( 1 \leq i \leq n \), by Theorem 4.2 we conclude that
\[
\left\{ M \right\}_x \cap \bigcup_{i=1}^{n} \left\{ M \right\}_{t_i} = \emptyset.
\]
Thus \( \bigcup_{i=1}^{n} \left\{ M \right\}_{t_i} \) is a complement of \( \left\{ M \right\}_x \). Since \( \left\{ \left\{ M \right\}_x : x \in A \right\} \) is a semilattice we conclude that finite unions of \( \left\{ \left\{ M \right\}_x : x \in A \right\} \) form a Boolean lattice;

see Varlet [6].

(2) implies (3): Let \( \bigcup_{i=1}^{n} \left\{ M \right\}_{t_i} \) be the complement of \( \left\{ M \right\}_x \).

Clearly we have \( xt_i = 0 \), \( 1 \leq i \leq n \), and so \( t_i \in (x)^* \). Furthermore,

\[
K(\left\{ M \right\}_x \cup \bigcup_{i=1}^{n} \left\{ M \right\}_{t_i}) = K(\pi(A))
\]

that is

\[
K(\left\{ M \right\}_x) \cap \bigcap_{i=1}^{n} K(\left\{ M \right\}_{t_i}) = K(\pi(A)).
\]

Hence by Corollary 4.1 and using \( K(\pi(A)) = (0) \) we have

\( (x)^* \cap \bigcap_{i=1}^{n} (t_i)^* = (0). \)

(3) implies (4): As \( xt_i = 0 \), \( 1 \leq i \leq n \), we have \( x \in (t_i)^* \).

Hence \( (x)^* \subseteq (t_i)^* = (t_i)^* \) for \( 1 \leq i \leq n \).
Thus \( (x)^{**} \subseteq \bigcap_{i=1}^{n} (t_i)^* \). Let \( \bigcap_{i=1}^{n} (t_i)^* \), then \( a t_i = 0 \), \( 1 \leq i \leq n \). If \( y \in (x)^* \), then \( x y = 0 \). Also, \( y a t_i = 0 \) and \( y a x = 0 \).

Hence by (3), \( y a = 0 \) and so \( a f(x)^{**} \), which proves the implication.

(4) implies (5): In view of Theorem 4.9, we need only to prove the basic open sets \( \{ \{ M \} : x \notin A \} \) in \( J^h \) are open in \( J^d \). For \( x \notin A \), there exist \( t_i, 1 \leq i \leq n \), in \( A \) such that \( (x)^{**} = \bigcap_{i=1}^{n} (t_i)^* \).

Hence using Corollary 4.2 and Corollary 4.3. We have

\[ H(x) = \bigcup_{i=1}^{n} \{ M : t_i \} \]. Taking complements in \( x(A) \), we get,

\[ \{ M \} = \bigcap_{i=1}^{n} H(t_i) \]. Thus \( \{ M \} \) is a finite intersection of open sets in \( J^d \) and so is open and we are through.

(5) implies (1): By (5), \( \{ \{ M \} : x \notin A \} \) will also be a basis for closed sets in \( (x(A), J^h) \). To prove (1), we shall show that every family of closed sets with the finite intersection property has nonempty intersection. Let \( \{ \{ M \} : x \notin J \} \) be a family of closed sets having the finite intersection property.

This implies that \( \bigcap_{x \in F} \{ M \} \neq \emptyset \) whenever, \( F \subseteq J \) is finite and so \( \prod_{x \in F} (x) \neq (0) \), where \( \prod(x) \) denotes the product of ideals
(x), \( x \notin F \). As \( A \) has no nonzero nilpotent elements, this further implies that \( \prod_{x \in F} x \neq 0 \) where \( \prod_x \) is the product of \( x \), \( x \notin F \). Let \( X \) be the multiplicative semigroup generated by \( J \). Clearly \( 0 \notin X \) and so by Proposition 2.1, \( X \) is contained in a maximal \( m \)-system say \( T \) not containing \( 0 \). But then \( C(T) \) is a minimal prime ideal not meeting \( J \). Thus, \( C(T) \in \bigcap_{x \in J} \{ M \} \). Hence (1) holds.

Q.E.D.

The proof of the next proposition uses usual techniques and so we omit it:

**PROPOSITION 4.1**: Let \( X \) and \( Y \) be rings with unity.

Let \( f : X \to Y \) be an onto homomorphism. Then \( f^{-1}(P) = \{ a \in X : f(a) \in P \} \) is a prime ideal of \( X \) for every prime ideal \( P \) of \( Y \).

**DEFINITION 4.3**: A ring \( A \) is said to be normal if every prime ideal contains a unique minimal prime ideal.

Let \( A, B \) be two normal semiprime rings with unity. Let \( \pi(A) \) and \( \pi(B) \) denote the minimal prime spectra of \( A \) and \( B \) respectively. For any onto homomorphism \( f : A \to B \), define the map \( f^* : \pi(B) \to \pi(A) \) by \( f^*(M) = [f^{-1}(M)]^m \) where \( M \in \pi(B) \) and \( [f^{-1}(M)]^m \) denotes the unique minimal prime ideal contained in \( f^{-1}(M) \).
Finally we prove the last theorem of this Chapter.

**Theorem 4.12**: The map $f^*$ defined as above is a continuous map.

**Proof**: By Proposition 4.1, and by normality of $A$, it follows that the map $f^*$ is well defined.

We first observe that $f^{-1}_*$

Let $Q \in \{ M \}$, then $x \in f^{-1}(Q)$ and so $x \notin f^*(Q)$.

Hence $Q \in f^{-1}(\{ M \})$. Thus $\{ M \} \subseteq f^{-1}(\{ M \})$.

Next, let $Q \notin f^{-1}(\{ M \})$ then $f(Q) \in \{ M \}$ and so $[f^{-1}(Q)]^m \subseteq \{ M \}$. We claim that $x \notin f^{-1}(Q)$. Suppose to the contrary, then $f(x) \in Q$. Since $Q$ is a minimal prime ideal, by Theorem 3.4, there exists $f(y) \in \mathcal{O}(Q)$ such that $f(x)f(y) = 0$. As $A$ and $B$ do not have nonzero nilpotent elements, it follows that $f((x)(y)) = 0$ and so $(x)(y) \subseteq f^{-1}(0)$.

Since $[f^{-1}(Q)]^m$ is a minimal prime ideal containing $f^{-1}(0)$, it follows that $(y) \subseteq [f^{-1}(Q)]^m$ which is a contradiction. Hence $x \notin f^{-1}(Q)$ implies $Q \notin \{ M \}$. Thus $f^*(\{ M \}) = \{ M \} f(x)$. 


This shows that \( f^* \) pulls back open sets onto open sets and so it is continuous:

**Q.E.D.**

This theorem establishes a functorial correspondence between the category of normal rings and their minimal spectra.
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CHAPTER IV

Space of prime strict

Ideals in a

Rickart $\star$-Ring
We discussed ideals, *-ideals and strict ideals in Rickart *-rings in the final section of Chapter-I. We concentrate, in this Chapter, on strict ideals. Recalling back, a strict ideal is nothing but a *-ideal for which both an element and its right projection are in it. It can be seen that the principal ideals generated by a central projection in \( A \) is a strict ideal. In fact, we shall be concerned here with the set of prime strict ideals in a Rickart *-ring when it carries the hull-kernel topology. We shall also investigate the set of minimal prime strict ideals with the induced hull-kernel topology.

\[ \square 1: \text{Introduction:} \quad \text{We denote by } \Sigma(A) \text{ the set of prime strict ideals of a Rickart *-ring } A, \text{ as was done in the introduction to Part-B.} \]

In Section-2 of this chapter, we characterize prime strict ideals of a semiprime Rickart *-ring in terms of a set complement of a \( p \)-set, a concept introduced by us. An
interesting analogue of separation Theorem is obtained. Much of the study that has been carried out in this chapter rests heavily on this separation theorem.

In Section-3, we study $\Sigma(A)$ as a topological space. We have already introduced the hull-kernel topology on $\Sigma(A)$ in the introduction to Part-B.

We relate the right annihilator of a subset of $A$ with its hulls and kernels. We show that the topological space $\Sigma(A)$ is a Hausdorff space. Furthermore, it is also shown to be compact, and this compactness is characterized in several ways.

We have discussed at length in Chapter-I the concept of generalized comparability (GC). In Section-4 of this Chapter, along with (GC) for projections, we also discuss (dominated) comparability of projections. Prime strict ideals of a general Rickart *-ring $A$ (which need not be semiprime) with (GC) are characterized in three different ways. Moreover, for such rings, a homeomorphism from minimal prime spectrum, $\pi(A)$, to the prime spectrum of the Boolean ring of central projections is characterized in three ways. Faithful projections in Baer *-rings are also characterized.
The condition that the Rickart \( * \)-ring is semiprime is not assumed in Section-5 also. In this section we show that there is a functorial correspondence between the category of Rickart \( * \)-rings and the category of their prime spectra.

Section-6 begins with building up the necessary jargon for the setting of sheaf theory. We then obtain a general sheaf representation theorem for a Rickart \( * \)-ring \( A \). Further, it is shown that if the Rickart \( * \)-ring \( A \) is semiprime, then under the condition of fullness, every section of the sheaf with compact support is explicitly exhibited.

\( \text{ï¿½2: Separation Theorem} \) : 

In this section we shall assume that \( A \) is a semiprime Rickart \( * \)-ring. This assumption implies that all the projections in \( A \) are central. With this in mind, and in order to obtain a natural but interesting characterization of prime strict ideals we introduce a novel concept of a \( p \)-set. Motivation to introduce this concept stemmed from Keimel [7].
**Definition 2.1**: A nonempty subset $F$ of $A$ will be called a $p$-set if and only if

(i) $0 \in F$ (ii) $x \in F$ if and only if $RP(x) \in F$ and

(iii) $F$ is closed under product of projections.

Now the characterization.

**Theorem 2.1**: An ideal $P$ of $A$ is prime strict if and only if the set complement of $P$, i.e. $A-P$, is a $p$-set.

**Proof**: If $P$ is a strict ideal then clearly $0 \notin A-P$ and $x \notin A-P$ if and only if $RP(x) \notin A-P$.

Now let $P$ be prime strict. Let $e, f$ be projections in $A-P$. We have noted that the projections in $A$ are central. Hence $effP$ implies $(eAf)(AfA) \subseteq P$ and as a consequence we have $efP$ or $fP$ which is a contradiction to the choice of $e$ and $f$. This proves that $A-P$ is a $p$-set.

Conversely, suppose that for an ideal $P$, $A-P$ is a $p$-set. Clearly $P$ is a strict ideal. Let $I, J$ be strict ideals of $A$ such that $IJ \subseteq P$ but $I \nsubseteq P$, $J \nsubseteq P$. Let $x \notin I-P$, $y \notin J-P$. Then $RP(x)$, $RP(y) \notin P$. Since $A-P$ is a $p$-set, this implies $RP(x)RP(y) \notin IJ \cap (A-P)$, a contradiction. Hence $P$ is prime strict.

Q.E.D.
This characterization of prime strict ideals will be put to use later. It may be recalled that following Berberian [1] one may show that the sum and product of any two strict ideals is a strict ideal.

Crucial to our approach is the following separation theorem. This lemma may remind one, several of its relatives in different settings; see for example, Lambek [9].

**Theorem 2.2 (Separation theorem):**

Let I be a strict ideal of A. M be a p-set of A with $M \cap I = \emptyset$. Then there exists a strict ideal Q of A maximal with the property that $M \cap Q = \emptyset$, $I \subseteq Q$. Moreover, this strict ideal is prime strict.

**Proof:** Let $\emptyset$ denote the set of all strict ideals K of A such that $I \subseteq K$ and $M \cap K = \emptyset$. Clearly $\emptyset$ is nonempty and it can be partially ordered under set inclusion. Let $\mathcal{C}$ be a chain in $\emptyset$ and $\emptyset = \bigcup \mathcal{C}$. It can be shown that $I \subseteq B$, $B \cap M = \emptyset$ and B is a strict ideal. Hence by Zorn's lemma, $\emptyset$ has a maximal element Q containing I.

To show that Q is prime strict. Let J, K be strict ideals of A such that $J \cap Q$, $K \cap Q$. Clearly $J + Q$, $K + Q$ are strict ideals of A containing I. Hence the maximality of Q implies that there exist $s \subseteq (J + Q) \cap M$, $t \subseteq (K + Q) \cap M$. 
Let \( s = j + q, \ t = k + r \) where \( j \in J, \ k \in K, \ q, r \in Q \). Clearly \( \text{RP}(s) \leq \text{RP}(j) \lor \text{RP}(q), \ \text{RP}(t) \leq \text{RP}(k) \lor \text{RP}(r) \). Also, \( \text{RP}(s), \ \text{RP}(t) \in M \) imply \( \text{RP}(s) \text{RP}(t) \in M \). Since all the projections of \( A \) are central,

\[
\text{RP}(j) \lor \text{RP}(q) = \text{RP}(j) + \text{RP}(q) - \text{RP}(j) \text{RP}(q)
\]

and

\[
\text{RP}(k) \lor \text{RP}(r) = \text{RP}(k) + \text{RP}(r) - \text{RP}(k) \text{RP}(r).
\]

By definition of the partial order, we have

\[
\text{RP}(s) = \text{RP}(s) \left[ \text{RP}(j) \lor \text{RP}(q) \right] \quad \text{and}
\]

\[
\text{RP}(t) = \text{RP}(t) \left[ \text{RP}(k) \lor \text{RP}(r) \right].
\]

Hence after simple calculations we conclude that if \( JK \subseteq Q \) then \( \text{RP}(s) \text{RP}(t) \in Q \cap M \), a contradiction. This proves that \( Q \) is prime strict.

Q.E.D.

Here are some consequences of the separation theorem that are worth mentioning.

**Remark 2.1:** Since for any nonzero projection \( e \), \( \{e\} \) is a p-set not meeting the strict ideal \((0)\), taking \( M = \{e\} \) in the above theorem ensures rich supply of prime strict ideals.
**Remark 2.2:** Let \( e \neq 1 \) be a nonzero projection of \( A \). Clearly \( A \) is a strict ideal not containing the \( p \)-set \( M = \{ 1 \} \). Hence the Theorem 2.2 shows that every nonzero projection \( e \neq 1 \) is contained in a prime strict ideal.

**Remark 2.3:** Since for any \( x \in A \), \( RP(x)(1-RP(x)) = 0 \) and \( A \) has no nonzero nilpotent elements, it can be shown that \( (RP(x))(1-RP(x)) = (0) \) where \( (RP(x)), (1-RP(x)) \) are strict ideals generated by \( RP(x), 1-RP(x) \) respectively. This implies that any prime strict ideal contains exactly one of \( RP(x), 1-RP(x) \) for every \( x \in A \).

**Remark 2.4:** Let \( a \in \bigcap \{ P : P \in \Sigma(A) \} \) where \( \Sigma(A) \) is the prime spectrum of \( A \). Clearly \( 1-RP(a) \notin P \) for any \( P \in \Sigma(A) \). Hence by Remark 2.2, we get \( 1-RP(a) = 1 \) i.e. \( a = 0 \). Thus intersection of all prime strict ideals is \( (0) \).

**Remark 2.5:** From Remark 2.3 we conclude that \( \Sigma(A) \) has no comparable elements. For if, \( I, J \in \Sigma(A) \) and \( I \subseteq J \). Let \( x \in J-I \) then \( RP(x) \notin I \) implies \( 1-RP(x) \notin I \subseteq J \), a contradiction. Hence \( I \notin J \).

This shows that in this case, both, the minimal spectrum \( \pi(A) \) and the prime spectrum \( \Sigma(A) \) coincide. Also from Theorem 2.1, it follows that no two \( p \)-sets are comparable.

Here is a consequence of Remark 2.5.
PROPOSITION 2.1: \( F \) is a p-set of \( A \), implies that for each \( x \in F \), there exists \( a \in F \) such that \( \text{RP}(x)\text{RP}(a) = 0 \).

Proof: If \( \text{RP}(x)\text{RP}(a) \neq 0 \) for all \( a \in F \) and for some \( x \in F \) then \( G = F \cup \{ x, \text{RP}(x) \} \cup \{ \text{RP}(x)\text{RP}(a) : a \in F \} \) is a p-set which properly contains \( F \). This contradicts the incomparability of p-sets.

Q.E.D.

We use this lemma to obtain a representation of the set theoretic complement of a p-set. We have already noted that in a ring without nonzero nilpotent elements, the notions of right and left annihilators coincide. We denote the annihilator of a nonempty subset \( I \) of \( A \) by \( \text{Ann}(I) \).

When \( I = \{ a \} \) we write \( \text{Ann}(I) = \text{Ann}(a) \). It is known that for any \( a \in A \),

\[
\text{Ann}(a) = \text{Ann}\left(\text{RP}(a)\right) = \text{Ann}\left(\text{RP}(\text{RP}(a))\right) = [1 - \text{RP}(a)]A,
\]

where \( \text{RP}(a) \) is the strict ideal generated by \( \text{RP}(a) \).

Now we have

PROPOSITION 2.2: For a p-set \( F \), \( A - F = \bigcup \{ \text{Ann}(a) : a \in F \} \).

Proof: Since \( \text{Ann}(a) = (1 - \text{RP}(a)) \), it follows that for every \( a \in F \), \( 1 - \text{RP}(a) \notin F \). If \( x \in \text{Ann}(a) \cap F \) then \( \text{RP}(x) \notin F \).

This together with \( \text{RP}(a) \cap F \) implies \( \text{RP}(x) \cap \text{RP}(a) = 0 \cap F \), a contradiction which shows that \( \text{Ann}(a) \cap F = \emptyset \). Hence 
\[ \bigcup \{ \text{Ann}(a) : a \in F \} \subseteq A - F. \] Let \( x \notin A - F \). Clearly \( x \in F \) and so by 
Proposition 2.1, there exists \( y \in F \) such that \( \text{RP}(y) \cap \text{RP}(x) = 0 \).
Hence \( xy = 0 \) i.e. \( x \notin \text{Ann}(y) \). Thus \( x \notin \bigcup \{ \text{Ann}(a) : a \in F \} \).
This shows that \( A - F \subseteq \bigcup \{ \text{Ann}(a) : a \in F \} \) which proves the result.

Q.E.D.

Next, we state, without proof, a companion of the separation theorem.

**Theorem 2.3**: Let \( X \) be a strict ideal of \( A \) and \( M \) be a 
nonempty subset of \( A \) which is closed under product of 
projections and with the property that \( M \cap X = \emptyset \). Then there 
exists a subset \( N \) of \( A \) closed under product of projections 
and maximal with respect to containing \( M \) and not meeting \( A \).

\[ \text{§ 3: Topological Considerations.} \]

In this section also \( A \) will denote a 
semiprime Rickart \( * \)-ring. Recall that Remark 2.5 
implies \( \pi(A) \) and \( \Sigma(A) \) coincide. Now we express anni-
hilator of a strict ideal \( I \) in terms of hulls and kernels.
THEOREM 3.1: For a strict ideal I of A, \( I = K(H(I)) \).

Proof: Since the p-set \( \{1\} \) is disjoint from I, by the separation Theorem 2.2, we conclude \( I \subseteq P \) for some \( \Phi \subseteq A \). Hence \( H(I) \not= \emptyset \). Clearly \( I \subseteq K(H(I)) \). Let \( a \in K(H(I)) \setminus I \).

Then \( RP(a) \not= I \). Hence by Theorem 2.2, there exists \( Q \subseteq A \) such that \( I \subseteq Q \) and \( a \notin Q \). But then \( a \in K(H(I)) \) implies \( a \in Q \), a contradiction. Hence \( I = K(H(I)) \).

Q.E.D.

Let us also note some pertinent observations.

COROLLARY 3.1: For any \( a, b \in A \),

\[
(RP(a)) \cap (RP(b)) = (RP(a) \cap RP(b)).
\]

Proof: As \( (RP(a)) \cap (RP(b)) \subseteq (RP(a)) \cap (RP(b)) \), it follows that whenever, \( (RP(a)) \cap (RP(b)) \subseteq P \) for \( \Phi \subseteq A \), then \( (RP(a)) \cap (RP(b)) \subseteq P \).

Let \( (RP(a) \cap RP(b)) \subseteq P \) for some \( \Phi \subseteq A \) but \( RP(a) \not\in P, RP(b) \not\in P \). Then \( 1 - RP(a), 1 - RP(b) \in P \) by Remark 2.3. This together with \( RP(a) \cap RP(b) \subseteq P \) implies \( RP(b) \in P \), a contradiction. Hence either \( RP(a) \) or \( RP(b) \not\in P \), i.e., \( (RP(a)) \cap (RP(b)) \subseteq P \). It follows from Theorem 3.1 that \( (RP(a)) \cap (RP(b)) = (RP(a) \cap RP(b)) \) as both the strict ideals have the same hull.

Q.E.D.
Using Theorem 3.1 we readily have

**Remark 3.1:** Putting $J = (RP(x))$ and using $\text{Ann}(x) = \text{Ann}(RP(x))$ we can write

$$\text{Ann}(x) = \text{Ann}(RP(x)) = K(\Sigma(A) - H(RP(x)))$$.

This, after some calculations implies

$$H(K(\{P\}_{x})) = \{P\}_{x}$$.

Thus the sets of the form $\{P\}_{x}$ are both open and closed in the hull-kernel topology $\tau^h$, on $\Sigma(A)$.

**Remark 3.2:** As $H(x) = \Sigma(A) - \{P\}_{x}$, by closedness of $\{P\}_{x}$ it follows that $H(x)$ is open in $(\Sigma(A), \tau^h)$. i.e. $\tau^h$ is finer than $\tau^d$.

**Remark 3.3:** As $\{P\}_{x} = H(\text{Ann}(x)) = H(eA)$ where $e$ is a projection such that $\text{Ann}(x) = eA$. Since all the projections of $A$ are central, we have $eA = (e)$. Thus $\{P\}_{x} = H(e)$. But $H(e)$ is open in the dual hull-kernel topology $\tau^d$ on $\Sigma(A)$. Thus in view of Remark 3.2, we have $\tau^h = \tau^d$. 
**Remark 3.4:** For any \( P \in \Sigma(A) \), by Remark 2.3 we have \( x \notin P \) if and only if \( 1 - \text{RP}(x) \notin P \). Hence we conclude that \( x \notin P \) if and only if \( \text{Ann}(x) \notin P \).

**Remark 3.5:** Since \( \{ P \} \subseteq \{ P \} \) implies \( K(\{ P \}) \subseteq K(\{ P \}) \), by Remark 3.4, it follows that \( \text{Ann}(y) \subseteq \text{Ann}(x) \). Hence \( \text{Ann}(\text{Ann}(x)) \subseteq \text{Ann}(\text{Ann}(y)) \). Retracing the steps we conclude that \( \{ P \} \subseteq \{ P \} \) if and only if \( \text{Ann}(\text{Ann}(x)) \subseteq \text{Ann}(\text{Ann}(y)) \).

Let us note some properties of the topological space \( \Sigma(A) \).

**Theorem 3.2:** The hull-kernel topology on \( \Sigma(A) \) is Hausdorff.

**Proof:** Let \( X, Y \) be distinct prime strict ideals of \( A \). Let \( a \in X - Y \). Clearly \( Y \in \{ P \} = \{ P \} \) and \( X \in \{ P \} \).

In view of Remark 2.3, it follows that \( \{ P \} \) and \( \{ P \} \) are disjoint.

Q.E.D.

Next, we have
THEOREM 3.3: The hull-kernel topology \( \mathcal{T}^h \) on \( \Sigma(A) \) is a compact space.

Proof: Let \( \Sigma(A) = \bigcup_{i \in I} \{ P \}_i X_i \) be an open cover of \( \Sigma(A) \) where \( X_i \) are strict ideals of \( A \). Then \( \Sigma(A) = \bigcup_{i \in I} \{ P \}_i X_i \) (by using B.8), where \( \Sigma X_i \) is the ideal sum of the strict ideals \( X_i \). But then \( \Sigma X_i = A \) and so \( 1 = \sum_{i \in I} a_i, a_i \in X_i \) if only finitely many \( a_i \neq 0 \). We may denote the set of \( i \in I \) for which \( a_i \neq 0 \), say by \( F \). Then \( 1 = \sum_{i \in F} a_i, a_i \in X_i \) and so \( A = \Sigma X_i \). Hence \( \Sigma(A) = \bigcup_{i \in F} \{ P \}_i X_i \) where \( F \) is a finite set.

Q.E.D.

We saw that \( \Sigma(A) \) is Hausdorff, compact and \( \mathcal{T}^h = \mathcal{T}^d \).

In fact, we obtain several equivalent formulations of compactness of \( \Sigma(A) \) in the next result.

THEOREM 3.4: The following statements are equivalent in a Rickart \( * \)-ring \( A \) without nonzero nilpotent elements.
1) \((\Sigma(A), \mathcal{T}^h)\) is compact.

2) Finite unions of \(\{ \{ P \} : x \in A \}\) form a Boolean lattice.

3) For any \(x \in A\), there exist \(t_i \in A\), \(1 \leq i \leq n\), such that \(t_i \notin \text{Ann}(x)\) and

\[
\text{Ann}(x) \cap \bigcap_{i=1}^{n} \text{Ann}(t_i) = (0).
\]

4) For any \(x \in A\), there exist \(t_i \in A\), \(1 \leq i \leq n\) such that

\[
\text{Ann}(\text{Ann}(x)) = \bigcap_{i=1}^{n} \text{Ann}(t_i).
\]

5) \(\mathcal{T}^h = \mathcal{T}^d\).

Proof: (1) implies (2): Since \(H(x)\) is closed in \((\Sigma(A), \mathcal{T}^h)\) and \(H(x) \cap H(\text{Ann}(x)) = \emptyset\), by compactness of \(\Sigma(A)\), we have

\[
H(x) \cap \{ H(t_i) : t_i \notin \text{Ann}(x), i=1, \ldots, n \} = \emptyset.
\]

On taking complements in \(\Sigma(A)\) we have

\[
\Sigma(A) = \{ \{ P \} : x \in A \} \cup \{ \{ P \} : t_i \} \cup \cdots \cup \{ \{ P \} : t_n \}.
\]

As \(t_i \notin \text{Ann}(x), 1 \leq i \leq n\), we conclude that

\[
\{ \{ P \} : x \} \cap \bigcup_{i=1}^{n} \{ \{ P \} : t_i \} = \emptyset.
\]
Thus \( \bigcup_{i=1}^{n} \{ P \}_{t_{1}} \) is a complement of \( \{ P \}_{x} \). Since

\[ \{ P \}_{x} : x \in A \]

is a semilattice, we conclude that finite unions of subsets of \( \{ P \}_{x} : x \in A \) form a Boolean lattice; see Varlet [11].

(2) implies (3): Let \( \bigcup_{i=1}^{n} \{ P \}_{t_{1}} \) be the complement of \( \{ P \}_{x} \). Clearly \( t_{1} \in \text{Ann}(x), 1 \leq i \leq n \). If \( a \in \text{Ann}(x) \bigcap \bigcap_{i=1}^{n} \text{Ann}(t_{1}) \), \( a \neq 0 \), then using Remark 3.1, we conclude that \( a \in \text{K}( \{ P \}_{x} \bigcup \bigcup_{i=1}^{n} \{ P \}_{t_{1}} \) . Then Remark 2.4 implies \( a = 0 \). Thus (3) holds.

(3) implies (4): Clearly \( x t_{i} = 0, 1 \leq i \leq n \) and so \( x \in \text{Ann}(t_{1}), 1 \leq i \leq n \). This further implies \( \text{Ann}(\text{Ann}(x)) \subseteq \bigcap_{i=1}^{n} \text{Ann}(t_{1}) \). Let \( a \in \bigcap_{i=1}^{n} \text{Ann}(t_{1}) \) then \( a t_{i} = 0, 1 \leq i \leq n \). Let \( y \in \text{Ann}(x) \), then

\( x y = 0 \) and so \( x y a = 0 \). Also \( y a t_{i} = 0 \). By (3) and using \( A \) is a semiprime ring we conclude \( y a = 0 \) i.e. \( a \in \text{Ann}(\text{Ann}(x)) \).

Hence \( \text{Ann}(\text{Ann}(x)) = \bigcap_{i=1}^{n} \text{Ann}(t_{1}) \).

(4) implies (5): In view of Remark 3.2 we need only show

\( \mathcal{J}^{h} \subseteq \mathcal{J}^{d} \). For any \( x \in A \), there exist \( t_{i} \in A \), \( 1 \leq i \leq n \) such that \( \text{Ann}(\text{Ann}(x)) = \bigcap_{i=1}^{n} \text{Ann}(t_{i}) \). Since \( \text{Ann}(x) = (1-\text{RP}(x)) \) and

\( \text{Ann}(\text{Ann}(x)) = \text{Ann}(1-\text{RP}(x)) = (\text{RP}(x)) \),
where \((RP(x))\) is the strict ideal generated by \(RP(x)\).

We conclude by using B.12 that \(H((RP(x))) = \bigcup_{i=1}^{n} H(Ann(t_i))\).

Hence Remark 3.1 implies

\[
H(RP(x)) = \bigcup_{i=1}^{n} H(\mathcal{P} \{ t_i \}) = \bigcup_{i=1}^{n} \mathcal{P} \{ t_i \}.
\]

As \(H(x) = H(RP(x))\) we conclude

\[
H(x) = \bigcup_{i=1}^{n} \mathcal{P} \{ t_i \}.
\]

Taking complements in \(\Sigma(A)\), we get \(\{ \mathcal{P} \} = \bigcap_{i=1}^{n} H(t_i)\).

Thus \(\{ \mathcal{P} \}_x\) is open in \(\mathcal{J}^d\). Hence \(\mathcal{J}^d = \mathcal{J}^h\).

(5) implies (1): We may take \(\{ H(x): x \in A \}\) as a basis for the closed sets of the hull-kernel topology \(\mathcal{J}^h\) on \(\Sigma(A)\).

Let \(\{ H(x): x \in B \}\) be a family of closed sets with finite intersection property and suppose \(\bigcap_{x \in B} H(x) = \emptyset\). This implies \(\bigcap_{x \in B} H((RP(x))) = \emptyset\) and using B.13, we have

\[
H(\Sigma (RP(x))) = \emptyset. \text{ Then B.10 implies } \Sigma (RP(x)) = A.
\]

Consequently, \(1 = \Sigma_{t_i \in J} t_i \in (RP(x_i)), x_i \in B\), and only finitely many \(t_i \neq 0\). Thus we may assume \(J\) is finite. This implies \(H(A) = \bigcap_{x \in J} H(RP(x_i)) = \bigcap_{x \in J} H(x_i) = \bigcap_{x \in K} H(x_i)\) where \(K\)
is a finite subset of \( B \). This contradicts the finite intersection property of \( \{ H(x) : x \in B \} \) and implies the compactness of \((\Sigma(A), \mathcal{T}^h)\).

Q.E.D.

As topologies are completely determined by any of their subbases we can show that each of the five properties mentioned in Theorem 3.4 is equivalent to each of the following two properties.

(6) : \( \{ H(x) : x \in A \} \) is a subbasis for the open sets of \((\Sigma(A), \mathcal{T}^h)\).

(7) : \( \{ \{ x \} : x \notin A \} \) is a subbasis for the open sets of \((\Sigma(A), \mathcal{T}^d)\).

§ 4. Comparability:

Gooderal [4] exploits the axiom of comparability in the context of regular rings and establishes some nontrivial results. For example, he shows that every regular ring satisfying the comparability axiom is prime (i.e. \((0)\) is a prime ideal). In fact, for such rings, Goodearl determines a nice relationship between prime ideals and their central idempotents.
We have already discussed Rickart *-rings with the
generalized comparability for elements and for projections as
well in Chapter-I. In this section we relate the general
comparability (GC) of projections to the prime spectrum,
minimal spectrum of a Rickart *-ring A. In fact, we succeed
in showing that every Rickart *-ring A with (dominated)
comparability condition is prime in the sense that every strict
ideal of A is prime strict. Incidentally, we also obtain an
interesting relationship between prime strict ideals and central
projections.

In this section, A is a Rickart *-ring which need not
be semiprime. As usual Σ(A) and π(A) denote the prime
spectrum and the minimal spectrum of A. B(A) denotes the
Boolean algebra of central projections of A. We assume all
the known properties of Σ(B(A)), the prime spectrum of B(A)
without proof.

We begin with

**DEFINITION 4.1:** Projections e, f ∈ A are said to be (dominated)
comparable if either e ≤ f or f ≤ e.

In the next proposition we give a relationship
between prime strict ideals and comparability of projections.
PROPOSITION 4.1: If any two projections of A are (dominated) comparable, then the set of strict ideals of A is linearly ordered and every strict ideal is prime strict.

Proof: Let I, J be two incomparable strict ideals. Let \( x \notin I \leq J, y \notin J \leq I \). Then \( RP(x) \subseteq I, RP(y) \subseteq J \). By comparability, either \( RP(x) \leq RP(y) \) or \( RP(y) \leq RP(x) \). In the first case \( x \notin J \) and in the second \( y \notin I \), a contradiction in either case.

Let \( P \) be a strict ideal which is not prime. Let I, J be strict ideals such that \( IJ \leq P \) but \( I \nsubseteq P, J \nsubseteq P \). By the earlier part, either \( I \leq J \) or \( J \leq P \) and \( P \leq I, P \leq J \). Let \( I \leq J \). Then \( P \leq I = I^2 \leq IJ \) a contradiction to \( I \nsubseteq P \). Hence \( P \) is prime strict.

Q.E.D.

This Proposition would be used repeatedly. In particular, it would or its method of proof would help us to obtain a nice characterization of prime strict ideals.

It can be easily shown that if \( A \) is a Rickart *-ring and \( I \) is a strict ideal then \( A/I \) is a Rickart *-ring.

In fact, as every strict ideal is restricted (i.e., generated by its projections) we have the following
**Lemma 4.1** (Berberian, [1, p.143]):

If $A$ has (GC) then every central projection in $A/I$ has the form $h+I$ where $h$ is a central projection in $A$.

This proposition shows that the canonical map $B(A) \to B(A/I)$ is onto.

In the next proposition we give some properties of Baer $^*$-rings. For the concept of central cover that is needed in the following Lemma, please refer to the first section of Chapter-I.

**Lemma 4.2** (Berberian [1, p.36]): If $A$ is a Baer $^*$-ring and $B$ is a Baer $^*$-subring of $A$. If $xbB$, then $C_B(x) \subseteq C_A(x)$ where $C_B(x)$ is the central cover of $x$ in $B$ and $C_A(x)$ is the central cover of $x$ in $A$.

We now connect (GC) and strict ideals.

**Proposition 4.2:** Let $A$ be a Baer $^*$-ring with the generalized comparability (GC). $J$ be a strict ideal of $A$. Let $S$ be a Baer $^*$-subring of $A$, such that $J \subseteq S$. Then $S$ has (GC) implies the following statements.
1) \( S/J \) satisfies the generalized comparability

2) The natural map \( B(S) \to B(S/J) \) is surjective.

3) Whenever \( x \in S \) and \( y \in J \) such that \( xAy = 0 \) there exists \( e \in B(S) \) for which \( ex = x \) and \( ey = 0 \).

**Proof:** If \( S \) satisfies (GC) then (1) holds and (2) holds by Lemma 4.1.

Now let \( x \in S \) and \( y \in J \) such that \( xAy = 0 \). By Lemma 2.1 of Chapter-I \( C_A(x)C_A(y) = 0 \). Applying (GC) to the projections \( C_S(x) \) and \( C_S(y) \), there exists a central projection \( e \in B(S) \) such that

\[
eC_S(x) \leq eC_S(y) \text{ and } \]

\[
(1-e)C_S(y) \leq (1-e)C_S(x).
\]

Clearly \( C_S(x) \leq C_A(x) \) and \( C_S(y) \leq C_A(y) \) by Lemma 4.2. Hence \( C_S(x)C_S(y) = 0 \).

Now it follows that \( eC_S(x) = 0 \) and \( (1-e)C_S(y) = 0 \). As a consequence we have \( ex = 0 \) and \( ey = y \). Then the projection \( f = 1-e \) has the required properties.

Q.E.D.
As was mentioned earlier we obtain three equivalent formulations of prime strict ideals of a Rickart \(*\)-ring. Interestingly, we show that a prime strict ideal \(P\) of \(A\) linearly orders the set of strict ideals of \(A/P\).

**Proposition 4.3**: Let \(A\) be a Rickart \(*\)-ring with (GC). Let \(P\) be a strict ideal of \(A\). Then the following statements are equivalent:

i) \(P\) is a prime strict ideal of \(A\).

ii) For all \(e \in B(A)\), either \(eP\) or \(1-eP\)

iii) \(P \cap B(A)\) is a maximal ideal of \(B(A)\).

iv) The set of strict ideals of \(A/P\) is linearly ordered.

Proof: (i) implies (ii) implies (iii) is clear.

(iii) implies (iv): \(A\) satisfies (GC) implies \(A/P\) satisfies (dominated)comparability and hence by Proposition 4.1, (iv) holds.

(iv) implies (i): This can be proved on the lines of the proof of Proposition 4.1.

\(\mathbf{Q.E.D.}\)
The following consequence is immediate.

**COROLLARY 4.1**: Let \( P, Q \) be strict ideals of a Rickart *-ring \( A \) with (GC) such that \( P \subset Q \). If \( P \) is prime strict then so is \( Q \).

The next consequence, however, needs a little elaboration.

**COROLLARY 4.2**: Let \( A \) be a Rickart *-ring and \( M \) be a maximal ideal of \( B(A) \). Then \( MA \) is a minimal prime strict ideal of \( A \).

**Proof**: Clearly \( MA \) is an ideal of \( A \). To show \( MA \) is proper and strict. Let \( e_1x_1 + \ldots + e_nx_n = 1 \) for some \( e_i \in M, x_i \in A \).
Let \( e = e_1 \vee \ldots \vee e_n \). Since \( ee_i = e_i \), it follows that \( e=1 \in M \), a contradiction. Hence \( MA \) is proper. Let \( x \in MA \). Then

\[
x = \sum_{i=1}^{k} f_i y_i, \quad f_i \in M, \quad y_i \in A.
\]

Hence \( RP(x) \leq \bigvee_{i=1}^{k} RP(f_i y_i) \leq \bigvee_{i=1}^{k} f_i \) (as \( f_i y_i = y_i f_i \)).

As \( \bigvee_{i=1}^{k} f_i \in M \), it follows that \( RP(x) \in MA \) i.e. \( MA \) is strict.
Clearly \( M \subseteq MA \cap B(A) \) and so by maximality of \( M \) we have \( MA \cap B(A) = M \). Hence by Proposition 4.3, \( MA \) is a prime strict ideal of \( A \).

Suppose \( P \subseteq MA \), where \( P \) is a prime strict ideal. Then for any \( \epsilon \in M \), we have \( 1 - \epsilon \in MA \) and so \( 1 - \epsilon \in P \) which implies \( \epsilon \in P \) i.e. \( M \subseteq P \). Hence \( MA \) is a minimal prime strict ideal.

Q.E.D.

We see that the presence of \((GC)\) on a Rickart *-ring deeply affects the minimal spectrum. In fact, the minimal spectrum of \( A \) and the prime spectrum, \( \Sigma(B(A)) \), of the Boolean algebra, \( B(A) \), of \( A \) exhibit the same structure in the presence of \((GC)\). All these considerations very naturally lead us to several characterizations of the compactness of the minimal spectrum. All these considerations are digressed in the next theorem that happens to be one of the important results of Chapter-IV.

**Theorem 4.1:** Let \( A \) be a Rickart *-ring with \((GC)\). Then the map \( \phi: \pi(A) \to \Sigma(B(A)) \) defined by \( \phi(P) = P \cap B(A) \) defines a continuous bijection. The inverse bijection is given by \( \phi^{-1}(M) = MA \). Further, the following conditions are equivalent.
i) $\phi$ is a homeomorphism

ii) $\pi(A)$ is homeomorphic to $\Sigma(B(A))$

iii) $\pi(A)$ is compact

iv) For all $x \in A$, the set $\{ e \in B(A) : ex = 0 \}$ is a principal ideal of $B(A)$.

**Proof:** By Corollary 4.2, the map $\psi : \Sigma(B(A)) \rightarrow \pi(A)$ is well defined. Also the map $\phi : \pi(A) \rightarrow \Sigma(B(A))$ given by $\phi(P) = P \cap B(A)$ is also well defined. Now $P \in \pi(A)$ implies $\phi(P) \in \Sigma(B(A))$ and so $\psi(\phi(P)) = \psi(P \cap B(A)) \in \pi(A)$ and $\psi(\phi(P)) \subseteq P$. Hence by minimality of $P$, $\psi(\phi(P)) = P$.

For any $M \in \Sigma(B(A))$ we have $\phi(\psi(M)) \in \Sigma(B(A))$ and $\phi(\psi(M)) = \phi(MA) = MA \cap B(A) \supseteq M$. Thus $\phi(\psi(M)) = M$ by maximality of $M$. Hence $\phi$ and $\psi$ are inverse bijections.

If $X$ is any open subset of $\Sigma(B(A))$ then

$X = \{ M \in \Sigma(B(A)) : Y \subseteq M \}$ for some $Y \subseteq B(A)$. Then $\phi^{-1}(X) = \{ P \in \pi(A) : Y \subseteq P \}$ is an open set in $\pi(A)$. Hence $\phi$ is continuous.

(i) **implies** (ii) **implies** (iii) is clear.
(iii) implies (i): Since \( \phi \) is a continuous bijection we need to show that \( \phi \) maps open sets onto open sets.

Let \( D = \{ P \}_{x} \) be an open set in \( \pi(A) \). We have
\[
\phi(D) = \{ M \in \Sigma(B(A)) : M = P \cap B(A), P \in D \}.
\]
Then \( 1 - C(x) \in M \) for each \( M \in \phi(A) \). Let \( Q = M \) for some \( M \in \Sigma(B(A)) \) be such that \( 1 - C(x) \in M \). Therefore \( C(x) \notin M \) and \( 1 - C(x) \notin Q \). If \( x \in Q \) then
\[
x = e_1 x_1 + \ldots + e_n x_n
\]
for some \( e_i \in M \), \( x_i \in A \). Let \( e = e_1 \ve_2 \ldots \ve_n \) then \( e \notin M \) and \( e \notin x \). Hence \( C(x) \notin e \) and so \( 1 - e \leq 1 - C(x) \).
Thus \( 1 - e \notin Q \). But then \( e, 1 - e \notin Q \), a contradiction. Hence \( x \notin Q \), i.e. \( Q \notin \{ P \}_{x} \) and \( Q \notin \{ P \}_{C(x)} \). Thus \( \phi(D) \) if and only if \( C(x) \notin M \). Thus \( \phi(D) = \{ M \in \Sigma(B(A)) : C(x) \notin M \} \) is an open set in \( \Sigma(B(A)) \). Hence \( \phi \) is a homeomorphism.

(i) implies (iv): For \( x \in A \), define \( J = \{ e \in B(A) : e = 0 \} \).
Let \( X = \{ P \in \Sigma(A) : x \in P \} \) and \( Y = \{ P \in \Sigma(A) : J \subseteq P \} \). Clearly
\[
X \cup Y = \pi(A).
\]
For if, \( P \in \Sigma(A) \) is such that \( x \notin P \), then \( e = 0 \) implies \( (1 - e)x = x \) and so \( C(x) \leq 1 - e \). Thus \( eC(x) = 0 \) and \( (e)(C(x)) = (0) \).
This implies \( (e) \subseteq P \). Hence \( J \subseteq P \) i.e. \( P \in Y \).

If there exists \( P \in X \cap Y \) then \( x \in P = \gamma(\phi(P)) \). Hence
\[
x = e_1 x_1 + \ldots + e_n x_n
\]
for some \( e_i \in \phi(P) \), \( x_i \in A \). Setting \( e = e_1 \ve_2 \ldots \ve_n \) we have \( e \in \phi(P) \subseteq P \) and \( (1 - e)x = 0 \). But then \( 1 - e \in J \subseteq P \) which is impossible.
Thus $X$ and $Y$ are disjoint.

Hence $X$ and $Y$ are clopen sets of $\pi(A)$.

Since $\varphi$ is a homeomorphism, $\varphi(Y)$ must be a clopen set of $\Sigma(B(A))$. Hence there exists $f \in B(A)$ for which
\[
\varphi(Y) = \{ M \in \Sigma(B(A)) : f M \}.
\]
On the other hand,
\[
\varphi(Y) = \{ M \in \Sigma(B(A)) : J \subseteq M \}.
\]
Hence we have
\[
J = \bigcap \varphi(Y) = \{ e \in B(A) : e \leq f \}.
\]

(iv) implies (i): We have to show that $\varphi(X)$ is closed for any closed set $X$ in $\pi(A)$. Now,
\[
X = \{ P \in \pi(A) : Y \subseteq P \} = \bigcap_{Y \subseteq A} \{ P \in \pi(A) : y \in P \} \text{ for some } Y \subseteq A.
\]
Hence it is sufficient to show that whenever, $x \in A$ and $W = \{ P \in \pi(A) : x \in P \}$ then $\varphi(W)$ is closed in $\Sigma(B(A))$.

By hypothesis,
\[
\{ e \in B(A) : e x = 0 \} = \{ e \in B(A) : e \leq f \} \text{ for some } f \in B(A).
\]

Let $1 - f \in P$ where $P \ni (x \in A)$. Then $x \in P$ and so $P \ni x$. Now, for $P \ni x$ we have $x \in \varphi(P)$. This implies $x = e_1 x_1 + \ldots + e_n x_n$ for some $e_1 \in \varphi(P)$, $x_1 \in A$. Let $e = e_1 \vee \ldots \vee e_n$. Then $e \in \varphi(P) \subseteq P$ and we have $(1 - e)x = 0$. This further implies that $1 - e \leq f$ i.e. $1 - f \leq e$ and so $1 - f \in \varphi(P) \subseteq P$. 


Thus $x \in P$ if and only if $1 - x \in P$. Hence $W = \{P \in (A) : 1 - x \in P\}$. Therefore, $\phi(W) = \{M \in (B(A)) : 1 - x \in M\}$ is closed in $\Sigma(B(A))$.

Q.E.D.

It will be shown that the central projections that annihilate elements of a Rickart $*$-ring $A$ form a principal ideal of $B(A)$; and it follows very easily. It is also to be noted that the presence of (GC) is not needed.

**PROPOSITION 4.4:** In a Rickart $*$-ring $A$, $\{e \in B(A) : e = 0\}$ is a principal ideal of $B(A)$.

**Proof:** The following chain of equivalences proves the result.

$e = 0 \iff eR(x) = 0 \iff R(x) \leq 1 - e$

$\iff C(x) = C(R(x)) \leq 1 - e$

$\iff e \leq 1 - C(x)$.

Q.E.D.

As is to be expected from our experience of commutative rings, we note pleasantly that maximality of strict ideals implies prime strictness.
PROPOSITION 4.5: Every maximal strict ideal of $A$ is prime strict.

Proof: Let $M$ be a maximal strict ideal of $A$. Let $I,J$ be strict ideals such that $IJ \subseteq M$ but $I \nsubseteq M$, $J \nsubseteq M$. By maximality of $M$ we have $M+I = M+J = A$. This implies $m+i = n+j = 1$ for some $m,n \in M$, $i \in I$, $j \in J$. But then $(m+i)(n+j) = 1 \in M$, a contradiction. 

Q.E.D.

We again resort back to general comparability. It is established in the following result that Rickart *-rings are simultaneously both pm-rings and normal rings. By a pm-ring we mean a ring with unity in which every prime ideal is contained in a unique maximal ideal; see DeMarco and Orsatti [3] and the concept of normal rings was touched upon in Chapter-III.

PROPOSITION 4.6: Let $A$ be a Rickart *-ring with (GC). Then $P$ is contained in a unique maximal strict ideal of $A$ and $P$ contains a unique minimal prime strict ideal of $A$.

Proof: Since the set of strict ideals of $A/P$ is linearly ordered by Proposition 4.3, it follows that $A/P$ has a unique maximal strict ideal $M/P$ and consequently, $M$ is the unique maximal strict ideal of $A$ which contains $P$. 
As \( P \cap B(A) \) is a maximal ideal of \( B(A) \), by Corollary-4.2, \( Q = [P \cap B(A)]A \) is a minimal prime strict ideal of \( A \). Obviously, \( Q \leq P \). If \( K \leq P \) is any minimal prime strict ideal, then
\( K \cap B(A) = P \cap B(A) \), by minimality of \( K \). Hence \( Q \) is unique.

Q.E.D.

So far our study in this Chapter pertained to general projections. Let us pick up some special projections. We have

**Definition 4.2:** A projection \( e \) in a \( * \)-ring is said to be faithful if \( C(e) = 1 \).

**Definition 4.3:** An ideal \( I \) of a ring \( A \) is called large if for any nonzero ideal \( J \) of \( A \), \( I \cap J \neq (0) \).

The comparability condition is not needed in the final result of this section. We pleasantly characterize faithful projections in terms of largeness of an ideal. In proving this we use the following result of Berberian \([1, p.35]\).

**Lemma 4.3:** If \( A \) is a Baer \( * \)-ring and \( J \) is a right ideal in \( A \), then \( R(J) = hA \), where \( h \) is a central projection.
PROPOSITION 4.7: Let e be a projection in a Baer *-ring A. Then the following statements are equivalent.

i) e is faithful.

ii) (e) is a large ideal of A.

iii) There is no x∈A such that xe=0.

iv) There is no x∈A such that ex=0.

Proof: (i) implies (ii): Suppose (e) ∩ J = (0) for some ideal J of A. Then (e)J ⊆ (e) ∩ J and J(e) ⊆ (e) ∩ J imply that (e) ⊆ R(J) where R(J) is the right annihilator of J in A.

By the Lemma 4.3, there exists a central projection h such that R(J) = hA. Then (e) ⊆ hA implies e ≤ h. Hence C(e) ≤ h. As C(e)=1 we conclude h = 1, i.e. R(J)=A. Thus J=(0).

(ii) implies (i): Now suppose that (e) is large. Consider the ideal (1-C(e)). Clearly (e) ∩ (1-C(e)) = (0) implies 1-C(e)=0. Hence e is faithful.

(i) implies (iii)

xa=0 ⇒ C(x)C(e)=0 ⇒ C(x) = 0 ⇒ x = 0.

(iii) implies (i): As 1-C(e) is a central projection, we have (1-C(e))Ae = 0. Hence 1-C(e)=0 i.e. e is faithful.
(i) implies (iv): \( eAx = 0 \implies C(x)C(x) = 0 \implies C(x) = 0 \implies x = 0. \)

(iv) implies (i): \( eA(1 - C(x)) = 0 \implies 1 - C(x) = 0 \implies C(x) = 1. \)

Q.E.D.

\[ \int \text{5: Functorial Representation:} \]

This section is the smallest in the Thesis. We exhibit here a functorial correspondence between the category of Rickart *-rings and their prime spectra. (The ring is not assumed to be semiprime). In that direction we have

**Proposition 5.1:** Let A and B be Rickart *-rings. Let \( f : A \to B \) be a *-isomorphism, and \( P \) be a prime strict ideal of \( B \). Then \( f^{-1}(P) \) is a prime strict ideal of \( A \).

**Proof:** For a *-isomorphism \( f : A \to B \), it can be easily shown that \( f(RP(x)) = RP(f(x)) \). Clearly \( f^{-1}(P) \) is an ideal of \( A \) whenever, \( P \) is an ideal of \( B \).

As \( P \) is a strict ideal, \( f(x) \in P \) implies \( RP(f(x)) = f(RP(x)) \in P \) and so whenever, \( xf^{-1}(P), RP(x) \in f^{-1}(P) \).

Thus \( f^{-1}(P) \) is a strict ideal.
Let $I, J$ be strict ideals of $A$ such that $IJ \subseteq \overline{f(P)}$ where $f$ is a prime strict ideal of $B$. Then $f(IJ) \subseteq P$, i.e. $f(I)f(J) \subseteq P$. Hence either $I \subseteq \overline{f(P)}$ or $J \subseteq \overline{f(P)}$. Thus $\overline{f(P)}$ is a prime strict ideal of $A$.

Q.E.D.

Thus we see that a $*$-isomorphism pulls back prime strict ideals into prime strict ideals. This is now used to produce a map from the minimal prime spectrum of a Rickart $*$-ring $A$ onto the minimal prime spectrum of another Rickart $*$-ring $B$, that is continuous.

**PROPOSITION 5.2:** Let $A$ and $B$ be Rickart $*$-rings. Let $f:A \to B$ be a $*$-isomorphism. Then the map $f^*: \pi(B) \to \pi(A)$ defined by $f^*(M) = \overline{f^{-1}(M)}$ is a continuous map.

**Proof:** Clearly $f^*$ is well defined.

We have

$$Q \ni f^{-1}(\{ M \}) \leftrightarrow x \ni f^*(Q) \ni \{ M \} \ni x$$

$$\leftrightarrow x \ni f^*(Q) \leftrightarrow x \ni \overline{f^{-1}(Q)} \leftrightarrow f(x) \ni Q \leftrightarrow Q \ni \{ M \} f(x).$$

Thus $f^*$ pulls back open sets onto open sets and hence it is continuous.

Q.E.D.
This establishes the stipulated functorial correspondence between the category of Rickart *-rings and their prime spectra.

\[ \mathcal{S} 6 : \text{Sheaf Representation} : \]

In the case of a noncommutative semiprime ring \( A \), Koh [6] gave a representation of \( A \) in terms of the ring of global sections of a sheaf over the prime ideal space of \( A \). It is highly known that other algebraic structures can be represented by sheaves; see e.g. Dauns and Hofmann [2], Oystaeyen [10], Keimel [7] etc. In fact, Keimel [7] formulated sheaf representation for lattice-ordered rings.

In this section we obtain a representation of a Rickart *-ring in terms of the sections of a sheaf over a dense subspace \( X \) of \( \Sigma(A) \).

Let \( A \) be a Rickart *-ring, \( X \) be a subspace of \( \Sigma(A) \), \( Y \) be any topological space and \( \alpha : X \to Y \) be a continuous map. For an open subset \( B \) of \( Y \), let \( O_B \) denote the set

\[ O_B = \cap \alpha^{-1}(B) = \cap \{ P \in \Sigma(A) : \alpha(P) \in B \} . \]
As an intersection of a family of strict ideals, \( Q_B \) is also a strict ideal. For each \( y \in Y \) we define

\[
Q_y = \bigcup \{ Q_B : B \text{ is open neighbourhood of } y \}.
\]

If \( V \subset B \) then \( \alpha^{-1}(V) \subset \alpha^{-1}(B) \) and so \( Q_B \subset Q_y \). Hence the collection of strict ideals \( Q_B \) where \( B \) runs through all open neighbourhoods of \( y \) is upwards directed and so \( Q_y \), the union of this collection, is also a strict ideal. This strict ideal \( Q_y \) is called the \( \alpha \)-germinal ideal associated with \( y \).

For each \( a \in A \), the coset \( \hat{a}(y) = a + Q_y \) is called the \( \alpha \)-germ of \( a \) in \( y \). A subspace \( X \) of \( E(A) \) is called full if \( \left\{ P \cap X : P \neq X \right\} \neq X \) for each strict ideal \( I \) of \( A \) i.e. every proper strict ideal \( I \) of \( A \) is contained in at least one \( P \cap X \).

With these notations we prove

**Proposition 6.1**: For \( a \in A \), the sets of the form

\[
\theta(a) = \{ y \in Y : a \notin Q_y \}
\]

are open in \( Y \).

**Proof**: If \( a \notin Q_y \) then \( a \notin Q_B \) for some open neighbourhood \( B \) of \( y \) and hence \( a \notin Q_z \) for all \( z \in B \) i.e. \( \hat{a}(z) = 0 \) for all \( z \in B \).
Thus,

$$\theta(a) = \{ y \in Y : a \notin O_y \} = \{ y \in Y : \hat{a}(y) = 0 \}$$

$$= \{ y \in Y : \hat{a}(B) = 0 \text{ where } B \text{ is open neighbourhood of } y \}$$

$$= U \{ B : \hat{a}(B) = 0 \}.$$ 

Therefore, $\theta(a)$ is open in $Y$.

Q.E.D.

Let $X$ be a subspace of $\Sigma(A)$ and $Y$ be any topological space that gives rise to $\alpha$-germinal ideals $O_y$ with $y \in Y$. We now show that the intersection of $O_y$ as $y$ runs over $Y$ is equal to the intersection of prime ideals in $X$.

**PROPOSITION 6.2**. For a subspace $X$ of $\Sigma(A)$

$$\bigcap \{ O_y : y \in Y \} = \bigcap \{ P : P \in X \}.$$ 

**Proof**: We have $O_B = \bigcap \alpha^{-1}(B) = \bigcap \{ P \in X : \alpha(P) \in B \}$, and so $O_B \subseteq P$ for every open neighbourhood $B$ of $y \in Y$ and $\alpha(P) = y$. Hence by definition of $O_y$, it follows that
\[ O_y = \bigcup \{ O_B : B \text{ is open neighbourhood of } y \} \subseteq P. \]

Thus \[ \bigcap \{ O_y : y \in B \} \subseteq \bigcap \{ P : \alpha(P) = y \} \]

\[ = \bigcap \{ P : \alpha(P) = B \} = \bigcap \alpha^{-1}(B) = O_B. \]

Conversely, \( O_B \subseteq O_y \) for each \( y \in B \), by definition of \( O_y \). Hence we have

\[ O_B = \bigcap \{ O_y : y \in B \} = \bigcap \{ P : \alpha(P) \in B \}. \]

In particular, if we take \( B = Y \) then

\[ \bigcap \{ O_y : y \in B \} = \bigcap \{ P : P \in X \}. \]

Q.E.D.

If we take \( X = Y = \Sigma(A) \) and \( \alpha \) is the identity map on \( \Sigma(A) \) where \( A \) is a semiprime Rickart *-ring, then we have

**Proposition 6.3:** For any \( Q \in \Sigma(A) \), the germinal ideal \( O_Q \) associated with \( Q \) may be expressed as \( O_Q = \bigcup \{ \text{Ann}(a) : a \in Q \} = Q. \)

**Proof:** By definition \( O_Q = \bigcup \{ O_B : B \text{ is open neighbourhood of } Q \} \).
Since the sets of the form \( \{ P_x \} \) \( \forall x \in A \) form an open basis for the topology on \( \Sigma(A) \), we may take the neighbourhood \( B \) of \( Q \) to be of the form \( \{ P_x \} \). Hence \( O_Q = \bigcup \{ O \} \{ P_x \} \). But by definitions, we have \( O_B = \bigcap \overline{\alpha}^{-1}(B) = \bigcap \{ P : P \in B \} \).

Hence \( O_Q = \bigcup \{ \bigcap \{ P \} \} : x \in Q \} \)

\[ = \bigcup \{ \text{Ann}(x) : x \in Q \} \text{ by Remark 3.1} \]

\[ = Q \text{ by Proposition 2.2}. \]

Q.E.D.

Now we shall give some definitions necessary for the sheaf representation.

A sheaf of sets over a set \( X \) is a triple \( F = (G, \beta, X) \) where \( G \) and \( X \) are topological spaces and \( \beta : G \to X \) is a surjective local homeomorphism. A section of the sheaf \( F \) is a continuous function \( f : X \to G \) such that \( \beta f \) is the identity map on \( X \).

We denote by \( \Gamma(X, F) \) the set of all sections of \( F \). The support of each section \( f \) is the set
\[ \text{supp}(f) = \{ x \in X : f(x) \neq 0 \} \]. We denote by \( \Gamma_k(X, F) \) the set of all sections \( f \in F \) with compact support. For each \( x \in X \), we may define a \( * \)-homomorphism

\[ g_x : \Gamma(X, F) \to \beta^{-1}(x) \text{ by } g_x(f) = f(x) \]

for each \( f \in \Gamma(X, F) \). We say that \( F \) is a \textbf{global sheaf} if \( g_x \) is surjective for each \( x \in X \).

Now let \( X \) be a subspace of \( X(A) \), \( Y \) be a topological space and \( \alpha : X \to Y \) be a continuous map. For \( y \in Y \) we have already introduced \( O_y \). Let \( E = \bigcup_{y \in Y} A/O_y \) and \( \beta : E \to Y \) be the canonical projection, \( A/O_y \to y \) and the map \( \hat{\alpha} : Y \to E \) be defined by \( \hat{\alpha}(y) = a + O_y \).

With these notations we prove

\textbf{Theorem 6.1 : General Representation Theorem.}

For a Rickart \( * \)-ring \( A \), the following statements hold:

1) The sets of the type \( \hat{\alpha}(B) \) where \( a \in A \), \( B \) an open set in \( Y \), form an open basis for the finest topology on \( E \).

2) \( F = (E, \beta, Y) \) is a global sheaf of Rickart \( * \)-rings.

3) \( a \to \hat{\alpha} \) is a \( * \)-homomorphism from \( A \) onto a \( * \)-subring \( \hat{A} = \{ \hat{\alpha} : a \in A \} \) of \( \Gamma(Y, F) \).

4) The \( * \)-homomorphic representation (3) is a \( * \)-isomorphism if and only if \( X \) is dense in \( X(A) \).
Proof (i) To prove the assertion, it is sufficient to show that if \( U, V \) are open sets in \( Y \) and \( a, b \in A \) then \( \hat{a}(U) \cap \hat{b}(V) \) is open in \( E \). By Proposition 6.4, we know that the set 
\[
\{ y \in Y : \hat{a}(y) = \hat{b}(y) \} = \{ y \in Y : a = b \} = \Theta(a-b) \text{ is open in } Y.
\]
Let \( W = \Theta(a-b) \cap U \cap V \), then \( W \) is also open in \( Y \). It can be shown that \( \hat{\Theta}(W) = \hat{\Theta}(W) = \hat{\Theta}(U) \cap \hat{\Theta}(V) \). Hence the required assertion.

With respect to this topology \( T \), clearly the functions \( \hat{a} : Y \to E \) are open and continuous. Hence \( T \) is finer than the topology \( T' \) on \( E \) which makes continuous all the functions \( \hat{a} \). (It is well known that \( T' \) is the finest topology). Hence \( T \) is the finest topology on \( E \).

(ii) Since the maps \( \hat{a} : Y \to E \) are open and continuous, \( \hat{a}^{-1} : \hat{a}(Y) \to Y \) is also continuous and \( \hat{a} \cdot \hat{a}^{-1} = \text{Id}_Y \), \( \hat{a}^{-1} \cdot \hat{a} = \text{Id}_{\hat{a}(Y)} \) show that \( \hat{a} \) is a local homeomorphism. Hence \( \beta : E \to Y \) is a global sheaf of Rickart *-rings.

(iii) We have \( \hat{A} = \{ \hat{a} : a \in A \} \). By defining \( \hat{a} + \hat{b} = (a + b)^{\wedge} \), \( (\hat{a})^* = (a^*)^{\wedge} \), \( \hat{a} \cdot \hat{b} = (ab)^{\wedge} \) it can be seen that \( \hat{A} \) is a *-subring of \( \Gamma(Y,E) \). Clearly, the map \( a \to \hat{a} \) is a *-homomorphism.

(iv) Suppose that the *-homomorphism \( f : A \to \hat{A} \) given by \( f(a) = \hat{a} \) is a *-isomorphism. Let \( \text{tf} \cap X \), \( \text{t} \neq 0 \). By Proposition 6.2,
$t \cap \{ Q_y : y \in Y \}$. Hence $\hat{t}(y) = 0$ for each $y \in Y$. But this contradicts the one-to-one property of $f$, proving that $\cap X = \{0\}$.

Conversely, suppose $\cap X = \{0\}$ but $f$ is not a $\ast$-isomorphism. Let $f(a) = f(b)$ for some $a, b \in A$ then $\hat{a} = \hat{b}$ and so $(a - b) y = 0$ for all $y \in Y$ shows that $a - b \in \cap X$. Thus $a = b$.

Hence $f$ is one-to-one. Hence $f$ is a $\ast$-isomorphism.

Q.E.D.

These considerations permit us to introduce

**DEFINITION 6.1**: The sheaf constructed in the preceding theorem will be denoted by $F(A, \alpha)$. It is called the sheaf of $\alpha$-germs associated with $A$.

**REMARK 6.1**: Assuming that $A$ is a semiprime Rickart $\ast$-ring, we can show that $rfP$ for all $ Pf\{ P \}$ if and only if $rs = 0$. If $X$ is a full subspace of $E(A)$ then also $rfP$ for every $ Pf\{ P \} \cap X$ if and only if $rs = 0$.

With the notations as in the General Representation Theorem 6.4, and assuming that $A$ is semiprime Rickart $\ast$-ring (so that all the projections of $A$ are central) we can show that every section of the sheaf $F$ with compact support has the form $\hat{a}$ for some $a \in A$. In that direction we have
THEOREM 6.2: If $X$ is full in $\Sigma(A)$ then every section $f$ of $F$ with compact support has the form $\hat{a}$ for some $a \in A$.

Proof: Let $f: Y \to E$ be a section. For each $y \in \text{supp}(f)$, there exists $r \in A$ such that $\hat{f}(y) = f(y)$. As these two sections coincide at a point $y$ there exists a neighbourhood $B$ of $y$ such that $\hat{f}(B) = f(B)$. Since $\text{supp}(f)$ is compact, there are open sets $B_i (i = 1, 2, \ldots, n)$ in $Y$ such that $\text{supp}(f) \subseteq B_1 \cup \ldots \cup B_n$ and $r_i \in A$ such that $\hat{r}_i(B_i) = f(B_i)$, $i = 1, 2, \ldots, n$. Let $B_0 = Y - \text{supp}(f)$ then it can be shown that $B_0$ is open.

Let $r_0 = 0$. Since $a: X \to Y$ is continuous, $\bar{a}^{-1}(B_i)$ are open in $X$ and $X = \bar{a}^{-1}(B_0) \cup \ldots \cup \bar{a}^{-1}(B_n)$.

Since $\bar{a}^{-1}(B_i)$ are open in $X$, we can write $\bar{a}^{-1}(B_i) = X \cap \bigcup_{i=0}^{n} s_i$ for some $s_i \in A$. Hence $X = \bigcup_{i=0}^{n} (X \cap \{P_i\}) = X \cap \bigcup_{i=0}^{n} s_i$, $s_i \in \bigcup_{i=0}^{n} (\text{RP}(s_i))$.

This shows that the strict ideal $\Sigma \left( \bigcup_{i=0}^{n} (\text{RP}(s_i)) \right)$ is not contained in any $P \in X$. As $X$ is full, this implies $\Sigma \left( \bigcup_{i=0}^{n} (\text{RP}(s_i)) \right) = A$.

We have $1 = \sum_{i=0}^{n} \text{RP}(s_i) t_i$ for some $t_i \in A$. Put $a = \sum_{i=0}^{n} r_i \text{RP}(s_i) t_i$. 


In order to show that \( \hat{a} = f \) it will be sufficient to show that

\[
a - r_i \in \mathcal{O}_{B_i} = \cap \left\{ P \in \Sigma(A) : \alpha(P) \in B_i \right\}
\]

This together with \( \hat{r}_i = f \) will imply \( \hat{a} = \hat{r}_i = f \).

Consider \( Q \in \mathcal{B}_i \cap \mathcal{B}_j \), \( i, j = 0, 1, \ldots, n \). Then

\[
Q \in X \cap \left\{ P \right\}_{i} \cap \left\{ P \right\}_{j}
\]

and so we have

\[
\hat{r}_i(\alpha(Q)) = f(\alpha(Q)) = \hat{r}_j(\alpha(Q))
\]

which implies \( r_i - r_j \in \mathcal{O}_{\alpha(Q)} \subseteq Q \).

As \( \left\{ P \right\}_{i} \cap \left\{ P \right\}_{j} = \left\{ P \right\}_{(RP(s_i))(RP(s_j))} \),

this implies \( RP(s_i)RP(s_j)(r_i - r_j) = 0 \) by Remark 6.1.

Hence \( RP(s_i)RP(s_j)t_x = RP(s_i)RP(s_j)r_j t_k \)

\[
0 \leq k \leq n, \ i, j = 0, 1, \ldots, n.
\]

From this, after simple calculations and using all projections of \( A \) are central, we conclude that \( a - r_j \in Q \).
If \( Q \in \alpha^{-1}(B_j) \) and \( Q \in \alpha^{-1}(B_1) \) then also it can be shown that \( a \in Q \). Thus

\[
a \cdot a_j \in \bigcap \{ \mathbf{P} : \mathbf{P} \cdot \mathbf{a}_j \equiv 0 \}, \quad 0 \leq j \leq n.
\]

Hence \( a = f \).

Q.E.D.

From this theorem it follows that if \( X \) is full in \( \Sigma(\Lambda) \)
then \( \Gamma_k F \subseteq \Lambda \).

If the topological space \( Y \) is chosen to be compact,
then each section of the sheaf \( F \) has a compact support,
since a closed subset of a compact set is compact. Thus
we have

**Corollary 6.1:** If \( \alpha \) is a continuous map from a full dense
subspace \( X \) of \( \Sigma(\Lambda) \) into a compact space \( Y \) then \( \Lambda \) is
\( * \)-isomorphic to the \( * \)-ring of all sections of the sheaf \( F \)
over \( Y \).

If we take \( X = Y = \Sigma(\Lambda) \) then we have

**Corollary 6.2:** \( \Lambda \) is \( * \)-isomorphic to the \( * \)-ring \( \Lambda \) of sections of
\( F(\Lambda, \text{Id}_{\Sigma(\Lambda)}) \) which has the property that \( \Gamma_k F(\Lambda, \text{Id}_{\Sigma(\Lambda)}) \subseteq \Lambda \).
§ 7: Concluding Remarks:

We proved the following corollary in Section 5.
"For any $a, b$ in a semiprime Rickart *-ring $A$,
$(\text{RP}(a)) \cap (\text{RP}(b)) = (\text{RP}(a)\text{RP}(b))"$. This corollary shows that
our semiprime Rickart *-rings are in fact, a.c. rings of
Henriksen and Jerison [5] because $\text{Ann}(a) = (1-\text{RP}(a))$ for
every $a \in A$.

Here by an a.c. ring we mean a semiprime ring
satisfying the following annihilator condition:

(a.c.) For $x, y \in A$, there exists $z \in A$ such that
$\text{Ann}(z) = \text{Ann}(x) \cap \text{Ann}(y)$.

The above corollary, also shows that every strict
ideal in a semiprime Rickart *-ring $A$ is a strongly Baer ideal
in the sense of Jayaram [6].

An ideal $I$ of $A$ is called Baer ideal if $x \in I$
implies $\text{Ann}(\text{Ann}(x)) \subseteq I$ and it is called strongly Baer if
for $x, y, z \in A$, $\text{Ann}(x) \cap \text{Ann}(y) = \text{Ann}(z)$ and $x, y \in I$ implies $z \in I$.

We have $\text{Ann}(x) = (1-\text{RP}(x))$ and $\text{Ann}(\text{Ann}(x)) = \text{Ann}((1-\text{RP}(x))) = (\text{RP}(x))$. Hence every strict ideal is a Baer
ideal also. The results of Jayaram [6] are obtained for a
commutative semiprime ring as against our considerations of a
noncommutative ring with involution.
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Apart from the material included in the four chapters of the Thesis, we succeeded in introducing the concepts of parallelogram law, GC, PC etc. in Rickart \(*\)-semigroups and have obtained several interesting results. But in such a study one has to take in consideration only closed projections. We are not satisfied with such a strong formulation, though others have studied Baer \(*\)-semigroups with only closed projections.

We are also successful in introducing hull-kernel topology on the set of \(p\)-semi-ideals (a nonempty subset \(P\) of projections is called a \(p\)-semi-ideal if \(e, f \in P\) and \(e \sim f\) implies \(f \in P\) and \(e \sim f\) implies \(f \in P\),) and obtain some interesting results. For example, we have shown that this topology is compact and non regular etc. But the results are yet to take definitive, fruitful and substantial formulations.

Our efforts to prove that every unitary element in a Baer \(*\)-ring is a product of at most four symmetries is also half way through. This problem says that in a properly infinite Von Neumann algebra every unitary element is the product of at most four symmetries. It is solved in a von Neumann algebra by Fillmore. [On products of symmetries. Canad. J. Math. 18 (1966), 897-900]. However, this is not solved even for general
$AW^*$-algebras. Our observations permit us to opine that most of the algebraic results with original formulation in von Neumann algebras have been proved or disproved at least for $AW^*$-algebras. The result under reference is the sole exception perhaps.

Following is the list of papers we have formulated that are based on the material not included in the Thesis.

1) p-semi-ideals and Restricted ideals in Rickart $*$-semigroups.

2) Prime p-semi-ideals in the poset of projections of a $*$-semigroup.

These have been communicated for publication.

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