EFFECTS OF AXIAL VARIATION OF VISCOSITY CAUSED BY ACCUMULATION OF RED CELLS IN THE ENTIRE REGION OF AN ARTERY WITH MILD STENOSIS ON FLOW CHARACTERISTICS
4.1 INTRODUCTION

Stenosis in arteries of humans is a common occurrence and hemodynamic factors play a significant role in the formation and proliferation of this disease. It is well known that at various locations in the arterial system, stenosis may develop due to abnormal intravascular growths. In the case of an artery, stenosis most commonly occurs in large distributing arteries such as coronary, renal, cerebral and femoral arteries. When the coronary artery is affected by a stenosis, critical flow conditions occur, such as flow separation, high wall shear stress and wall compression, which are believe to be the significant factors at the onset of coronary heart diseases. The narrowing of an artery caused by stenosis increases the vascular resistance as described by Poiseuille’s equation, which says that resistance is inversely proportional to the radius to the fourth power. Therefore, if the radius (or diameter) of a vascular segment is reduced by one-half, the resistance within that narrowed segment increases by 16-fold.

In this chapter we consider the case of axial variation of blood viscosity in the whole artery caused by red cells accumulation with mild stenosis in the artery. We therefore study the effect of axial variation of viscosity in the whole artery on resistance to flow of blood and shear stress in a stenotic artery.

4.2 ASSUMPTIONS

The following assumptions are made in the study:

- There are three zones of flow i.e., inlet, stenotic and outlet zone.
- Accumulation of red blood cells just after the inlet zone and before the region upto the maximum height of stenosis.
• The viscosity of blood is axial co-ordinate dependent. Also it is assumed that blood viscosity increases upto the maximum height of stenosis in the whole artery due to accumulation of red cells after which it decreases.

• The flowing blood is incompressible, homogeneous and Newtonian in character and motion of the flowing blood is steady and laminar.

• Radial velocity in the stenotic region is very small in comparison to the axial velocity (young 1968).

• The stenosis developed in the artery is axially symmetric and depends upon the axial distance z.

• The maximum height of the stenosis is much less as compared to the length and unobstructed radius of the artery i.e. stenosis is mild.

4.3 DEVELOPMENT OF THE MODEL:

The physical configuration of the stenosis in an artery is shown in figure (4.1)
The radius of the artery depends upon the geometry of the stenosis and can be written as follows, young (1968), Shukla et. al. (1979)

\[
\frac{R(z)}{R_0} = 1 - \frac{\delta}{2R_0} \left[ 1 + \cos \left( \frac{2\pi}{L_0} (z - L_1 - \frac{L_0}{2}) \right) \right], \quad L_1 \leq z \leq L_1 + L_0
\]

\[= 1; \text{ elsewhere} \quad 0 \leq z \leq L_1 \text{ and } L_1 + L_0 \leq z \leq L
\]

(4.3.1)

Where \( L_0 \) is the length of the stenosis and \( \delta \) is the maximum height of the stenosis assumed to be much smaller in comparison to the radius of the artery (\( \delta \ll R_0 \)).

Now consider the laminar and steady flow of the fluid whose viscosity varies along the axial direction. Assuming that the inertial and entrance effects are negligible, the one dimensional flow equation, where viscosity is the function of \( z \), is given by

\[
0 = -\frac{dp}{dz} + \mu(z) \frac{d}{dr} \left[ \frac{dw}{dr} \right]
\]

(4.3.2)

Where \( w \) is the axial velocity, \( p \) is the fluid pressure and \( \mu(z) \) is the viscosity which is the function of \( z \).

To see the effects of axial viscosity variation, we consider that the viscosity variation along the axial direction is given by

\[
\mu(z) = \mu_0 \left( 1 + \gamma \frac{z}{L_1} \right), \quad 0 \leq z \leq L_1
\]

\[= \mu_20 \left( \frac{R(z)}{R_0} \right)^{-\alpha}, \quad L_1 \leq z \leq L_1 + L_0
\]

\[= \mu_20 \left[ 1 - \beta \left( \frac{z - (L_1 + L_0)}{L - (L_1 + L_0)} \right) \right], \quad L_1 + L_0 \leq z \leq L
\]
where $\beta = \frac{\mu_{20} - \mu_{30}}{\mu_{20}} \ll 1$, $\gamma = \frac{\mu_{20} - \mu_{0}}{\mu_{0}} \ll 1$ and $\alpha = 0, 1, 2, 3, \ldots$ \hspace{1cm} (4.3.3)

$\mu_0$ is the viscosity of the fluid at $z=0$, $\mu_{20}$ is the viscosity of the fluid at $z=L_1$ and $z=L_1+L_0$, $\mu_{30}$ is the viscosity of the fluid at $z=L$, $\alpha$ is any arbitrary constant parameter which is the index of viscosity variation in the stenotic region. $\beta$ and $\gamma$ are the indexes of viscosity variation in the inlet and outlet region simultaneously.

The boundary conditions associated with equation (4.3.2) are given as follows:

$$\frac{dw}{dr} = 0 \text{ at } r=0$$
$$w = 0 \text{ at } r = R(z) \hspace{1cm} (4.3.4)$$

**4.4 ANALYSIS OF THE MODEL**

**4.4.1 METHOD OF SOLUTION**

Now solving equation (4.3.2) with and using equation (4.3.4) we get

$$w = \frac{1}{4} \int_{0}^{R(z)} \left( r^2 - R^2(z) \right) \frac{d\mu}{dz} \frac{dp}{dz} dr$$

or

$$Q = \int_{0}^{R(z)} 2\pi rdr \left( \int_{0}^{r} r^2 \left( -\frac{dw}{dr} \right) dr \right)$$

or

$$Q = -\frac{\pi}{8} \frac{dp}{dz} I(z) \hspace{1cm} (4.4.1.3)$$

where

$$I(z) = \frac{R^4(z)}{\mu(z)} \hspace{1cm} (4.4.1.4)$$

and $Q$ is a constant.
The pressure gradient can be obtained from equation (4.4.1.3) as

$$\frac{dp}{dz} = -\frac{8Q}{\pi I(z)}$$  \hspace{1cm} (4.4.1.5)

which on integration gives

$$p = \frac{-8Q}{\pi} \int_{0}^{z} \frac{dz}{I(z)} \hspace{1cm} (4.4.1.6)$$

Now solving equation (4.4.1.6) along with the condition \( p = p_0 \) at \( z=0 \) and \( p = p_L \) at \( z=L_0 \), we have

$$\Delta p = p_0 - p_L = \frac{8Q}{\pi} \int_{0}^{L} \frac{dz}{I(z)} = \frac{8QF}{\pi}$$  \hspace{1cm} (4.4.1.7)

where

$$F = \int_{0}^{L} \frac{dz}{I(z)}$$  \hspace{1cm} , \( i=1, 2, 3. \)  \hspace{1cm} (4.4.1.8)

The peripheral resistance \( \lambda \) is defined as follows [Burton (1968) and Young (1968)]

$$\lambda = \frac{\Delta p}{Q} = \frac{8F}{\pi}$$  \hspace{1cm} (4.4.1.9)

or

$$\lambda = \frac{8}{\pi} \int_{0}^{L} \frac{dz}{I(z)}$$  \hspace{1cm} (4.4.1.10)

The equation (4.4.1.10) gives the effect of viscosity variation on the resistance to flow on the wall of artery. The result is valid for any viscosity function of \( z \).

Now let us consider three zones of flow, inlet region (0 to \( L_1 \)), stenotic region (\( L_1 \) to \( L_1+L_0 \)) and outlet region (\( L_1+L_0 \) to \( L \)), (see figure 4.1), then from equation (4.4.1.10) we get

$$\lambda = \frac{8}{\pi} \left[ \int_{0}^{L_1} \frac{dz}{I(z)} + \int_{L_1}^{L_1+L_0} \frac{dz}{I(z)} + \int_{L_1+L_0}^{L} \frac{dz}{I(z)} \right]$$  \hspace{1cm} (4.4.1.11)
where \( I(z) \) is given by equation (4.4.1.4). Thus for the three zones of flow, we have

**Region I:** Non Stenotic region (inlet zone) \( 0 \leq z \leq L_0 \)

\[
I(z) = I_0 = \frac{R_0^4}{\mu_0} \left( 1 - \gamma \frac{z}{L_0} \right)
\]

(4.4.1.12)

**Region II:** Non Stenotic region (outlet zone) \( L_0 \leq z \leq L \)

\[
I(z) = I_0 = \frac{R_0^4}{\mu_0} \left[ 1 + \beta \left( \frac{z-(L_0+L_0)}{L-(L_0+L_0)} \right) \right]
\]

(4.4.1.13)

**Region III:** Stenotic region \( L_0 \leq z \leq L_0 + L \)

\[
I(z) = \frac{R_0^4}{\mu_0} \left( \frac{R(z)}{R_0} \right)^{4+\alpha}
\]

(4.4.1.14)

Now from equation (4.4.1.11) we have

\[
\lambda = \frac{8\mu L}{\pi R_0^4} \left[ \frac{1}{4} \left( 1 - \frac{L_0}{L} \right) \left\{ 1 + (1-\beta)(1+\gamma) + 2(1+\gamma) \right\} \right. \\
\left. + \frac{(1+\gamma)^{L_0+L_0}}{L} \int_{L_0}^{L_0+L_0} \frac{dz}{\left( \frac{R(z)}{R_0} \right)^{4+\alpha}} \right]
\]

(4.4.1.15)

The equation (4.4.1.15) gives the effect of viscosity variation on the peripheral resistance for any value of \( \alpha \). If \( \alpha = 0 \), and \( \mu_0 = \mu_20 = \mu_30 \), the result is same as given by Young (1968), i.e. for constant viscosity case. It is clear from equation (4.4.1.15) that the peripheral resistance increases as \( \alpha \) increases. i.e. viscosity of fluid increases upto the maximum height of stenosis for fixed stenosis size. Further the variation of peripheral resistance \( \lambda \), with the stenosis size shows that the resistance increases as the height of stenosis increases. Also if \( \mu_0 = \mu_30 \), we have
\[ \lambda = \frac{8\mu_0}{\pi R_0^4} \left[ \frac{L - L_0}{2} \left( 1 + \frac{\mu_20}{\mu_0} \right) + \frac{\mu_20}{\mu_0} \int_{l_i}^{l_f} \frac{dz}{R(z) \left( \frac{R(z)}{R_0} \right)^4} \right] \]  

(4.4.1.16)

Case I: \( \alpha = 0 \)

Now let us consider the case \( \alpha = 0 \) i.e. viscosity is constant in the stenotic region. In such a case from equation (4.4.1.16) we have

\[ \lambda = \frac{8\mu_0 L}{\pi R_0^4} \left[ 1 - \frac{L_0}{L} \right] \left\{ 1 + (1 - \beta)(1 + \gamma) + 2(1 + \gamma) \right\} \int_{l_i}^{l_f} \frac{dz}{R(z) \left( \frac{R(z)}{R_0} \right)^4} \]  

(4.4.1.17)

or

\[ \lambda = \frac{8\mu_0 L}{\pi R_0^4} \left[ \frac{1}{4} \left( 1 - \frac{L_0}{L} \right) \right] \left\{ 1 + (1 - \beta)(1 + \gamma) + 2(1 + \gamma) \right\} \]  

\[ \quad + (1 + \gamma) \frac{L_0}{L} \left( 1 - \frac{\delta}{2R_0} \right) \left( 1 - \frac{\delta}{R_0} + \frac{5\delta^2}{8R_0^2} \right) \left( 1 - \frac{\delta}{R_0} \right)^{-\frac{3}{2}} \]  

(4.4.1.18)

or

\[ \bar{\lambda} = \frac{\lambda \pi R_0^4}{8\mu_0 L} = \left[ \frac{1}{4} \left( 1 - \frac{L_0}{L} \right) \right] \left\{ 1 + (1 - \beta)(1 + \gamma) + 2(1 + \gamma) \right\} \]  

\[ \quad + (1 + \gamma) \frac{L_0}{L} \left( 1 - \frac{\delta}{2R_0} \right) \left( 1 - \frac{\delta}{R_0} + \frac{5\delta^2}{8R_0^2} \right) \left( 1 - \frac{\delta}{R_0} \right)^{-\frac{3}{2}} \]  

(4.4.1.19)
Case II: when $\alpha = 1$

Then from equation (4.4.1.16), we get the peripheral resistance in this case as follows:

$$
\lambda = \frac{8\mu_0 L}{\pi R_0^4} \left[ \frac{1}{4} \left( 1 - \frac{L_0}{L} \right) \left\{ 1 + (1 - \beta)(1 + \gamma) + 2(1 + \gamma) \right\} + \frac{(1 + \gamma)^{L_0}}{L} \int_{L_1}^{L_0} \frac{dz}{\left( \frac{R(z)}{R_0} \right)^{3}} \right]
$$

or

$$
\lambda = \frac{8\mu_0 L}{\pi R_0^4} \left[ \frac{1}{4} \left( 1 - \frac{L_0}{L} \right) \left\{ 1 + (1 - \beta)(1 + \gamma) + 2(1 + \gamma) \right\} + (1 + \gamma) \frac{L_0}{4L} \left( 1 - \frac{\delta}{R_0} \right)^{-\frac{3}{2}} \left\{ 4 \left( 1 - \frac{\delta}{2R_0} \right)^{4} + 3 \frac{\delta^2}{R_0^2} \left( 1 - \frac{\delta}{2R_0} \right)^{2} + \frac{3\delta^2}{32R_0^4} \right\} \right]
$$

(4.4.1.20)

Case III: when $\alpha = 2$

Again from equation (4.4.1.16), we get the peripheral resistance in this case as follows:

$$
\lambda = \frac{8\mu_0 L}{\pi R_0^4} \left[ \frac{1}{4} \left( 1 - \frac{L_0}{L} \right) \left\{ 1 + (1 - \beta)(1 + \gamma) + 2(1 + \gamma) \right\} + \frac{(1 + \gamma)^{L_0}}{L} \int_{L_1}^{L_0} \frac{dz}{\left( \frac{R(z)}{R_0} \right)^{6}} \right]
$$

(4.4.1.21)
or \( \lambda = \frac{8\mu L}{\pi R^6} \left[ \frac{1}{4} \left( 1 - \frac{L_0}{L} \right) \{1 + (1 - \beta)(1 + \gamma) + 2(1 + \gamma)\}^2 \right] \)

\[ + (1 + \gamma) \frac{L_0}{4L} \left[ 1 - \frac{\delta}{2R_0} \left( 1 - \frac{\delta}{R_0} \right)^{-\frac{3}{2}} \right] \left[ 4 \left( 1 - \frac{\delta}{2R_0} \right)^4 + \frac{5\delta^2}{R_0^2} \left( 1 - \frac{\delta}{2R_0} \right)^2 + \frac{25\delta^4}{32R_0^4} \right] \]

or \( \lambda = \frac{\lambda \pi R^4}{8\mu L} = \left[ \frac{1}{4} \left( 1 - \frac{L_0}{L} \right) \{1 + (1 - \beta)(1 + \gamma) + 2(1 + \gamma)\}^2 \right] \)

\[ + (1 + \gamma) \frac{L_0}{4L} \left[ 1 - \frac{\delta}{2R_0} \left( 1 - \frac{\delta}{R_0} \right)^{-\frac{3}{2}} \right] \left[ 4 \left( 1 - \frac{\delta}{2R_0} \right)^4 + \frac{5\delta^2}{R_0^2} \left( 1 - \frac{\delta}{2R_0} \right)^2 + \frac{25\delta^4}{32R_0^4} \right] \]

(4.4.1.23)

The shearing stress at the wall is given by

\[ \tau_w = -\mu(z) \frac{dw}{dr} \bigg|_{r=R(z)} \]

Noting that \( \frac{dw}{dr} = \frac{1}{2} \frac{dp}{dz} \frac{r}{\mu(z)} \); \( \frac{dp}{dz} = -\frac{8Q}{\pi R(z)} \)

We have

\[ \tau_w = \frac{4QR(z)}{\pi I(z)} \]

(4.4.1.25)

which gives the shear stress at the wall at maximum height of stenosis

\[ \tau_w \bigg|_{z=L_0+\frac{L_0}{2}} = \frac{4Q}{\pi} \frac{R(z)}{I(z)} \bigg|_{z=L_0+\frac{L_0}{2}} \]

(4.4.1.26)

The equation (4.4.1.26) gives the effect of viscosity variation on the shear stress on the wall of artery. The result is valid for any viscosity function of \( z \).
Substituting the value of \( I(z) \) from equation (4.4.1.14) in equation (4.4.1.25), we get the shear stress at the wall

\[
\tau_w = \frac{4\mu_0 Q}{\pi R_0^3} \left( \frac{R(z)}{R_0} \right)^{-3-\alpha}
\]

Also from equation (4.4.1.26) the shear stress at the maximum height of stenosis is given by

\[
\tau_{w} = \frac{4\mu_0 Q}{\pi R_0^3} \left( \frac{R(z)}{R_0} \right)^{-3-\alpha} = \frac{4\mu_0 Q}{\pi R_0^3} \left( 1 - \frac{\delta}{R_0} \right)^{-3-\alpha}
\]

or

\[
\tau_{w} = \frac{\tau_w \pi R_0^3}{4\mu_0 Q} = (1 + \gamma) \left( 1 - \frac{\delta}{R_0} \right)^{-3-\alpha}
\]

If \( \alpha = 0 \), the result is same as given by young (1968). \textit{i.e.} for constant viscosity case. It is clear from equation (4.4.1.28) that the wall shear stress increases as \( \alpha \) and \( \gamma \) increases. \textit{i.e.} viscosity of fluid increases up to the maximum height of stenosis for fixed stenosis size. Further the variation of wall shear stress \( \tau \), with the stenosis size shows that the wall shear stress increases as the height of stenosis increases.

**Case I: \( \alpha = 0 \)**

Now let us consider the case \( \alpha = 0 \) i.e. viscosity is constant in the stenotic region. In such a case from equation (4.4.1.27) and equation (4.4.1.28) we have

\[
\tau_{w} = \frac{4\mu_0 Q}{\pi R_0^3} \left( \frac{R(z)}{R_0} \right)^{-3} = \frac{4\mu_0 Q}{\pi R_0^3} \left( 1 - \frac{\delta}{R_0} \right)^{-3}
\]

or

\[
\tau_{w} = \frac{\tau_{w} \pi R_0^3}{4\mu_0 Q} = (1 + \gamma) \left( 1 - \frac{\delta}{R_0} \right)^{-3}
\]
Case I: \( \alpha = 1 \), then from equation (4.4.1.27) and equation (4.4.1.28) we have

\[
\tau_w = \frac{4\mu_0 Q R_0}{\pi R_0^3} \left( \frac{R(z)}{R_0} \right)^{-4} = \frac{4\mu_0 Q}{\pi R_0^3} \left( 1 - \frac{\delta}{R_0} \right)^{-4}
\]

(4.4.1.31)

or

\[
\tau_w = \tau_w \pi R_0^3 = (1 + \gamma) \left( 1 - \frac{\delta}{R_0} \right)^{-4}
\]

(4.4.1.32)

Case I: \( \alpha = 2 \), then from equation (4.4.1.27) and equation (4.4.1.28) we have

\[
\tau_w = \frac{4\mu_0 Q R_0}{\pi R_0^3} \left( \frac{R(z)}{R_0} \right)^{-5} = \frac{4\mu_0 Q}{\pi R_0^3} \left( 1 - \frac{\delta}{R_0} \right)^{-5}
\]

(4.4.1.33)

or

\[
\tau_w = \tau_w \pi R_0^3 = (1 + \gamma) \left( 1 - \frac{\delta}{R_0} \right)^{-5}
\]

(4.4.1.34)

4.5 DISCUSSION AND RESULTS:

The expression for equation (4.4.1.16) is plotted in figures\(^{2}\) (4.2) to (4.11). Figures (4.2)-(4.4) show the variation of resistance to flow with height of stenosis for different values of index of viscosity variation in the outlet region when the index of viscosity variation in the inlet and stenotic region is constant and the length of stenosis is 10% of the length of artery. It is shown that as the value of \( \beta \) increases the resistance to flow decreases. Similarly figures (4.5)-(4.7) show the variation of resistance to flow with height of stenosis for different values of index of viscosity variation in the inlet region when the index of viscosity variation in the stenotic and outlet region is constant. It is shown that for a fixed value of height of stenosis, as the value of \( \gamma \) increases the resistance to flow
increases. Figure (4.8) shows the variation of resistance to flow with height of stenosis for different values of index of viscosity variation in the stenotic region when the index of viscosity variation in the inlet and outlet region is constant. It is shown that as the value of $\alpha$ increases the resistance to flow increases. Figures (4.9)-(4.11) show the variation of resistance to flow with height of stenosis for different values of length of stenosis and for various combinations of index of viscosity variation in the three regions. It is shown that resistance to flow increases as the length of stenosis increases. Also it is shown in figures (4.2)-(4.11) that for a fixed value of $\alpha, \beta$ and $\gamma$, the resistance to flow increases as the height of stenosis increases. The expression for equation (4.4.1.28) is plotted in figures (4.12) to (4.15). Figures (4.12)-(4.14) show the variation of wall shear stress with height of stenosis for different values of index of viscosity variation in the inlet region when the index of viscosity variation in the stenotic region is constant. Figure (4.15) shows the variation of wall shear stress with height of stenosis for different values of index of viscosity variation in the stenotic region when the index of viscosity variation in the inlet region is constant. It is shown in figures (4.12)-(4.15) that for a fixed value of height of stenosis the wall shear stress increases as the value of $\alpha$ and $\gamma$ increases. Also it is shown that for a fixed value of $\alpha$ and $\gamma$ the wall shear stress increases as the height of stenosis increases.
Variation of resistance to flow with height of stenosis for different values of index of viscosity variation in the inlet zone

Figure 4.2

Variation of resistance to flow with height of stenosis for different values of index of viscosity variation in the inlet zone

Figure 4.3
Variation of resistance to flow with height of stenosis for different combination of index of viscosity variation in the inlet zone

Figure 4.4

Variation of resistance to flow with height of stenosis for different values of index of viscosity variation in the outlet region

Figure 4.5
Variation of resistance to flow with height of stenosis for different values of index of viscosity variation in the outlet region

Figure 4.6

Variation of resistance to flow with height of stenosis for different values of index of viscosity variation in the outlet region

Figure 4.7
Variation of resistance to flow with height of stenosis for different values of index of viscosity variation in the stenotic region

Figure 4.8

Variation of resistance to flow with height of stenosis for different values of length of stenosis

Figure 4.9
Variation of resistance to flow with height of stenosis for different values of length of stenosis

Figure 4.10

Figure 4.11
Variation of wall shear stress with height of stenosis for different values of index of viscosity variation in the inlet region.

Figure 4.12

Variation of wall shear stress with height of stenosis for different values of index of viscosity variation in the inlet region.

Figure 4.13
Variation of wall shear stress with height of stenosis for different values of index of viscosity variation in the inlet region

Figure 4.14

Variation of wall shear stress with height of stenosis for different values of index of viscosity variation in the stenotic region

Figure 4.15