Chapter -II

FIVE DIMENSIONAL
PLANE GRAVITATIONAL WAVES
IN
BIMETRIC THEORY OF RELATIVITY

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2.1 Introduction

Several attempts were made to modify the general theory of relativity with little success. However, Professor Rosen (1940, 54, 63, 66, 70, 73, 74, 75, 78, 79, 80, 83, 85, 89) proposed the bimetric theory of relativity where the metric tensors at each point of the space-time are defined: One is a Riemannian metric tensor $g_{ij}$, which describes gravitation and other is the background flat space-time metric $f_{ij}$, which enters into the field equations and interacts with $g_{ij}$ but also has no direct interaction with matter. Accordingly, at each space-time point, we have two line elements

$$ds^2 = g_{ij} \, dx^i \, dx^j,$$

and

$$d\sigma^2 = f_{ij} \, dx^i \, dx^j.$$  

(2.1.1) \hspace{5cm} (2.1.2)

In Rosen’s bimetric theory of relativity, the plane gravitational waves $g_{ij}$ in four dimensions are the solutions of the field equations $N^I_j = 0$ which retain Takeno’s (1961) format. This work has been extended by Adhav and Karade (1994) in five and six dimensional space-times. Mete and Adhav (2002) have generalized this result in n-dimensions. Thengne and Karade (2001) reformulated the definition of Takeno (1961) in four dimensional space-time. Using this definition, it is shown by Katore and Karade (2001) that the solutions ($g_{ij}$) of plane gravitational waves obtained from Rosen’s field equations $N^I_j = 0$ can be expressed in Takeno’s format.
This chapter is the extension of the work done by Katore and Karade (2001). It is shown that the plane gravitational wave solutions in five dimensional space-time for the field equations \( N_j^i = 0 \) in bimetric theory of relativity are given by \( g_{ij} \) satisfying the equations

\[
S \mu_j^i + R \zeta_j^i = 0 ,
\]

which can also be written in Takeno’s format as,

\[
\overline{W}_\alpha \mu_j^i + \overline{W}_\alpha \zeta_j^i = 0 = \phi_\alpha \mu_j^i + \phi_\alpha \zeta_j^i .
\]

2.2 Plane Gravitational Waves

The plane gravitational wave is a non-flat solution \( g_{ij} \) of the field equations

\[
N_j^i = 0 \quad i, j = 1, 2, 3, 4, 5.
\]

in an empty region of space-time such that

\[
g_{ij} = g_{ij}(Z)
\]

where

\[
Z = Z(x^i) , 
\]

\[
x^i = (x^1, x^2, x^3, x^4, x^5)
\]

in some suitable coordinate system so that,

\[
g^{ij} Z_{ij} Z_{ij} = 0 , \quad Z_{ij} = \frac{\partial Z}{\partial x^i}
\]

\[
Z = Z(x^1, x^2, x^3, x^4, x^5) , \quad Z_{ij} \neq 0 \quad i = 1, 2, 3, 4, 5 ,
\]

where

\[
N_j^i = \frac{1}{2} f^{ab} \left( g^{-1} \frac{\partial g_{ij}}{\partial x^a} \right) |^\beta
\]

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\[ N = N_i^i ; \quad i = 1, 2, 3, 4, 5 \]
\[ \kappa = \frac{\bar{\varepsilon}}{\sqrt{\bar{f}}} , \quad g = \det (g_{ij}) , \quad f = \det (f_{ij}) \]

and the bar (\(\bar{\cdot}\)) stands for \(f\)-covariant differentiation.

The signature convention is that of Takeno's (1961),
\[
\begin{vmatrix}
g_{aa} < 0 , \\
| \begin{vmatrix}
g_{aa} & g_{a\beta} \\
g_{\beta a} & g_{\beta\beta} \\
\end{vmatrix} > 0 , \\
\end{vmatrix}
\begin{vmatrix}
g_{11} & g_{12} & g_{13} & g_{14} \\
g_{21} & g_{22} & g_{23} & g_{24} \\
g_{31} & g_{32} & g_{33} & g_{34} \\
g_{41} & g_{42} & g_{43} & g_{44} \\
\end{vmatrix} < 0 , \\
\begin{vmatrix}
g_{51} & g_{52} & g_{53} & g_{54} \\
\end{vmatrix} > 0 , \\
g_{55} < 0 . \tag{2.2.5}
\]

There is no summation for \(\alpha\) and \(\beta\) (\(\alpha, \beta = 1, 2, 3, 4\)) and accordingly we have,
\[ g = \det (g_{ij}) < 0 . \]

### 2.3 Existence of Plane Gravitational Waves

From equation (2.2.3), we have
\[ g^{11}(Z,1)^2 + g^{12}(Z,1)(Z,2) + g^{13}(Z,1)(Z,3) + g^{14}(Z,1)(Z,4) + g^{15}(Z,1)(Z,5) + g^{21}(Z,2)(Z,1) + g^{22}(Z,2)^2 + g^{23}(Z,2)(Z,3) + g^{24}(Z,2)(Z,4) + g^{25}(Z,2)(Z,5) + g^{31}(Z,3)(Z,1) + g^{32}(Z,3)(Z,2) + g^{33}(Z,3)^2 + g^{34}(Z,3)(Z,4) + g^{35}(Z,3)(Z,5) + g^{41}(Z,4)(Z,1) + g^{42}(Z,4)(Z,2) + g^{43}(Z,4)(Z,3) + g^{44}(Z,4)^2 + g^{45}(Z,4)(Z,5) + g^{51}(Z,5)(Z,1) + g^{52}(Z,5)(Z,2) + g^{53}(Z,5)(Z,3) + g^{54}(Z,5)(Z,4) + g^{55}(Z,5)^2 = 0 \]

dividing both sides by \((Z,5)^2\), we get
\[
g^{11}\phi_1^2 + 2g^{12}\phi_1\phi_2 + 2g^{13}\phi_1\phi_3 + 2g^{14}\phi_1\phi_4 + 2g^{15}\phi_1 + g^{22}\phi_2^2 + 2g^{23}\phi_2\phi_3 + 2g^{24}\phi_2\phi_4 + 2g^{25}\phi_2 + g^{33}\phi_3^2 + 2g^{34}\phi_3\phi_4 + 2g^{35}\phi_3 + g^{44}\phi_4^2 + 2g^{45}\phi_4 + g^{45} = 0 \ ,
\]

where

\[
\frac{Z_1}{Z_5} = \phi_1 \ , \quad \frac{Z_2}{Z_5} = \phi_2 \ , \quad \frac{Z_3}{Z_5} = \phi_3 \ , \quad \frac{Z_4}{Z_5} = \phi_4 \ , \quad \frac{Z_5}{Z_5} = \phi_5 = 1.
\]

(2.3.2)

Solving Lagrange’s partial D. E. \[
\frac{Z_1}{Z_5} = \phi_1 \quad \text{we get},
\]

[For details, refer Appendix (2.3.1)]

\[
\omega_1(Z) = x^5 + \phi_1 x^1
\]

\[
= t + \phi_1 x^1 \ ,
\]

(2.3.3)

where

\[
\phi_1 = \phi_1(Z) \ , \quad \omega_1 = \omega_1(Z) \ , \quad x^5 = t.
\]

Differentiating (2.3.3) with respect to t (\(=x^5\)), we get

\[
\bar{\omega}_1 Z_5 = 1 + x^1 \frac{\phi_1}{\phi_1} Z_5
\]

\[
\Rightarrow \quad Z_5 = \frac{1}{\omega_1 - x^1 \frac{\phi_1}{\phi_1}}
\]

\[
\Rightarrow \quad Z_5 = \frac{1}{M_1} \ ,
\]

(2.3.4)

where

\[
M_1 = \omega_1 - x^1 \frac{\phi_1}{\phi_1} \ .
\]

(2.3.5)

The bar (\(\bar{\cdot}\)) over a letter means derivative with respect to Z.

Differentiating (2.3.3) with respect to \(x^1\), we get
\[ \overline{\omega_1} Z_{11} = \phi_1 + x^1 \overline{\phi_1} Z_{11} \]

\[ \Rightarrow \quad Z_{11} = \frac{\overline{\omega_1}}{\omega_1 - x^1 \overline{\phi_1}} \]

\[ \Rightarrow \quad Z_{11} = \frac{\phi_1}{M_1} \quad \text{(2.3.6)} \]

Solving Lagrange's partial D. E. \[ \frac{Z_{12}}{Z_{15}} = \phi_2 \]

we obtain the solution as

\[ \omega_2(Z) = t + \phi_2 x^2 \quad \text{(2.3.7)} \]

Differentiating (2.3.7) with respect to \( x^2 \) and \( t \), we get

\[ Z_{12} = \frac{\phi_2}{M_2} \quad \text{and} \quad Z_{15} = \frac{1}{M_2} \quad \text{(2.3.8)} \]

where

\[ M_2 = \omega_2 - x^2 \overline{\phi_2} \quad \text{(2.3.9)} \]

Similarly from partial D. E. \[ \frac{Z_{23}}{Z_{25}} = \phi_3 \]

we get, the solution as

\[ \omega_3(Z) = t + \phi_3 x^3 \quad \text{(2.3.10)} \]

Differentiating (2.3.10) with respect to \( x^3 \) and \( t \), we get

\[ Z_{23} = \frac{\phi_3}{M_3} \quad \text{and} \quad Z_{25} = \frac{1}{M_3} \quad \text{(2.3.11)} \]

where

\[ M_3 = \omega_3 - x^3 \overline{\phi_3} \quad \text{(2.3.12)} \]

From partial D. E. \[ \frac{Z_{34}}{Z_{35}} = \phi_4 \]

we get, solution as
\[ \omega_4(Z) = t + \phi_4 x^4 . \]  \hspace{1cm} (2.3.13)

Differentiating (2.3.13) with respect to \( x^4 \) and \( t \), we get
\[ Z_{14} = \frac{\phi_4}{M_4} \quad \text{and} \quad Z_{15} = \frac{1}{M_4} , \]  \hspace{1cm} (2.3.14)

where
\[ M_4 = \omega_4 - x^4 \phi_4 . \]  \hspace{1cm} (2.3.15)

In general, the solution of D. E. \( \frac{Z_\alpha}{Z_{n_5}} = \phi_\alpha \) is given by
\[ \omega_\alpha(Z) = t + \phi_\alpha x^\alpha . \]  \hspace{1cm} (2.3.16)

Differentiating (2.3.16) with respect to \( x^\alpha \) and \( t \), we get
\[ Z_{n_\alpha} = \frac{\phi_\alpha}{M_\alpha} \quad \text{and} \quad Z_{n_5} = \frac{1}{M_\alpha} , \]  \hspace{1cm} (2.3.17)

where
\[ M_\alpha = \omega_\alpha - x^\alpha \phi_\alpha , \quad \alpha = 1, 2, 3, 4 . \]  \hspace{1cm} (2.3.18)

Let
\[ M_\alpha = \omega_\alpha - x^\alpha \phi_\alpha = R . \]  \hspace{1cm} (2.3.19)

Using equation (2.3.19), equations (2.3.6), (2.3.8), (2.3.11) and (2.3.14) reduces to
\[ Z_{11} = \frac{\phi_1}{R} , \quad Z_{12} = \frac{\phi_2}{R} , \quad Z_{13} = \frac{\phi_3}{R} , \quad Z_{14} = \frac{\phi_4}{R} , \quad Z_{15} = \frac{1}{R} . \]  \hspace{1cm} (2.3.20)

Differentiating (2.3.5) with respect to \( t \ (= x^5) \), we get
\[ M_{15} = \omega_5 Z_{15} - \phi_5 Z_{15} x^1 \]
\[ = (\omega_5 - \phi_5 x^1) Z_{15} , \quad \left( \text{put} \quad \omega_5 - \phi_5 x^1 = N_1 \right) \]
\[ = N_1 Z_{15} , \]
\[
\frac{N_1}{M_1} \quad \text{[using (2.3.5)]}
\]

\[
\frac{N_1}{R} \quad \text{[using (2.3.14)]}
\]

(2.3.21)

Similarly differentiating (2.3.9), (2.3.12) and (2.3.15), we obtain

\[
M_{2,5} = \frac{N_2}{M_2} = \frac{N_2}{R}, \quad M_{3,5} = \frac{N_3}{M_3} = \frac{N_3}{R}, \quad M_{4,5} = \frac{N_4}{M_4} = \frac{N_4}{R},
\]

(2.3.22)

where

\[
N_2 = \omega_2 - x^2 \phi_2, \quad N_3 = \omega_3 - x^3 \phi_3, \quad N_4 = \omega_4 - x^4 \phi_4.
\]

(2.3.23)

Let \( N_\alpha = \omega_\alpha - x^\alpha \phi_\alpha = S, \quad \alpha = 1, 2, 3, 4 \)

(2.3.24)

Differentiating (2.3.5) with respect to \( x^1 \), we get

\[
M_{1,1} = \omega_1 Z_{11} - \left( \phi_1 + \phi_1 Z_{11} x^1 \right)
\]

\[
= (\omega_1 - \phi_1 x^1) Z_{11} - \phi_1
\]

\[
= N_1 Z_{11} - \phi_1
\]

\[
= \frac{N_1}{M_1} \phi_1 - \phi_1
\]

\[
= \frac{S}{R} \phi_1 - \phi_1.
\]

Using (2.3.19), we get

\[
R_{11} = \frac{S}{R} \phi_1 - \phi_1.
\]

(2.3.25)

Similarly, differentiating (2.3.9) with respect to \( x^2 \), we get

\[
M_{2,2} = \frac{N_2}{M_2} \phi_2 - \phi_2 = \frac{S}{R} \phi_2 - \phi_2.
\]

Using (2.3.19), we get

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\[ R_{\alpha} = \frac{S}{R} \phi_2 - \bar{\phi}_2 \, . \]  

(2.3.26)

Differentiating (2.3.12) with respect to \( x^3 \), we get

\[ M_{3,3} = \frac{N_3}{M_3} \phi_3 - \bar{\phi}_3 = \frac{S}{R} \phi_3 - \bar{\phi}_3 \, . \]

Using (2.3.19), we get

\[ R_{3} = \frac{S}{R} \phi_3 - \bar{\phi}_3 \, . \]  

(2.3.27)

Differentiating (2.3.15) with respect to \( x^4 \), we get

\[ M_{4,4} = \frac{N_4}{M_4} \phi_4 - \bar{\phi}_4 = \frac{S}{R} \phi_4 - \bar{\phi}_4 \, . \]

Using (2.3.19), we get

\[ R_{4} = \frac{S}{R} \phi_4 - \bar{\phi}_4 \, . \]  

(2.3.28)

where a bar (\(-\)) over a letter denotes the derivative with respect to \( Z \) and

\[ R = -x^1 \phi_1 = -x^2 \phi_2 = -x^3 \phi_3 = -x^4 \phi_4 \neq 0 \]

\[ S = -x^1 \bar{\phi}_1 = -x^2 \bar{\phi}_2 = -x^3 \bar{\phi}_3 = -x^4 \bar{\phi}_4 \neq 0 \, . \]  

(2.3.29)

We have the field equations as,

\[ N^i_j = 0 \]

\[ \Rightarrow f^{a\beta}(\underline{g}^{hi} \underline{g}_{\beta j \alpha})_{\beta} = 0 \, . \]

Assuming \( f_{ij} \) as Lorentz metric (-1, -1, -1, -1, 1) the \( f \)-covariant derivative becomes the ordinary partial derivative

\[ \therefore f^{a\beta}(\underline{g}^{hi} \underline{g}_{\beta j \alpha})_{,\beta} = 0 \]
\[ \Rightarrow f^{11}(g^{\text{hi}}g_{hj,1})_{,1} + f^{22}(g^{\text{hi}}g_{hj,2})_{,2} + f^{33}(g^{\text{hi}}g_{hj,3})_{,3} + f^{44}(g^{\text{hi}}g_{hj,4})_{,4} \\
+ f^{55}(g^{\text{hi}}g_{hj,5})_{,5} = 0 \]

\[ \Rightarrow f^{11}(g^{\text{hi}}g_{hj}Z_{,1}^1)_{,1} + f^{22}(g^{\text{hi}}g_{hj}Z_{,2}^2)_{,2} + f^{33}(g^{\text{hi}}g_{hj}Z_{,3}^3)_{,3} + f^{44}(g^{\text{hi}}g_{hj}Z_{,4}^4)_{,4} \\
+ f^{55}(g^{\text{hi}}g_{hj}Z_{,5}^5)_{,5} = 0 \]

Substituting values from equation (2.3.20), we get

\[ f^{11}(g^{\text{hi}}g_{hj} \frac{\phi_1}{R})_{,1} + f^{22}(g^{\text{hi}}g_{hj} \frac{\phi_2}{R})_{,2} + f^{33}(g^{\text{hi}}g_{hj} \frac{\phi_3}{R})_{,3} + f^{44}(g^{\text{hi}}g_{hj} \frac{\phi_4}{R})_{,4} \\
+ f^{55}(g^{\text{hi}}g_{hj} \frac{1}{R})_{,5} = 0 \]

\[ \Rightarrow f^{11}(\frac{g^{\text{hi}}g_{hj} \phi_1^2}{R^2} + g^{\text{hi}}g_{hj} \phi_1 \phi_2 + g^{\text{hi}}g_{hj} \phi_1 \phi_3)_{,1} \\
+ f^{22}(\frac{g^{\text{hi}}g_{hj} \phi_2^2}{R^2} + g^{\text{hi}}g_{hj} \phi_2 \phi_1 + g^{\text{hi}}g_{hj} \phi_2 \phi_3)_{,2} \\
+ f^{33}(\frac{g^{\text{hi}}g_{hj} \phi_3^2}{R^2} + g^{\text{hi}}g_{hj} \phi_3 \phi_1 + g^{\text{hi}}g_{hj} \phi_3 \phi_2)_{,3} \\
+ f^{44}(\frac{g^{\text{hi}}g_{hj} \phi_4^2}{R^2} + g^{\text{hi}}g_{hj} \phi_4 \phi_1 + g^{\text{hi}}g_{hj} \phi_4 \phi_2)_{,4} \\
+ f^{55}(\frac{g^{\text{hi}}g_{hj} 1}{R^2} + g^{\text{hi}}g_{hj} \frac{1}{R} + g^{\text{hi}}g_{hj} \frac{1}{R})_{,5} = 0 \]
\[
\Rightarrow \left( -1 \right) \left[ g_{h_1 i} g_{h_2 j} + g_{h_2 i} g_{h_1 j} R^2 \left( \frac{\phi_1^2}{R^2} - \frac{\phi_2 R_{1}}{R^2} \right) \right] \\
+ \left( -1 \right) \left[ g_{h_1 i} g_{h_2 j} + g_{h_2 i} g_{h_1 j} R^2 \left( \frac{\phi_2^2}{R^2} - \frac{\phi_3 R_{2}}{R^2} \right) \right] \\
+ \left( -1 \right) \left[ g_{h_1 i} g_{h_2 j} + g_{h_2 i} g_{h_1 j} R^2 \left( \frac{\phi_3^2}{R^2} - \frac{\phi_4 R_{3}}{R^2} \right) \right] \\
+ \left( -1 \right) \left[ g_{h_1 i} g_{h_2 j} + g_{h_2 i} g_{h_1 j} R^2 \left( \frac{\phi_4^2}{R^2} - \frac{\phi_4 R_{4}}{R^2} \right) \right] \\
+ \left( 1 \right) \left[ g_{h_1 i} g_{h_2 j} + g_{h_2 i} g_{h_1 j} R^2 \left( \frac{R_{15}}{R^2} \right) \right] = 0.
\]

\[
\uparrow \left( \frac{\phi_1^2}{R^2} - \frac{\phi_2^2}{R^2} - \frac{\phi_3^2}{R^2} - \frac{\phi_4^2}{R^2} \right) \left[ g_{h_1 i} g_{h_2 j} + g_{h_2 i} g_{h_1 j} \right] \\
\left[ \left( \frac{R_{1} \phi_1 - \phi_1 R_{1}}{R^2} \right) + \left( \frac{R_{2} \phi_2 - \phi_2 R_{2}}{R^2} \right) + \left( \frac{R_{3} \phi_3 - \phi_3 R_{3}}{R^2} \right) \right] \\
- \left[ \left( R_{4} \phi_4 - \phi_4 R_{4} \right) + \left( R_{15} \right) \right] \] \\
\Rightarrow \left[ 1 - \left( \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \right) \right] \left[ g_{h_1 i} g_{h_2 j} + g_{h_2 i} g_{h_1 j} \right] \\
\left[ \left( \frac{R_{1} \phi_1 - \phi_1 R_{1}}{R^2} \right) + \left( \frac{R_{2} \phi_2 - \phi_2 R_{2}}{R^2} \right) + \left( \frac{R_{3} \phi_3 - \phi_3 R_{3}}{R^2} \right) \right] \\
- \left[ \left( R_{4} \phi_4 - \phi_4 R_{4} \right) + R_{15} \right] g_{h_1 i} g_{h_2 j} = 0.
\]

Substituting

\[
Z_{1a} = \frac{\phi_{1a}}{R}, \quad R_{1a} = \frac{S}{R} - \frac{\phi_{1}}{R_1}, \quad \text{for } \alpha = 1, 2, 3, 4
\]

and \( Z_{15} = \frac{1}{R} \), \( R_{15} = \frac{S}{R} \)

we get,
\[ \Rightarrow \left[ 1 - \left( \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \right) \right] \left( g_{hi}^{hi} g_{hi} + g_{hi}^{hi} g_{hi} \right) \\
- \left[ \left( 2 \phi_1 \phi_1 - \frac{S}{R} \phi_1^2 \right) + \left( 2 \phi_2 \phi_2 - \frac{S}{R} \phi_2^2 \right) + \left( 2 \phi_3 \phi_3 - \frac{S}{R} \phi_3^2 \right) \right] g_{hi}^{hi} g_{hi} = 0 \ . \]

\[ \Rightarrow \ R \left[ 1 - \left( \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \right) \right] \left( g_{hi}^{hi} g_{hi} + g_{hi}^{hi} g_{hi} \right) \\
- 2 R \left[ \phi_1 \phi_1 + \phi_2 \phi_2 + \phi_3 \phi_3 + \phi_4 \phi_4 \right] g_{hi}^{hi} g_{hi} \\
+ S \left[ -1 + \left( \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \right) \right] g_{hi}^{hi} g_{hi} = 0 \ . \]

\[ \Rightarrow \ S \left[ \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 - 1 \right] g_{hi}^{hi} g_{hi} \\
+ R \left[ 1 - \left( \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \right) \right] \left( g_{hi}^{hi} g_{hi} + g_{hi}^{hi} g_{hi} \right) \\
- 2 R \left[ \phi_1 \phi_1 + \phi_2 \phi_2 + \phi_3 \phi_3 + \phi_4 \phi_4 \right] g_{hi}^{hi} g_{hi} = 0 \ . \]

\[ \Rightarrow \ S \left[ \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 - 1 \right] g_{hi}^{hi} g_{hi} \\
+ R \frac{d}{dz} \left\{ \left[ 1 - \left( \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \right) \right] g_{hi}^{hi} g_{hi} \right\} = 0 \ . \]

The above equation reduces to,

\[ S \mu_j^i + R \zeta_j^i = 0 \ , \quad (2.3.30) \]

where

\[ \mu_j^i = \left[ \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 - 1 \right] g_{hi}^{hi} g_{hi} \]

\[ \zeta_j^i = \frac{d}{dz} \left\{ \left[ 1 - \left( \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \right) \right] g_{hi}^{hi} g_{hi} \right\} \ . \]
Using equations (2.3.29), the equation (2.3.30) reduces to

\[ (\omega_\alpha - \phi_\alpha x^\alpha) \mu_j^i + (\omega_\alpha - \phi_\alpha x^\alpha) \zeta_j^i = 0 \]

\[ \Rightarrow \quad \omega_\alpha \mu_j^i - \phi_\alpha x^\alpha \mu_j^i + \omega_\alpha \zeta_j^i - \phi_\alpha x^\alpha \zeta_j^i = 0 \]

\[ \Rightarrow \quad (\omega_\alpha \mu_j^i + \omega_\alpha \zeta_j^i) - (\phi_\alpha \mu_j^i + \phi_\alpha \zeta_j^i) x^\alpha = 0 \]

\[ \Rightarrow \quad \omega_\alpha \mu_j^i + \omega_\alpha \zeta_j^i = 0 \quad \text{and} \quad \phi_\alpha \mu_j^i + \phi_\alpha \zeta_j^i = 0 \]

i.e., \[ W_\alpha \mu_j^i + W_\alpha \zeta_j^i = 0 = \phi_\alpha \mu_j^i + \phi_\alpha \zeta_j^i \] (2.3.31)

which is again in the Takeno’s format.

2.4 Conclusion:

The plane gravitational wave solutions in five dimensional space-times of the field equations \( N_j^l = 0 \) in bimetric theory of relativity are given by \( g_{ij} = g_{ij}(Z) \), where \( Z_{si} \neq 0 \), \( i = 1, 2, 3, 4, 5 \) are characterized by the equations (2.2.2), (2.2.3), (2.3.16), (2.3.19), (2.3.24), (2.3.29), (2.3.30) and (2.3.31).
Appendix [2.3.1]

The Lagranges partial D. E. is \[ \frac{Z_{11}}{Z_{s5}} = \phi_1 \]

\[ \Rightarrow \quad Z_{1} = \phi_1 Z_{s5} \]

\[ \Rightarrow \quad \frac{\partial Z}{\partial x^1} = \phi_1 \frac{\partial Z}{\partial x^5} \]

\[ \Rightarrow \quad \frac{\partial Z}{\partial x^1} - \phi_1 \frac{\partial Z}{\partial x^5} = 0 \]

The D. E. is of type \( pP + qQ = R \).

The auxiliary equation is

\[ \frac{dx^1}{1} = \frac{dx^5}{-\phi_1} = \frac{dZ}{0} \]

\[ \Rightarrow \quad \frac{dx^1}{1} = \frac{dx^5}{-\phi_1} = \phi_1 \]

\[ \Rightarrow \quad -\phi_1 \, dx^1 = dx^5 \quad \text{and} \quad dZ = 0. \]

Integrating, we get

\[ x^5 + \phi_1 \, x^1 = \text{constant} \quad \text{and} \quad Z = \text{constant}. \]

Its complete solution is

\[ \omega_1(Z) = x^5 + \phi_1 x^1, \]

where

\[ \phi_1 = \phi_1(Z), \quad \omega_1 = \omega_1(Z). \]