Chapter 4

Fuzzy \(\ell\)-ideal in \(\ell\)-near ring

In this chapter, we study about the fuzzy \(\ell\)-ideal of the \(\ell\)-near ring. The concept of fuzzy \(\ell\)-ideal and level \(\ell\)-ideal of a \(\ell\)-near ring were introduced and their properties are studied. Further, the characterization theorem of fuzzy \(\ell\)-ideal, a procedure to construct a fuzzy \(\ell\)-ideal from any given ascending chain of \(\ell\)-ideals of a \(\ell\)-near ring, the necessary and sufficient condition for equality of two level \(\ell\)-ideals. Also established the image of a fuzzy \(\ell\)-ideal and the pre-image of a fuzzy \(\ell\)-ideal under the \(\ell\)-homomorphism are also fuzzy \(\ell\)-ideals.

**Definition 4.1.** Let \((L_N, +, \cdot, \vee, \wedge)\) be a \(\ell\)-near ring and \(\mu\) be a fuzzy subset of \(L_N\). Then \(\mu\) is called a fuzzy \(\ell\)-ideal of \(L_N\) or fuzzy sub \(\ell\)-near ring ideal, if it satisfies the following:

(i) \(\mu(x - y) \geq \min\{\mu(x), \mu(y)\}\);

(ii) \(\mu(xy) \geq \min\{\mu(x), \mu(y)\}\);

(iii) \(\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}\);

(iv) \(\mu(x \wedge y) \geq \max\{\mu(x), \mu(y)\}\);

(v) \(\mu(x) = \mu(y + x - y)\);
(vi) $\mu(xy) \geq \mu(y)$;

(vii) $\mu((x + i)y - xy) \geq \mu(i)$, for all $x, y, i \in \mathcal{L}_N$.

**Example 4.1.** Let $\mathcal{L}_N = \{m, n\}$ be a set of two different symbols with four binary operations are defined as follows:

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and $m \cdot n = n$.

Then $(\mathcal{L}_N, +, \cdot, \vee, \wedge)$ is a $\ell$-near ring.

We define a fuzzy subset $\mu : \mathcal{L}_N \to [0, 1]$ by $\mu(n) \leq \mu(m)$. Then $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$.

**Proposition 4.1.** If $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$, then $\mu(x + y) = \mu(y + x)$, for all $x, y \in \mathcal{L}_N$.

**Proof:**

Assume that $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$.

To prove that $\mu(x + y) = \mu(y + x)$.

Let $x, y \in \mathcal{L}_N$ be arbitrary and put $z = x + y$. Then, $\mu(x + y) = \mu(z) = \mu(-x + z + x) = \mu(-x + x + y + x) = \mu(y + x)$.

**Proposition 4.2.** If $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$, then $\mu(1) \leq \mu(x) \leq \mu(0)$, for all $x \in \mathcal{L}_N$.

**Proof:**

Given that $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$.

To prove that $\mu(1) \leq \mu(x) \leq \mu(0)$, for all $x \in \mathcal{L}_N$.

We take $x = x, y = 1$, then $\mu(x.1) \geq \mu(1)$.

$$\implies \mu(x) \geq \mu(1).$$

Also $\mu(0) = \mu(x - x) \geq \min\{\mu(x), \mu(x)\} \geq \mu(x)$.

$$\implies \mu(0) \geq \mu(x).$$

Hence $\mu(1) \leq \mu(x) \leq \mu(0)$. 

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Proposition 4.3. If \( \mu \) is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \), then \( \mu(x) = \mu(-x) \), for all \( x \in \mathcal{L}_N \).

Proof:
Given that \( \mu \) is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \).
To prove that \( \mu(x) = \mu(-x) \), for all \( x \in \mathcal{L}_N \).
Let \( x \in \mathcal{L}_N \) be arbitrary.
Then \( -x = 0 + (-x) = 0 - x \).
\[ \implies \mu(-x) = \mu(0 - x) \geq \min\{\mu(0), \mu(x)\} = \mu(x), \text{ by proposition 4.2.} \]
\[ \implies \mu(-x) \geq \mu(x). \quad (1) \]
Again \( \mu(x) = \mu(-(x)) \geq \mu(-x) \), by (1).
\[ \implies \mu(x) \geq \mu(-x). \quad (2) \]
Hence \( \mu(x) = \mu(-x) \), by (1) and (2).

Proposition 4.4. If \( \mu \) is a fuzzy \( \ell \)-ideal of a \( \ell \)-near ring \( \mathcal{L}_N \), then \( \mu(x + y) \geq \min\{\mu(x), \mu(y)\} \), for all \( x, y \in \mathcal{L}_N \).

Proof:
Given that \( \mu \) is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \).
To prove that \( \mu(x + y) \geq \min\{\mu(x), \mu(y)\} \), for all \( x, y \in \mathcal{L}_N \).
Let \( x, y \in \mathcal{L}_N \) be arbitrary. Then,
\[ \mu(x + y) = \mu(x - (-y)) \]
\[ \geq \min\{\mu(x), \mu(-y)\} \]
\[ = \min\{\mu(x), \mu(y)\}, \text{ by proposition 4.3.} \]
Hence \( \mu(x + y) \geq \min\{\mu(x), \mu(y)\} \), for all \( x, y \in \mathcal{L}_N \).

Proposition 4.5. Let \( \mu \) be a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \). If \( \mu(x - y) = \mu(0) \), then \( \mu(x) = \mu(y) \), for all \( x, y \in \mathcal{L}_N \).
Proof:

Given that $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$ and assume that $\mu(x - y) = \mu(0)$.

To prove that $\mu(x) = \mu(y)$, for all $x, y \in \mathcal{L}_N$.

Since $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$, then we have

\[
\mu(x) = \mu(y + x - y) \\
= \mu(y + (x - y)) \\
\geq \min\{\mu(y), \mu(x - y)\}, \text{ by proposition 4.4} \\
= \min\{\mu(y), \mu(0)\}, \text{ by assumption} \\
= \mu(y), \text{ by proposition 4.2}
\]

$\mu(x) \geq \mu(y)$.

Also, $\mu(y) = \mu(x - x + y) \\
= \mu(x - (x - y)) \\
\geq \min\{\mu(x), \mu(x - y)\} \\
= \min\{\mu(x), \mu(0)\}, \text{ by assumption} \\
= \mu(x), \text{ by proposition 4.2}

$\mu(y) \geq \mu(x)$.

Hence $\mu(x) = \mu(y)$.

Proposition 4.6. If $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$, then $\mu(x) \geq \mu(y)$ whenever $x \leq y$, for all $x, y \in \mathcal{L}_N$.

Proof:

Given that $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$ and let $x \leq y$.

To prove that $\mu(x) \geq \mu(y)$, for all $x, y \in \mathcal{L}_N$.

Here $x \leq y$, then $x = x \land y$ and $y = x \lor y$.

$\implies \mu(x) = \mu(x \land y) \geq \max\{\mu(x), \mu(0)\} \geq \mu(y)$. 

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Proposition 4.7. Let $\mu$ be a fuzzy $\ell$-ideal of $\mathcal{L}_N$. If $\mu(x - y) = 1$, then $\mu(x) = \mu(y)$, for all $x, y \in \mathcal{L}_N$.

Proof:
Given that $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$ and assume that $\mu(x - y) = 1$.
To prove that $\mu(x) = \mu(y)$, for all $x, y \in \mathcal{L}_N$.
Let $x, y \in \mathcal{L}_N$ be arbitrary.
Since $\mathcal{L}_N$ is a $\ell$-near ring and $x - y \in \mathcal{L}_N$, then by proposition 4.2, we have $\mu(x - y) \leq \mu(0)$ and thus $1 \leq \mu(0)$, by assumption.
But $\mu(0) \leq 1$, since $\operatorname{Im}(\mu) \in [0, 1]$ and so $\mu(0) = 1$.
Hence $\mu(x - y) = \mu(0)$, by assumption.
Therefore $\mu(x) = \mu(y)$, by proposition 4.5.

Proposition 4.8. Let $\mu$ be a fuzzy $\ell$-ideal of $\mathcal{L}_N$. If $\mu(x) < \mu(y)$, then $\mu(x + y) = \mu(x)$, for all $x, y \in \mathcal{L}_N$.

Proof:
Given that $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$.
Assume that $\mu(x) < \mu(y)$. \hfill (1)
To prove that $\mu(x + y) = \mu(x)$, for all $x, y \in \mathcal{L}_N$.
Let $x, y \in \mathcal{L}_N$ be arbitrary.
$\implies \mu(x + y) \geq \min\{\mu(x), \mu(y)\}$, by proposition 4.4.
$\implies \mu(x + y) \geq \mu(x)$, by (1). \hfill (2)
Since $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$, then
\[
\mu(x) = \mu(y + x - y) \\
= \mu((y + x) - y) \\
\geq \min\{\mu(y + x), \mu(y)\} \\
= \mu(y + x) \\
= \mu(x + y), \text{by proposition } 4.1
\]
\[ \mu(x) \geq \mu(x + y). \]  
(3)

Hence \( \mu(x) = \mu(x + y) \), by (2) and (3).

**Proposition 4.9.** Let \( \mu \) and \( \sigma \) be any two fuzzy \( \ell \)-ideals of \( \mathcal{L}_N \). If \( \mu(x) < \sigma(x) \) and \( \mu(y) < \sigma(y) \), then \( \mu(x + y) < \sigma(x + y) \), for some \( x, y \in \mathcal{L}_N \).

**Proof:**

Given that \( \mu \) and \( \sigma \) are two fuzzy \( \ell \)-ideals of \( \mathcal{L}_N \).

Assume that \( \mu(x) < \sigma(x) \) and \( \mu(y) < \sigma(y) \).  
(1)

To prove that \( \mu(x + y) < \sigma(x + y) \), for some \( x, y \in \mathcal{L}_N \).

By proposition 4.4, we have \( \mu(x + y) \geq \min\{\mu(x), \mu(y)\} \) and \( \sigma(x + y) \geq \min\{\sigma(x), \sigma(y)\} \), for all \( x, y \in \mathcal{L}_N \).  
(2)

To prove that \( \mu(x + y) < \sigma(x + y) \).

**Case (i)**

Let \( \min\{\mu(x), \mu(y)\} = \mu(x) \) and \( \min\{\sigma(x), \sigma(y)\} = \sigma(x) \).

\[ \implies \mu(x) < \mu(y) \text{ and } \sigma(x) < \sigma(y). \]

\[ \implies \mu(x + y) = \mu(x) \text{ and } \sigma(x + y) = \sigma(x), \text{ by proposition 4.8.} \]

\[ \implies \mu(x + y) < \sigma(x + y), \text{ by (1) and (3).} \]

**Case (ii)**

Let \( \min\{\mu(x), \mu(y)\} = \mu(y) \) and \( \min\{\sigma(x), \sigma(y)\} = \sigma(y) \).

\[ \implies \mu(y) < \mu(x) \text{ and } \sigma(y) < \sigma(x). \]

\[ \implies \mu(x + y) = \mu(y) \text{ and } \sigma(x + y) = \sigma(y), \text{ by proposition 4.8.} \]

\[ \implies \mu(x + y) < \sigma(x + y), \text{ by (1) and (4).} \]

**Case (iii)**

Let \( \min\{\mu(x), \mu(y)\} = \mu(x) \), min\{\sigma(x), \sigma(y)\} = \sigma(y) \).

\[ \implies \mu(x) < \mu(y) \text{ and } \sigma(y) < \sigma(x). \]

\[ \implies \mu(x + y) = \mu(x) \text{ and } \sigma(x + y) = \sigma(y), \text{ by proposition 4.8.} \]

\[ \implies \mu(x + y) < \sigma(x + y), \text{ by (1) and (5).} \]

From (1) and (5), we have \( \mu(x) < \mu(y) < \sigma(y) \implies \mu(x) < \sigma(y) \).

\[ \implies \mu(x + y) < \sigma(x + y), \text{ by (6).} \]
Case (iv)
Let \( \min\{\mu(x), \mu(y)\} = \mu(y), \min\{\sigma(x), \sigma(y)\} = \sigma(x) \).
\[ \implies \mu(y) < \mu(x) \text{ and } \sigma(x) < \sigma(y). \quad (7) \]
\[ \implies \mu(x + y) = \mu(y) \text{ and } \sigma(x + y) = \sigma(x), \text{ by proposition 4.8.} \quad (8) \]
From (1) and (7), we have \( \mu(y) < \mu(x) < \sigma(x) \).
\[ \implies \mu(y) < \sigma(x). \]
\[ \implies \mu(x + y) < \sigma(x + y), \text{ by (8).} \]
Thus for all \( x, y \in L_N \), \( \mu(x + y) < \sigma(x + y) \) in all cases.

**Theorem 4.1.** Let \( S \) be any nonempty proper subset of \( L_N \). If \( \mu \) is a fuzzy \( \ell \)-ideal of \( L_N \), defined by \( \mu(x) = \begin{cases} g & \text{if } x \in S \\ h & \text{if } x \in L_N - S \end{cases} \), where \( g, h \in [0, 1] \) with \( g > h \), then \( S \) is a \( \ell \)-ideal of \( L_N \).

**Proof:**
Given that \( \mu \) is a fuzzy \( \ell \)-ideal of \( L_N \) and \( \mu(x) = \begin{cases} g & \text{if } x \in S \\ h & \text{if } x \in L_N - S \end{cases} \) where \( g, h \in [0, 1] \) with \( g > h \).
To prove that \( S \) is a \( \ell \)-ideal of \( L_N \).
It is enough to prove the following:

(i) \( x, y \in S \Rightarrow x - y, x \lor y \in S; \)

(ii) \( n + x - n \in S, \text{ for all } x \in S \text{ and } n \in L_N; \)

(iii) \( n \in L_N \text{ and } x \in S \Rightarrow nx \in S; \)

(iv) \( (n + i)n' - nn' \in S, \text{ for all } n, n' \in L_N \text{ and } i \in S; \)

(v) \( n \in N \text{ and } x \in S \text{ with } n \leq x \Rightarrow n \in S. \)

For (i)
Let \( x, y \in S \). Then \( \mu(x) = \mu(y) = g \) and \( \min\{\mu(x), \mu(y)\} = g \).

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\[ \Rightarrow \text{all the values of } \mu(x - y) \text{ and } \mu(x \lor y) \text{ are greater than or equal to } g. \]

But \( \mu \) has only two values \( g \) and \( h \) with \( g > h \). So, all values of \( \mu(x - y) \) and \( \mu(x \lor y) \) are equal to \( g \). Thus \( x - y, x \lor y \in S \).

**For (ii)**

Let \( x \in S \). Then \( \mu(x) = g \).
\[ \Rightarrow \mu(x) = \mu(n + x - n), \text{ since } \mu \text{ is a fuzzy } \ell\text{-ideal and for all } n \in \mathcal{L}_N. \]
\[ \Rightarrow \mu(n + x - n) = g. \]
\[ \Rightarrow n + x - n \in S. \]

**For (iii)**

Let \( x \in S \) and \( n \in \mathcal{L}_N \). Then \( \mu(x) = g \).
\[ \Rightarrow \mu(nx) \geq \mu(x), \text{ since } \mu \text{ is a fuzzy } \ell\text{-ideal.} \]
\[ \Rightarrow \mu(nx) = g. \]
\[ \Rightarrow nx \in S. \]

**For (iv)**

Let \( n, n' \in \mathcal{L}_N \) and \( i \in S \). Then \( \mu(i) = g \).
\[ \Rightarrow \mu((n + i)n' - nn') \geq \mu(i), \text{ since } \mu \text{ is a fuzzy } \ell\text{-ideal.} \]
\[ \Rightarrow \mu((n + i)n' - nn') \geq g. \]
\[ \Rightarrow (n + i)n' - nn' \in S. \]

**For (v)**

Let \( x \in S \) and \( n \leq x \). Then \( \mu(x) = g \).

As \( n \leq x \), then \( \mu(n) \geq \mu(x) \), by proposition 4.6.
\[ \Rightarrow \mu(n) = g. \]
\[ \Rightarrow n \in S. \]

Hence \( S \) is a \( \ell \)-ideal of \( \mathcal{L}_N \).

**Proposition 4.10.** If \( \mu \) is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \), then the level subsets \( \mu_t \), \( t \in [0, 1] \) are \( \ell \)-ideals of \( \mathcal{L}_N \).
Proof:

Given that $\mu$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$ and let $\mu_t = \{x \in \mathcal{L}_N : \mu(x) \geq t\}$ where $t \in [0, 1]$. To prove that the level subsets $\mu_t$ are $\ell$-ideals of $\mathcal{L}_N$.

We known that $\mu(0) \geq \mu(x)$, for all $x \in \mathcal{L}_N$.

$\implies \mu(0) \geq t$, for all $t \in [0, 1]$.

$\implies 0 \in \mu_t$ for all $t$.

$\implies \mu_t \neq \phi$.

Now we prove that the level subsets $\mu_t$, $t \in [0, 1]$ are $\ell$-ideal of $\mathcal{L}_N$.

It is enough to prove that $\mu_t$ satisfies the following:

(i) $x, y \in \mu_t \implies x \land y \in \mu_t$;

(ii) $n + x - n \in \mu_t$, for all $x \in \mu_t$ and $n \in \mathcal{L}_N$;

(iii) $n \in \mathcal{L}_N$ and $x \in \mu_t \implies nx \in \mu_t$;

(iv) $(n + i)n' - nn' \in \mu_t$, for all $n, n' \in \mathcal{L}_N$ and $i \in \mu_t$;

(v) $n \in \mathcal{L}_N$ and $x \in \mu_t$ with $n \preceq x \implies n \in \mu_t$.

For (i)

Let $x, y \in \mu_t$. Then $\mu(x) \geq t$, $\mu(y) \geq t$ and $\min\{\mu(x), \mu(y)\} \geq t$.

$\implies \mu(x \land y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}$, since $\mu$ is a fuzzy $\ell$-ideal.

$\implies \mu(x \land y) \geq t$, $\mu(x \lor y) \geq t$.

$\implies x \land y, x \lor y \in \mu_t$.

For (ii)

Let $x \in \mu_t$. Then $\mu(x) \geq t$.

$\implies \mu(x) = \mu(n + x - n)$, since $\mu$ is a fuzzy $\ell$-ideal.

$\implies \mu(n + x - n) \geq t$.

$\implies n + x - n \in \mu_t$. 

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For (iii)
Let \( x \in \mu_t \) and \( n \in \mathcal{L}_N \). Then \( \mu(x) \geq t \).
\[ \implies \mu(nx) \geq \mu(x), \text{ since } \mu \text{ is a fuzzy } \ell\text{-ideal.} \]
\[ \implies \mu(nx) \geq t. \]
\[ \implies nx \in \mu_t. \]

For (iv)
Let \( n, n' \in \mathcal{L}_N \) and \( i \in \mu_t \). Then \( \mu(i) \geq t \).
\[ \implies \mu((n + i)n' - nn') \geq \mu(i), \text{ since } \mu \text{ is a fuzzy } \ell\text{-ideal.} \]
\[ \implies \mu((n + i)n' - nn') \geq t. \]
\[ \implies ((n + i)n' - nn') \in \mu_t. \]

For (v)
Let \( x \in \mu_t \) and \( n \leq x \). Then \( \mu(x) \geq t \).
\[ \implies \mu(n) \geq \mu(x), \text{ by proposition 4.6.} \]
\[ \implies \mu(n) \geq t. \]
\[ \implies n \in \mu_t. \]

Hence the level subsets \( \mu_t \) are \( \ell\text{-ideal of } \mathcal{L}_N \).

**Theorem 4.2. Characterization Theorem** A fuzzy subset \( \mu \) of a \( \ell\text{-near ring } \mathcal{L}_N \), is a fuzzy \( \ell\text{-ideal of } \mathcal{L}_N \) if and only if the level subset \( \mu_t \), where \( t \in \text{Im}(\mu) \subseteq [0, 1] \) is a \( \ell\text{-ideal of } \mathcal{L}_N \).

**Proof:**
Assume that \( \mu \) is a fuzzy \( \ell\text{-ideal of } \mathcal{L}_N \).

To prove that the level subset \( \mu_t \) where \( t \in \text{Im}(\mu) \) is an \( \ell\text{-ideal of } \mathcal{L}_N \).

By the proposition 4.10, we get the proof of this part.

Conversely, assume that the level subset \( \mu_t, t \in \text{Im}(\mu) \) is an \( \ell\text{-ideal of } \mathcal{L}_N \).

To prove that \( \mu \) is a fuzzy \( \ell\text{-ideal of } \mathcal{L}_N \).

It is enough to prove \( \mu \) satisfies the following axioms:

(i) \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \);
(ii) \(\mu(xy) \geq \min\{\mu(x), \mu(y)\}\);

(iii) \(\mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}\);

(iv) \(\mu(x \land y) \geq \max\{\mu(x), \mu(y)\}\);

(v) \(\mu(x) = \mu(y + x - y)\);

(vi) \(\mu(xy) \geq \mu(y)\);

(vii) \(\mu((x + i)y - xy) \geq \mu(i)\) for all \(x, y, i \in \mathcal{L}_N\);

Let \(x, y \in \mathcal{L}_N\) be arbitrary.

For (i), (ii), (iii)

Let \(\min\{\mu(x), \mu(y)\} = t_1\).

\[\implies\] either \(\mu(x) = t_1\) and \(\mu(y) \geq \mu(x) = t_1\) or \(\mu(y) = t_1\) and \(\mu(x) \geq \mu(y) = t_1\).

\[\implies\] \(\mu(x) \geq t_1, \mu(y) \geq t_1\).

\[\implies\] \(x, y \in \mu_t\).

\[\implies\] \(x - y, xy, x \lor y \in \mu_t\), since \(\mu_t\) is a \(\ell\)-ideal.

\[\implies\] \(\mu(x - y) \geq t_1, \mu(xy) \geq t_1, \mu(x \lor y) \geq t_1\).

\[\implies\] \(\mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \mu(xy) \geq \min\{\mu(x), \mu(y)\}\).

\[\mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}\).

For (iv)

Let \(\max\{\mu(x), \mu(y)\} = t_2\).

Case (i) Let \(\max\{\mu(x), \mu(y)\} = \mu(x)\).

\[\implies\] \(\mu(x) = t_2\).

\[\implies\] \(x \land y \in \mu_t\), since \(\mu_t\) is a \(\ell\)-ideal of \(\mathcal{L}_N\) and \(x \land y \leq x\), as \(x \in \mu_t\).

\[\implies\] \(\mu(x \land y) \geq t_2 = \max\{\mu(x), \mu(y)\}\).

\[\implies\] \(\mu(x \land y) \geq \max\{\mu(x), \mu(y)\}\).

Case (ii) Let \(\max\{\mu(x), \mu(y)\} = \mu(y)\).

\[\implies\] \(\mu(y) = t_2\).
\[ \Rightarrow x \land y \in \mu_{t_2}, \text{ since } \mu_{t_2} \text{ is a } \ell\text{-ideal of } \mathcal{L}_N \text{ and } y \in \mu_{t_2}, \text{ as } x \land y \leq y. \]

\[ \Rightarrow \mu(x \land y) \geq t_2 = \max\{\mu(x), \mu(y)\}. \]

\[ \Rightarrow \mu(x \land y) \geq \max\{\mu(x), \mu(y)\}. \]

For (v)

Let \( t \in [0, 1] \) such that \( \mu(x) = t \).

\[ \Rightarrow x \in \mu_t, \]

\[ \Rightarrow x = y + x - y, \text{ since } (\mu_t, +) \text{ is a normal subgroup of } (\mathcal{L}_N, +). \]

\[ \Rightarrow \mu(x) = \mu(y + x - y). \]

For (vi)

Let \( t_1 \in [0, 1] \) such that \( t_1 = \mu(y) \).

\[ \Rightarrow y \in \mu_{t_1}, \]

\[ \Rightarrow xy \in \mu_{t_1}, \text{ since } \mu_{t_1} \text{ is a } \ell\text{-ideal and } x \in \mathcal{L}_N, \]

\[ \Rightarrow \mu(xy) \geq t_1 = \mu(y). \]

\[ \Rightarrow \mu(xy) \geq \mu(y). \]

For (vii)

Let \( i \in \mathcal{L}_N \) and \( t_2 \in [0, 1] \) such that \( \mu(i) = t_2 \).

\[ \Rightarrow (x + i)y - xy \in \mu_{t_2}, \text{ since } \mu_{t_2} \text{ is a } \ell\text{-ideal of } \mathcal{L}_N. \]

\[ \Rightarrow \mu((x + i)y - xy) \geq t_2 = \mu(i). \]

\[ \Rightarrow \mu((x + i)y - xy) \geq \mu(i). \]

Hence \( \mu \) is a \( \ell\)-ideal of \( \mathcal{L}_N \).

**Theorem 4.3.** If \( I \) is a \( \ell\)-ideal of \( \mathcal{L}_N \), then there exists a fuzzy \( \ell\)-ideal \( \mu \) of \( \mathcal{L}_N \) such that \( \mu_t = I \), for any \( t \in (0, 1) \).

**Proof:**

Assume that \( I \) is a \( \ell\)-ideal of \( \mathcal{L}_N \).

To prove that there exists a fuzzy \( \ell\)-ideal \( \mu \) of \( \mathcal{L}_N \) such that \( \mu_t = I \).

Let \( \mu : \mathcal{L}_N \rightarrow [0, 1] \) be a fuzzy set defined by \( \mu(x) = \begin{cases} t & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases} \)

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where \( t \) is a fixed number in \((0, 1)\). Then clearly \( \mu_t = I \).

Next we prove \( \mu \) is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \).

It is enough to prove \( \mu \) satisfies the following conditions:

\begin{itemize}
  \item [(i)] \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \);
  \item [(ii)] \( \mu(xy) \geq \min\{\mu(x), \mu(y)\} \);
  \item [(iii)] \( \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\} \);
  \item [(iv)] \( \mu(x \land y) \geq \max\{\mu(x), \mu(y)\} \);
  \item [(v)] \( \mu(x) = \mu(y + x - y) \);
  \item [(vi)] \( \mu(xy) \geq \mu(y) \);
  \item [(vii)] \( \mu((x + i)y - xy) \geq \mu(i) \), for all \( x, y, i \in \mathcal{L}_N \).
\end{itemize}

Let \( x, y, i \in \mathcal{L}_N \) be arbitrary.

For \((i), (ii), (iii), (iv)\)

Case (i) \quad \text{Let } x, y \in I.

Then \( \mu(x) = \mu(y) = t \) and \( \min\{\mu(x), \mu(y)\} = \max\{\mu(x), \mu(y)\} = t \).

Since \( I \) is a \( \ell \)-ideal of \( \mathcal{L}_N \), then \( x - y, xy, x \lor y \in I \) and \( x \land y \in I \), as \( x \land y \leq x \).

\[ \Rightarrow \mu(x - y) = \mu(xy) = \mu(x \lor y) = \mu(x \land y) = t . \]

\[ \Rightarrow \mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \mu(xy) \geq \min\{\mu(x), \mu(y)\}; \]

\[ \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}, \mu(x \land y) \geq \max\{\mu(x), \mu(y)\} \].

Case (ii) \quad \text{Let } x, y \notin I.

Then \( \mu(x) = \mu(y) = 0 \) and \( \min\{\mu(x), \mu(y)\} = \max\{\mu(x), \mu(y)\} = 0 \).

Here \( x \lor y, x \land y, x - y, xy \) may either belong to \( I \) or to \( \mathcal{L}_N \).

So their images under \( \mu \) will be either \( t \) or \( 0 \).

If \( \mu(x - y) = t \), then \( \mu(x - y) = t \geq 0 = \min\{\mu(x), \mu(y)\} \);

If \( \mu(x - y) = 0 \), then \( \mu(x - y) = 0 = \min\{\mu(x), \mu(y)\} \);
If \( \mu(xy) = t \), then \( \mu(xy) = t \geq 0 = \min\{\mu(x), \mu(y)\} \);
If \( \mu(xy) = 0 \), then \( \mu(xy) = 0 = \min\{\mu(x), \mu(y)\} \);
If \( \mu(x \lor y) = t \), then \( \mu(x \lor y) = t \geq 0 = \min\{\mu(x), \mu(y)\} \);
If \( \mu(x \lor y) = 0 \), then \( \mu(x \lor y) = 0 = \min\{\mu(x), \mu(y)\} \);
If \( \mu(x \land y) = t \), then \( \mu(x \land y) = t \geq 0 = \max\{\mu(x), \mu(y)\} \);
If \( \mu(x \land y) = 0 \), then \( \mu(x \land y) = 0 = \max\{\mu(x), \mu(y)\} \).

Case (iii) Let \( x \in I \) and \( y \notin I \).
Then \( \mu(x) = t \), \( \mu(y) = 0 \) and \( \min\{\mu(x), \mu(y)\} = 0 \), \( \max\{\mu(x), \mu(y)\} = t \).
Here \( x - y, xy, x \lor y \) may either belong to \( I \) or to \( \mathcal{L}_N \). So their images under \( \mu \) will be either \( t \) or \( 0 \). Then by case (ii), we have \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \), \( \mu(xy) \geq \min\{\mu(x), \mu(y)\} \), \( \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\} \).
Since \( I \) is a \( t \)-ideal of \( \mathcal{L}_N \) and as \( x \land y \leq x \), then we have \( x \land y \in I \).
\[ \implies \mu(x \land y) = t = \max\{\mu(x), \mu(y)\} \]

For (v)
Assume that \( \mu(x) > \mu(y + x - y) \), for some \( x, y \in \mathcal{L}_N \).
\[ \implies \mu(x) = t \) and \( \mu(y + x - y) = 0 \), since \( \mu \) is two valued.
\[ \implies x \in I \) and \( y + x - y \notin I \).
Which is a contradiction to \( (I, +) \) is a normal subgroup of \( (\mathcal{L}_N, +) \).
Therefore \( \mu(x) \leq \mu(y + x - y) \). \hfill (1)

Suppose that \( \mu(y + x - y) > \mu(x) \).
\[ \implies \mu(x) = 0 \) and \( \mu(y + x - y) = t \), since \( \mu \) is two valued.
\[ \implies x \notin I \) and \( y + x - y \in I \).
Since \( (I, +) \) is a normal subgroup of \( (\mathcal{L}_N, +) \), then we can write
\( x = x + (y + x - y) - x \) and thus \( x \in I \).
Which is a contradiction to \( x \notin I \).
Therefore \( \mu(x) \geq \mu(y + x - y) \). \hfill (2)
Hence \( \mu(x) = \mu(y + x - y) \), by (1) and (2).
For (vi)
Assume that \( \mu(xy) < \mu(y) \), for some \( x, y \in \mathcal{L}_N \).
\[ \implies \mu(y) = t \text{ and } \mu(xy) = 0, \text{ since } \mu \text{ is two valued.} \]
\[ \implies y \in I \text{ and } xy \notin I. \]
Which is a contradiction to \( I \) is a \( \ell \)-ideal.
Hence \( \mu(xy) \geq \mu(y) \).

For (vii)
Assume that \( \mu((x+i)y - xy) < \mu(i) \), for some \( x, y, i \in \mathcal{L}_N \).
\[ \implies \mu((x+i)y - xy) = 0 \text{ and } \mu(i) = t, \text{ since } \mu \text{ is two valued.} \]
\[ \implies i \in I \text{ and } ((x+i)y - xy) \notin I. \]
Which is a contradiction to \( I \) is a \( \ell \)-ideal.
\[ \implies \mu((x+i)y - xy) \geq \mu(i). \]
Hence \( \mu \) is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \).

**Definition 4.2.** Let \( \mu \) be a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \). Then the \( \ell \)-ideal \( \mu_t \) of \( \mu, t \in [0,1] \) with \( t \leq \mu(0) \) is called a level \( \ell \)-ideal of \( \mu \).

**Proposition 4.11.** Two level \( \ell \)-ideals \( \mu_{t_1} \) and \( \mu_{t_2} \) with \( t_1 < t_2 \) of a fuzzy \( \ell \)-ideal \( \mu \) of \( \mathcal{L}_N \) are equal if and only if there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

**Proof:**
Given that \( \mu_{t_1} \) and \( \mu_{t_2} \) are two level \( \ell \)-ideals of a fuzzy \( \ell \)-ideal \( \mu \) of \( \mathcal{L}_N \).
Assume that \( \mu_{t_1} \) and \( \mu_{t_2} \) are equal with \( t_1 < t_2 \).
To prove that there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).
Suppose that there is an \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).
\[ \implies \mu(x) \leq t_1 \text{ and } \mu(x) \geq t_2. \]
\[ \implies x \notin \mu_{t_1} \text{ and } x \in \mu_{t_2}. \]
Which is a contradiction to \( \mu_{t_1} = \mu_{t_2} \).
Therefore there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

Conversely, assume that there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

To prove that \( \mu_{t_1} \) and \( \mu_{t_2} \) are equal.

Let \( \mu_{t_1} = \{ x \in \mathcal{L}_N : \mu(x) \geq t_1 \} \) and \( \mu_{t_2} = \{ x \in \mathcal{L}_N : \mu(x) \geq t_2 \} \) with \( t_1 < t_2 \).

Then clearly \( \mu_{t_2} \subseteq \mu_{t_1} \). It is enough to prove that \( \mu_{t_1} \subseteq \mu_{t_2} \).

Let \( x \in \mu_{t_1} \). Then \( \mu(x) \geq t_1 \). Also \( \mu(x) \geq t_2 \), since \( \mu(x) \not\in t_2 \) and so \( x \in \mu_{t_2} \).

Hence \( \mu_{t_1} \subseteq \mu_{t_2} \).

**Theorem 4.4.** Let \( \mu \) be any fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \). Let \( \mu_{t_1} \) and \( \mu_{t_2} \) be any two level \( \ell \)-ideals of \( \mu \). Then

1. \( \mu_{t_1} \cup \mu_{t_2} = \{ x \in \mathcal{L}_N : \mu(x) \geq \min\{t_1, t_2\} \} \);

2. \( \mu_{t_1} \cap \mu_{t_2} = \{ x \in \mathcal{L}_N : \mu(x) \geq \max\{t_1, t_2\} \} \).

are also level \( \ell \)-ideals of \( \mu \).

**Proof:**

Given that \( \mu \) is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \) and let \( \mu_{t_1} \) and \( \mu_{t_2} \) be any two level \( \ell \)-ideals of \( \mathcal{L}_N \). To prove that (1) and (2) are level \( \ell \)-ideals of \( \mu \).

*For (1)*

It is enough to prove following axioms:

(i) \( x, y \in \mu_{t_1} \cup \mu_{t_2} \Rightarrow x - y, x \lor y \in \mu_{t_1} \cup \mu_{t_2} \);

(ii) \( n + x - n \in \mu_{t_1} \cup \mu_{t_2} \), for all \( x \in \mu_{t_1} \cup \mu_{t_2} \) and \( n \in \mathcal{L}_N \);

(iii) \( n \in \mathcal{L}_N, x \in \mu_{t_1} \cup \mu_{t_2} \Rightarrow nx \in \mu_{t_1} \cup \mu_{t_2} \);

(iv) \( (n + i)n' - nn' \in \mu_{t_1} \cup \mu_{t_2} \), for all \( i \in \mu_{t_1} \cup \mu_{t_2} \) and \( n, n' \in \mathcal{L}_N \);

(v) \( x \in \mu_{t_1} \cup \mu_{t_2} \) and \( n \in \mathcal{L}_N \) with \( n \leq x \Rightarrow n \in \mu_{t_1} \cup \mu_{t_2} \).
For (i)

Let \( x, y \in \mu_1 \cup \mu_2 \).

Then \( x, y \in \mathcal{L}_N \) such that \( \mu(x) \geq \min\{t_1, t_2\} \) and \( \mu(y) \geq \min\{t_1, t_2\} \).

If \( \min\{t_1, t_2\} = t_1 \), then \( x, y \in \mathcal{L}_N \) such that \( \mu(x) \geq t_1 \) and \( \mu(y) \geq t_1 \).

\[ \implies x, y \in \mu_1. \]

\[ \implies x - y, x \lor y \in \mu_1, \text{ since } \mu_1 \text{ is a } \ell\text{-ideal of } \mathcal{L}_N. \]

\[ \implies x - y, x \lor y \in \mathcal{L}_N \text{ such that } \mu(x - y) \geq t_1, \mu(x \lor y) \geq t_1. \]

\[ \implies x - y, x \lor y \in \mathcal{L}_N \text{ such that } \mu(x - y) \geq \min\{t_1, t_2\}, \mu(x \lor y) \geq \min\{t_1, t_2\}. \]

\[ \implies x - y, x \lor y \in \mu_1 \cup \mu_2. \]

If \( \min\{t_1, t_2\} = t_2 \), then \( x, y \in \mathcal{L}_N \) such that \( \mu(x) \geq t_2 \) and \( \mu(y) \geq t_2 \).

\[ \implies x, y \in \mu_2. \]

\[ \implies x - y, x \lor y \in \mu_2, \text{ since } \mu_2 \text{ is a } \ell\text{-ideal of } \mathcal{L}_N. \]

\[ \implies x - y, x \lor y \in \mathcal{L}_N \text{ such that } \mu(x - y) \geq t_2, \mu(x \lor y) \geq t_2. \]

\[ \implies x - y, x \lor y \in \mathcal{L}_N \text{ such that } \mu(x - y) \geq \min\{t_1, t_2\}, \mu(x \lor y) \geq \min\{t_1, t_2\}. \]

\[ \implies x - y, x \lor y \in \mu_1 \cup \mu_2. \]

For (ii)

Let \( x \in \mu_1 \cup \mu_2 \). Then \( x \in \mathcal{L}_N \) such that \( \mu(x) \geq \min\{t_1, t_2\} \).

If \( \min\{t_1, t_2\} = t_1 \), then \( \mu(x) \geq t_1 \).

\[ \implies x \in \mu_1 \text{ such that } x = n + x - n, \text{ since } \mu_1 \text{ is a } \ell\text{-ideal and for all } n \in \mathcal{L}_N. \]

\[ \implies \mu(x) = \mu(n + x - n). \]

\[ \implies \mu(n + x - n) \geq t_1 = \min\{t_1, t_2\}. \]

\[ \implies n + x - n \in \mu_1 \cup \mu_2. \]

If \( \min\{t_1, t_2\} = t_2 \), then \( \mu(x) \geq t_2 \).

\[ \implies x \in \mu_2 \text{ such that } x = n + x - n, \text{ since } \mu_2 \text{ is a } \ell\text{-ideal and for all } n \in \mathcal{L}_N. \]

\[ \implies \mu(x) = \mu(n + x - n). \]

\[ \implies \mu(n + x - n) \geq t_2 = \min\{t_1, t_2\}. \]

\[ \implies n + x - n \in \mu_1 \cup \mu_2. \]
For (iii) 
Let \( x \in \mu_1 \cup \mu_2 \) and \( n \in \mathcal{L}_N \).
Then \( x \in \mathcal{L}_N \) such that \( \mu(x) \geq \min\{t_1, t_2\} \).
If \( \min\{t_1, t_2\} = t_1 \), then \( \mu(x) \geq t_1 \).
\[ \implies x \in \mu_1. \]
\[ \implies nx \in \mu_1, \text{ since } \mu_1 \text{ is a } \ell \text{-ideal of } \mathcal{L}_N. \]
\[ \implies \mu(nx) \geq t_1. \]
\[ \implies \mu(nx) \geq \min\{t_1, t_2\}. \]
\[ \implies nx \in \mu_1 \cup \mu_2. \]
If \( \min\{t_1, t_2\} = t_2 \), then \( \mu(x) \geq t_2 \).
\[ \implies x \in \mu_2. \]
\[ \implies nx \in \mu_2, \text{ since } \mu_2 \text{ is a } \ell \text{-ideal of } \mathcal{L}_N. \]
\[ \implies \mu(nx) \geq t_2. \]
\[ \implies \mu(nx) \geq \min\{t_1, t_2\}. \]
\[ \implies nx \in \mu_1 \cup \mu_2. \]

For (iv) 
Let \( i \in \mu_1 \cup \mu_2 \) and \( n, n' \in \mathcal{L}_N \).
Then \( i \in \mathcal{L}_N \) such that \( \mu(i) \geq \min\{t_1, t_2\} \).
If \( \min\{t_1, t_2\} = t_1 \), then \( \mu(i) \geq t_1 \).
\[ \implies i \in \mu_1. \]
\[ \implies (n + i)n' - nn' \in \mu_1, \text{ since } \mu_1 \text{ is a } \ell \text{-ideal of } \mathcal{L}_N. \]
\[ \implies \mu((n + i)n' - nn') \geq t_1. \]
\[ \implies \mu((n + i)n' - nn') \geq \min\{t_1, t_2\}. \]
\[ (n + i)n' - nn' \in \mu_1 \cup \mu_2. \]
If \( \min\{t_1, t_2\} = t_2 \), then \( \mu(i) \geq t_2 \).
\[ \implies i \in \mu_2. \]
\[ \implies (n + i)n' - nn' \in \mu_2, \text{ since } \mu_2 \text{ is a } \ell \text{-ideal of } \mathcal{L}_N. \]
\[ \Rightarrow \mu((n + i)n' - nn') \geq t_2. \]
\[ \Rightarrow \mu((n + i)n' - nn') \geq \min\{t_1, t_2\}. \]
\[ \Rightarrow (n + i)n' - nn' \in \mu_1 \cup \mu_2. \]

For (v)

Let \( x \in \mu_1 \cup \mu_2 \) and \( n \in \mathcal{L}_N \) with \( n \leq x \).
Then \( x \in \mathcal{L}_N \) such that \( \mu(x) \geq \min\{t_1, t_2\} \).
If \( \min\{t_1, t_2\} = t_1 \), then \( \mu(x) \geq t_1 \).
\[ \Rightarrow x \in \mu_1 \text{ with } n \leq x. \]
\[ \Rightarrow \mu(n) \geq \mu(x), \text{ by proposition 4.6.} \]
\[ \Rightarrow \mu(n) \geq t_1. \]
\[ \Rightarrow n \in \mu_1 \cup \mu_2. \]
If \( \min\{t_1, t_2\} = t_2 \), then \( \mu(x) \geq t_2 \).
\[ \Rightarrow x \in \mu_2 \text{ with } n \leq x. \]
\[ \Rightarrow \mu(n) \geq \mu(x), \text{ by proposition 4.6.} \]
\[ \Rightarrow \mu(n) \geq t_2. \]
\[ \Rightarrow n \in \mu_1 \cup \mu_2. \]

Hence \( \mu_1 \cup \mu_2 \) is a level \( \ell \)-ideal of \( \mathcal{L}_N \).

For (2)

It is enough to prove following axioms:

(i) \( x, y \in \mu_1 \cap \mu_2 \Rightarrow x - y, x \lor y \in \mu_1 \cap \mu_2; \)

(ii) \( n + x - n \in \mu_1 \cap \mu_2 \), for all \( x \in \mu_1 \cap \mu_2 \) and \( n \in \mathcal{L}_N; \)

(iii) \( n \in \mathcal{L}_N, x \in \mu_1 \cap \mu_2 \Rightarrow nx \in \mu_1 \cap \mu_2; \)

(iv) \( (n + i)n' - nn' \in \mu_1 \cap \mu_2 \), for all \( i \in \mu_1 \cup \mu_2 \) and \( n, n' \in \mathcal{L}_N; \)

(v) \( x \in \mu_1 \cap \mu_2 \) and \( n \in \mathcal{L}_N \) with \( n \leq x \Rightarrow n \in \mu_1 \cap \mu_2. \)
For(i)
Let \(x, y \in \mu_{t_1} \cap \mu_{t_2}\).
Then \(x, y \in \mathcal{L}_N\) such that \(\mu(x) \geq \max\{t_1, t_2\}\) and \(\mu(y) \geq \max\{t_1, t_2\}\).
If \(\max\{t_1, t_2\} = t_1\), then \(x, y \in \mathcal{L}_N\) such that \(\mu(x) \geq t_1\) and \(\mu(y) \geq t_1\).
\(\implies x, y \in \mu_{t_1}\).
\(\implies x - y, x \lor y \in \mu_{t_1}\), since \(\mu_{t_1}\) is a \(\ell\)-ideal of \(\mathcal{L}_N\).
\(\implies x - y, x \lor y \in \mathcal{L}_N\) such that \(\mu(x - y) \geq t_1, \mu(x \lor y) \geq t_1\).
\(\implies x - y, x \lor y \in \mathcal{L}_N\) such that \(\mu(x - y) \geq \max\{t_1, t_2\}, \mu(x \lor y) \geq \max\{t_1, t_2\}\).
\(\implies x - y, x \lor y \in \mu_{t_1} \cap \mu_{t_2}\).
If \(\max\{t_1, t_2\} = t_2\), then \(x, y \in \mathcal{L}_N\) such that \(\mu(x) \geq t_2\) and \(\mu(y) \geq t_2\).
\(\implies x, y \in \mu_{t_2}\).
\(\implies x - y, x \lor y \in \mu_{t_2}\), since \(\mu_{t_2}\) is a \(\ell\)-ideal of \(\mathcal{L}_N\).
\(\implies x - y, x \lor y \in \mathcal{L}_N\) such that \(\mu(x - y) \geq t_2, \mu(x \lor y) \geq t_2\).
\(\implies x - y, x \lor y \in \mathcal{L}_N\) such that \(\mu(x - y) \geq \max\{t_1, t_2\}, \mu(x \lor y) \geq \max\{t_1, t_2\}\).
\(\implies x - y, x \lor y \in \mu_{t_1} \cap \mu_{t_2}\).
For(ii)
Let \(x \in \mu_{t_1} \cap \mu_{t_2}\). Then \(x \in \mathcal{L}_N\) such that \(\mu(x) \geq \max\{t_1, t_2\}\).
If \(\max\{t_1, t_2\} = t_1\), then \(\mu(x) \geq t_1\).
\(\implies x \in \mu_{t_1}\) such that \(x = n + x - n\), since \(\mu_{t_1}\) is a \(\ell\)-ideal and for all \(n \in \mathcal{L}_N\).
\(\implies \mu(x) = \mu(n + x - n)\).
\(\implies \mu(n + x - n) \geq t_1 = \max\{t_1, t_2\}\).
\(\implies n + x - n \in \mu_{t_1} \cap \mu_{t_2}\).
If \(\max\{t_1, t_2\} = t_2\), then \(\mu(x) \geq t_2\).
\(\implies x \in \mu_{t_2}\) such that \(x = n + x - n\), since \(\mu_{t_2}\) is a \(\ell\)-ideal and for all \(n \in \mathcal{L}_N\).
\(\implies \mu(x) = \mu(n + x - n)\).
\(\implies \mu(n + x - n) \geq t_2 = \max\{t_1, t_2\}\).
\(\implies n + x - n \in \mu_{t_1} \cap \mu_{t_2}\).
For (iii)
Let \( x \in \mu_{t_1} \cap \mu_{t_2} \) and \( n \in \mathcal{L}_N \).

Then \( x \in \mathcal{L}_N \) such that \( \mu(x) \geq \max\{t_1, t_2\} \).

If \( \max\{t_1, t_2\} = t_1 \), then \( \mu(x) \geq t_1 \).

\[ \Rightarrow x \in \mu_{t_1}, \]

\[ \Rightarrow nx \in \mu_{t_1}, \text{ since } \mu_{t_1} \text{ is a } \ell \text{-ideal of } \mathcal{L}_N. \]

\[ \Rightarrow \mu(nx) \geq t_1. \]

\[ \Rightarrow \mu(nx) \geq \max\{t_1, t_2\}. \]

\[ \Rightarrow nx \in \mu_{t_1} \cap \mu_{t_2}. \]

If \( \max\{t_1, t_2\} = t_2 \), then \( \mu(x) \geq t_2 \).

\[ \Rightarrow x \in \mu_{t_2}, \]

\[ \Rightarrow nx \in \mu_{t_2}, \text{ since } \mu_{t_2} \text{ is a } \ell \text{-ideal of } \mathcal{L}_N. \]

\[ \Rightarrow \mu(nx) \geq t_2. \]

\[ \Rightarrow \mu(nx) \geq \max\{t_1, t_2\}. \]

\[ \Rightarrow nx \in \mu_{t_1} \cap \mu_{t_2}. \]

For (iv)
Let \( i \in \mu_{t_1} \cap \mu_{t_2} \) and \( n, n' \in \mathcal{L}_N \).

Then \( i \in \mathcal{L}_N \) such that \( \mu(i) \geq \max\{t_1, t_2\} \).

If \( \max\{t_1, t_2\} = t_1 \), then \( \mu(i) \geq t_1 \).

\[ \Rightarrow i \in \mu_{t_1}, \]

\[ \Rightarrow (n + i)n' - nn' \in \mu_{t_1}, \text{ since } \mu_{t_1} \text{ is a } \ell \text{-ideal of } \mathcal{L}_N. \]

\[ \Rightarrow \mu((n + i)n' - nn') \geq t_1. \]

\[ \Rightarrow \mu((n + i)n' - nn') \geq \max\{t_1, t_2\}. \]

\[ \Rightarrow (n + i)n' - nn' \in \mu_{t_1} \cap \mu_{t_2}. \]

If \( \max\{t_1, t_2\} = t_2 \), then \( \mu(i) \geq t_2 \).

\[ \Rightarrow i \in \mu_{t_2}, \]

\[ \Rightarrow (n + i)n' - nn' \in \mu_{t_2}, \text{ since } \mu_{t_2} \text{ is a } \ell \text{-ideal of } \mathcal{L}_N. \]
\[\mu((n+i)n' - nn') \geq t_2.\]
\[\mu((n+i)n' - nn') \geq \max\{t_1, t_2\}.\]
\[(n+i)n' - nn' \in \mu_1 \cap \mu_2.\]

For \(n\)

Let \(x \in \mu_1 \cap \mu_2\) and \(n \in \mathcal{L}_N\) with \(n \leq x\).
Then \(x \in \mathcal{L}_N\) such that \(\mu(x) \geq \max\{t_1, t_2\}\).
If \(\max\{t_1, t_2\} = t_1\), then \(\mu(x) \geq t_1\).
\[x \in \mu_1\text{ with }n \leq x.\]
\[\mu(n) \geq \mu(x),\text{ since }\mu_1\text{ is a }\ell\text{-ideal of }\mathcal{L}_N\text{ and by proposition 4.6.}\]
\[\mu(n) \geq t_1.\]
\[\mu(n) \geq \max\{t_1, t_2\}.\]
\[n \in \mu_1 \cap \mu_2.\]
If \(\max\{t_1, t_2\} = t_2\), then \(\mu(x) \geq t_2\).
\[x \in \mu_2\text{ with }n \leq x.\]
\[\mu(n) \geq \mu(x),\text{ since }\mu_2\text{ is a }\ell\text{-ideal of }\mathcal{L}_N\text{ and by proposition 4.6.}\]
\[\mu(n) \geq t_2.\]
\[\mu(n) \geq \max\{t_1, t_2\}.\]
\[n \in \mu_1 \cap \mu_2.\]

Hence \(\mu_1 \cap \mu_2\) is a level \(\ell\)-ideal of \(\mathcal{L}_N\).

**Theorem 4.5.** Let \(\mu\) be any fuzzy \(\ell\)-ideal of \(\mathcal{L}_N\). If \(\text{Im } (\mu) = \{t_1, t_2, t_3, \ldots t_n\}\) with \(t_1 > t_2 > t_3 > \cdots > t_n\), then we have the following chain of level \(\ell\)-ideals of \(\mu\). That is \(\mu_1 \subseteq \mu_2 \subseteq \mu_3 \subseteq \cdots \subseteq \mu_n = \mathcal{L}_N\).

**Proof:**

Given that \(\mu\) is any fuzzy \(\ell\)-ideal of \(\mathcal{L}_N\) and let \(\text{Im } (\mu) = \{t_1, t_2, t_3, \ldots t_n\}\) with \(t_1 > t_2 > t_3 > \cdots > t_n\). \(\text{(1)}\)

To prove that \(\mu_1 \subseteq \mu_2 \subseteq \mu_3 \subseteq \cdots \subseteq \mu_n = \mathcal{L}_N\).

Let \(x_1 \in \mu_1\) be arbitrary. Then \(x_1 \in \mathcal{L}_N\) such that \(\mu(x_1) \geq t_1\).
\[ \implies \mu(x_1) > t_2, \text{ by (1)}. \]
\[ \implies x_1 \in \mu_{t_2}. \]
\[ \implies \mu_1 \subseteq \mu_{t_2}. \]
Again \( x_2 \in \mu_{t_2} \). Then \( x_2 \in \mathcal{L}_N \) such that \( \mu(x_2) \geq t_2 \).
\[ \implies \mu(x_2) > t_3, \text{ by (1)}. \]
\[ \implies x_2 \in \mu_{t_3}. \]
\[ \implies \mu_2 \subseteq \mu_{t_3}. \]
\[ \implies \mu_1 \subseteq \mu_{t_2} \subseteq \mu_{t_3}. \]
Proceeding like this, we get \( \mu_1 \subseteq \mu_{t_2} \subseteq \mu_{t_3} \subseteq \cdots \subseteq \mu_{t_n} = \mathcal{L}_N \).

**Theorem 4.6.** Two fuzzy \( \ell \)-ideals \( \mu \) and \( \sigma \) of a \( \ell \)-near ring \( \mathcal{L}_N \) whose images are of finite cardinality are equal if and only if \( \text{Im}(\mu) = \text{Im}(\sigma) \) and \( \mathcal{F}_\mu = \mathcal{F}_\sigma \).

**Proof:**

Given that \( \mu \) and \( \sigma \) are two fuzzy \( \ell \)-ideals of a \( \ell \)-near ring \( \mathcal{L}_N \), whose images are of finite cardinality.

Assume that \( \mu \) and \( \sigma \) are equal. That is \( \mu(x) = \sigma(x) \) for all \( x \in \mathcal{L}_N \).

To prove that \( \text{Im}(\mu) = \text{Im}(\sigma) \) and \( \mathcal{F}_\mu = \mathcal{F}_\sigma \).

Let \( x \in \mathcal{L}_N \) be arbitrary and \( \mu(x) \in \text{Im}(\mu) \).

\[ \implies \mu(x) = \sigma(x) \in \text{Im}(\sigma), \text{ by assumption}. \]
\[ \implies \mu(x) \in \text{Im}(\sigma). \]
\[ \implies \text{Im}(\mu) \subseteq \text{Im}(\sigma). \]

Similarly, \( \text{Im}(\sigma) \subseteq \text{Im}(\mu) \).

Hence \( \text{Im}(\mu) = \text{Im}(\sigma) \).

Let \( \mu_t \in \mathcal{F}_\mu \) be arbitrary and \( t \leq \mu(0) \).

\[ \implies x \in \mu_t \text{ such that } \mu(x) \geq t, t \in \text{Im}(\mu). \]
\[ \implies \sigma(x) \geq t, \text{ since } \mu(x) = \sigma(x). \]
\[ \implies x \in \sigma_t. \]
\[ \implies \mu_t \subseteq \sigma_t. \]
Similarly, \( \sigma_i \subseteq \mu_i \).
\[
\implies \mu_i = \sigma_i \in \mathcal{F}_\sigma.
\]
\[
\implies \mathcal{F}_\mu \subseteq \mathcal{F}_\sigma.
\]
Similarly, \( \mathcal{F}_\sigma \subseteq \mathcal{F}_\mu \).

Hence \( \mathcal{F}_\mu = \mathcal{F}_\sigma \).

Conversely, assume that \( \text{Im}(\mu) = \text{Im}(\sigma) \) and \( \mathcal{F}_\mu = \mathcal{F}_\sigma \).

To prove that \( \mu \) and \( \sigma \) are equal.

Suppose that \( \mu(x) \neq \sigma(x) \), for some \( x \in \mathcal{L}_N \).

Then the cardinalities of \( \text{Im}(\mu) \) and \( \text{Im}(\sigma) \) are not equal and \( \mathcal{F}_\mu \neq \mathcal{F}_\sigma \).

Which is a contradiction. Hence \( \mu \) and \( \sigma \) are equal.

**Theorem 4.7.** If \( \{ \mu_i : i \in \Lambda \} \) is the collection of all fuzzy \( \ell \)-ideals of \( \mathcal{L}_N \), then \( \bigcap_{i \in \Lambda} \mu_i \) is a fuzzy \( \ell \)-ideals of \( \mathcal{L}_N \), where \( (\bigcap_{i \in \Lambda} \mu_i)(x) = \inf_{i \in \Lambda} \mu_i(x) \), for all \( x \in \mathcal{L}_N \), where \( \Lambda \) is any index set.

**Proof:**

Given that \( \{ \mu_i : i \in \Lambda \} \) is the collection of fuzzy \( \ell \)-ideals of \( \mathcal{L}_N \).

To prove that \( (\bigcap_{i \in \Lambda} \mu_i)(x) = \inf_{i \in \Lambda} \mu_i(x) \), is a fuzzy \( \ell \)-ideals of \( \mathcal{L}_N \), for all \( x \in \mathcal{L}_N \).

Let \( x, y \in \mathcal{L}_N \) be arbitrary. Then

\[
(\bigcap_{i \in \Lambda} \mu_i)(x - y) = \inf_i (\mu_i(x) - \mu_i(y))
\]
\[
\geq \inf_i (\min_i \mu_i(x), \min_i \mu_i(y))
\]
\[
= \min_i \{ \inf_i \mu_i(x), \inf_i \mu_i(y) \}
\]
\[
= \min_i \{(\cap \mu_i)(x), (\cap \mu_i)(y)\}.
\]

\[
(\bigcap_{i \in \Lambda} \mu_i)(xy) = \inf_i (\mu_i(xy))
\]
\[
\geq \inf_i (\min_i \mu_i(x), \min_i \mu_i(y))
\]
\[
= \min_i \{ \inf_i \mu_i(x), \inf_i \mu_i(y) \}
\]
\[
= \min_i \{(\cap \mu_i)(x), (\cap \mu_i)(y)\}.
\]
\[
(\bigcap_{i \in \Lambda} \mu_i)(x \lor y) = \inf_i (\mu_i(x) \lor \mu_i(y)) \\
\geq \inf_i (\min\{\mu_i(x), \mu_i(y)\}) \\
= \min\{\inf_i \mu_i(x), \inf_i \mu_i(y)\} \\
= \min\{(\bigcap_i \mu_i)(x), (\bigcap_i \mu_i)(y)\}.
\]
\[
(\bigcap_{i \in \Lambda} \mu_i)(x \land y) = \inf_i (\mu_i(x) \land \mu_i(y)) \\
\geq \inf_i (\max\{\mu_i(x), \mu_i(y)\}) \\
= \max\{\inf_i \mu_i(x), \inf_i \mu_i(y)\} \\
= \max\{(\bigcap_i \mu_i)(x), (\bigcap_i \mu_i)(y)\}.
\]
\[
(\bigcap_{i \in \Lambda} \mu_i)(x) = \inf_i (\mu_i(x)) \\
= \inf_i (\mu_i(x + y - y)) \\
= (\bigcap_i \mu_i)(x + y - y).
\]
\[
(\bigcap_{i \in \Lambda} \mu_i)(xy) = \inf_i (\mu_i(xy)) \\
\geq \inf_i (\mu_i(y)) \\
= (\bigcap_i \mu_i)(y).
\]
\[
(\bigcap_{i \in \Lambda} \mu_i)((x + z)y - xy) = \inf_i (\mu_i((x + z)y - xy)) \\
\geq \inf_i (\mu_i(z)) \\
= (\bigcap_i \mu_i)(z).
\]

Hence \(\bigcap_{i \in \Lambda} \mu_i\) is a fuzzy \(\ell\)-ideal of \(\mathcal{L}_N\).

**Theorem 4.8.** If \(\{\mu_i : i \in \Lambda\}\) is the collection of fuzzy \(\ell\)-ideals of \(\mathcal{L}_N\) such that \(\mu_1 \subseteq \mu_2 \subseteq \cdots \mu_{i-1} \subseteq \mu_i \subseteq \mu_{i+1} \subseteq \cdots\), then \(\bigcup_{i \in \Lambda} \mu_i\) is a fuzzy \(\ell\)-ideal of \(\mathcal{L}_N\).

**Proof:**

Given that \(\{\mu_i : i \in \Lambda\}\) is the collection of fuzzy \(\ell\)-ideals of \(\mathcal{L}_N\) such that \(\mu_1 \subseteq \mu_2 \subseteq \cdots \mu_{i-1} \subseteq \mu_i \subseteq \mu_{i+1} \subseteq \cdots\).
To prove that $\bigcup_{i \in \Lambda} \mu_i$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$.

For any $n, m \in \Lambda$, then there exists $i \in \Lambda$ such that $\mu_n(x) \leq \mu_i(x)$ and $\mu_m(y) \leq \mu_i(y)$.

\[ \Rightarrow \min\{\mu_n(x), \mu_m(y)\} \leq \min\{\mu_i(x), \mu_i(y)\}. \]  

(1)

Since $\mu_i$ is a fuzzy $\ell$-ideal of $\mathcal{L}_N$, we have $\mu_i(x - y) \geq \min\{\mu_i(x), \mu_i(y)\}$  

(2)

$\mu_i(xy) \geq \min\{\mu_i(x), \mu_i(y)\}$ and  

$\mu_i(x \lor y) \geq \min\{\mu_i(x), \mu_i(y)\}$  

(3)

(4)

From (1),(2),(3) and (4) we have $\mu_i(x - y) \geq \min\{\mu_n(x), \mu_m(y)\}$.

$\mu_i(xy) \geq \min\{\mu_n(x), \mu_m(y)\}$ and $\mu_i(x \lor y) \geq \min\{\mu_n(x), \mu_m(y)\}$.

Therefore $\min\{(\bigcup_{i \in \Lambda} \mu_i)(x), (\bigcup_{i \in \Lambda} \mu_i)(y)\} = \min\{\sup(\mu_i)(x), \sup(\mu_i)(y)\}$  

(5)

\[ = \sup(\min\{\mu_i(x), \mu_i(y)\}). \]

But $\min\{(\bigcup_{i \in \Lambda} \mu_i)(x), (\bigcup_{i \in \Lambda} \mu_i)(y)\} \leq \sup(\mu_i(x - y))$  

(2) and (5)

$((\bigcup_{i \in \Lambda} \mu_i)(x - y)) \geq \min\{(\bigcup_{i \in \Lambda} \mu_i)(x), (\bigcup_{i \in \Lambda} \mu_i)(y)\}$.  

From (3) and (5), we have

\[ \min\{(\bigcup_{i \in \Lambda} \mu_i)(x), (\bigcup_{i \in \Lambda} \mu_i)(y)\} = \sup(\min\{\mu_i(x), \mu_i(y)\}) \]

\[ \leq \sup(\mu_i(xy)) \]

(2)

$= ((\bigcup_{i \in \Lambda} \mu_i)(xy))$  

(2)

$((\bigcup_{i \in \Lambda} \mu_i)(xy)) \geq \min\{(\bigcup_{i \in \Lambda} \mu_i)(x), (\bigcup_{i \in \Lambda} \mu_i)(y)\}$.  

From (4) and (5), we have,

\[ \min\{(\bigcup_{i \in \Lambda} \mu_i)(x), (\bigcup_{i \in \Lambda} \mu_i)(y)\} = \sup(\min\{\mu_i(x), \mu_i(y)\}) \]

\[ \leq \sup(\mu_i(x \lor y)) \]

(2)

$= ((\bigcup_{i \in \Lambda} \mu_i)(x \lor y))$  

(2)

From (3) and (5), we have,
\[(\bigcup_{i \in \Lambda} \mu_i)(x \vee y) \geq \min\{\bigcup_{i \in \Lambda} \mu_i(x), (\bigcup_{i \in \Lambda} \mu_i(y)\}\cdot (\bigcup_{i \in \Lambda} \mu_i)(x \wedge y)\]
\[= \sup_{i}(\mu_i)(x \wedge y)\]
\[\geq \sup_{i}\max\{\mu_i(x), \mu_i(y)\}\]
\[= \max\{\sup_{i} \mu_i(x), \sup_{i} \mu_i(y)\}\]
\[= \max\{(\bigcup_{i \in \Lambda} \mu_i)(x), (\bigcup_{i \in \Lambda} \mu_i)(y)\}\].

\[(\bigcup_{i \in \Lambda} \mu_i)(x) = \sup_{i}(\mu_i(x))\]
\[= \sup_{i}(\mu_i(y + x - y))\]
\[= (\bigcup_{i} \mu_i)(y + x - x).\]

\[(\bigcup_{i \in \Lambda} \mu_i)(xy) = \sup_{i}(\mu_i(xy))\]
\[\geq \sup_{i}(\mu_i(y))\]
\[= (\bigcup_{i} \mu_i)(y).\]

\[(\bigcup_{i \in \Lambda} \mu_i)((x + z)y - xy) = \sup_{i}(\mu_i((x + z)y - xy))\]
\[\geq \sup_{i}(\mu_i(z))\]
\[= (\bigcup_{i} \mu_i)(z).\]

Hence \(\bigcup_{i \in \Lambda} \mu_i\) is a fuzzy \(\ell\)-ideal of \(\mathcal{L}_N\).

**Theorem 4.9.** A \(\ell\)-near ring \(\ell\)-homomorphic preimage of a fuzzy \(\ell\)-ideal is a fuzzy \(\ell\)-ideal of \(\mathcal{L}_N\).

**Proof:**

Given that \(f : \mathcal{L}_N \to \mathcal{L}'_N\) is a \(\ell\)-near ring \(\ell\)-homomorphism and let \(\nu\) be a fuzzy \(\ell\)-ideal of \(\mathcal{L}'_N\).

To prove that a \(\ell\)-near ring \(\ell\)-homomorphic preimage of a fuzzy \(\ell\)-ideal is a fuzzy \(\ell\)-ideal of \(\mathcal{L}_N\).

\[f^{-1}(\nu(x - y)) = \nu(f(x - y))\]
\[= \nu(f(x) - f(y))\]

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\[\begin{align*}
\geq & \quad \min\{\nu(f(x)), \nu(f(y))\} \\
& = \min\{f^{-1}(\nu(x)), f^{-1}(\nu(y))\} \\
\end{align*}\]

\[f^{-1}(\nu(xy)) = \nu(f(xy)) = \nu(f(x)f(y)) \geq \min\{\nu(f(x)), \nu(f(y))\} = \min\{f^{-1}(\nu(x)), f^{-1}(\nu(y))\}.\]

\[f^{-1}(\nu(x \lor y)) = \nu(f(x \lor y)) = \nu(f(x) \lor f(y)) \geq \min\{\nu(f(x)), \nu(f(y))\} = \min\{f^{-1}(\nu(x)), f^{-1}(\nu(y))\}.\]

\[f^{-1}(\nu(x \land y)) = \nu(f(x \land y)) = \nu(f(x) \land f(y)) \geq \max\{\nu(f(x)), \nu(f(y))\} = \max\{f^{-1}(\nu(x)), f^{-1}(\nu(y))\}.\]

\[f^{-1}(\nu(y + x - y)) = \nu(f(y + x - y)) = \nu(f(y) + f(x) - f(y)) = \nu(f(x)) = f^{-1}(\nu(x)).\]

\[f^{-1}(\nu(xy)) = \nu(f(xy)) = \nu(f(x)f(y)) \geq \nu(f(y)) = f^{-1}(\nu(y)).\]

\[f^{-1}(\nu((x + i)y - xy)) = \nu(f((x + i)y - xy)) = \nu((f(x) + f(i))f(y) - f(x)f(y))\]

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\[ \geq \nu(f(i)) \]
\[ = f^{-1}(\nu(i)). \]

Hence \( \ell \)-near ring \( \ell \)-homomorphic preimage of a fuzzy \( \ell \)-ideal is a fuzzy \( \ell \)-ideal.

**Theorem 4.10.** The \( \ell \)-homomorphic image of a fuzzy \( \ell \)-ideal of a \( \ell \)-near ring \( \mathcal{L}_N \) having the sup property is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \).

**Proof:**

Given that \( f : \mathcal{L}_N \to \mathcal{L}_N' \) is a \( \ell \)-homomorphism and let \( \mu \) be a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \) with the sup property and let \( \nu \) be the image of \( \mu \) under \( f \).

To prove \( \nu \) is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \).

Given \( f(x), f(y), f(i) \in f(\mathcal{L}_N) \) and let \( x_0 \in f^{-1}(f(x)), y_0 \in f^{-1}(f(y)) \) and \( i_0 \in f^{-1}(f(i)) \) be such that \( \mu(x_0) = \sup_{t \in f^{-1}(f(x))} \mu(t), \mu(y_0) = \sup_{t \in f^{-1}(f(y))} \mu(t) \)
and \( \mu(i_0) = \sup_{t \in f^{-1}(f(i))} \mu(t) \). Then,

\[ \nu(f(x) - f(y)) = \sup_{t \in f^{-1}(f(x) - f(y))} \mu(t) \]
\[ \geq \mu(x_0 - y_0) \]
\[ = \min\{\mu(x_0), \mu(y_0)\} \]
\[ = \min\{\sup_{t \in f^{-1}(f(x))} \mu(t), \sup_{t \in f^{-1}(f(y))} \mu(t)\} \]
\[ = \min\{\nu(f(x)), \nu(f(y))\}. \]

\[ \nu(f(x)f(y)) = \sup_{t \in f^{-1}(f(x)f(y))} \mu(t) \]
\[ \geq \mu(x_0y_0) \]
\[ = \min\{\mu(x_0), \mu(y_0)\} \]
\[ = \min\{\sup_{t \in f^{-1}(f(x))} \mu(t), \sup_{t \in f^{-1}(f(y))} \mu(t)\} \]
\[ = \min\{\nu(f(x)), \nu(f(y))\}. \]
\[
\begin{align*}
\nu(f(x) \lor f(y)) &= \sup_{t \in f^{-1}(f(x) \lor f(y))} \mu(t) \\
&\geq \mu(x_0 \lor y_0) \\
&\geq \min\{\mu(x_0), \mu(y_0)\} \\
&= \min\{\sup_{t \in f^{-1}(f(x))} \mu(t), \sup_{t \in f^{-1}(f(y))} \mu(t)\} \\
&= \min\{\nu(f(x)), \nu(f(y))\}
\end{align*}
\]
\[
\begin{align*}
\nu(f(x) \land f(y)) &= \sup_{t \in f^{-1}(f(x) \land f(y))} \mu(t) \\
&\geq \mu(x_0 \land y_0) \\
&\geq \max\{\mu(x_0), \mu(y_0)\} \\
&= \max\{\sup_{t \in f^{-1}(f(x))} \mu(t), \sup_{t \in f^{-1}(f(y))} \mu(t)\} \\
&= \max\{\nu(f(x)), \nu(f(y))\}.
\end{align*}
\]
\[
\begin{align*}
\nu(f(x + y - y)) &= \nu(f(y) + f(x) - f(y)) \\
&= \sup_{t \in f^{-1}(f(y) + f(x) - f(y))} \mu(t) \\
&\geq \mu(y_0 + x_0 - y_0) \\
&= \mu(x_0) \\
&= \nu(f(x)).
\end{align*}
\]
\[
\begin{align*}
\nu(f(xy)) &= \sup_{t \in f^{-1}(f(x)f(y))} \mu(t) \\
&\geq \mu((x_0)(y_0)) \\
&\geq \mu(f(y_0)) \\
&= \sup_{t \in f^{-1}(f(y))} \mu(t) \\
&= \nu(f(y)).
\end{align*}
\]
\[
\begin{align*}
\nu(f((x + i)y - xy)) &= \nu((f(x) + f(i))f(y) - f(x)f(y)) \\
&= \sup_{t \in f^{-1}((f(x) + f(i))f(y) - f(x)f(y))} \mu(t)
\end{align*}
\]
\[ \nu((x_0 + i_0)y_0 - x_0y_0)) \geq \nu(i_0) \\
= \sup_{t \in f^{-1}(f(i))} \mu(t) \\
= \nu(f(i)). \]

Hence \( \nu \) is a fuzzy \( \ell \)-ideal of \( \mathcal{L}_N \).

**Theorem 4.11.** Let \( f : \mathcal{L}_N \to \mathcal{L}'_N \) be a onto \( \ell \)-homomorphism and let \( \mu \) and \( \nu \) be a fuzzy \( \ell \)-ideals of \( \mathcal{L}_N \) and \( \mathcal{L}'_N \) respectively, such that \( \text{Im}(\mu) = \{t_0, t_1, \ldots, t_n\} \) with \( t_0 > t_1 > \ldots t_n \) and \( \text{Im}(\nu) = \{s_0, s_1, \ldots, s_m\} \) with \( s_0 > s_1 > \ldots s_m \). Then,

1. \( \text{Im}(f(\mu)) \subset \text{Im}(\mu) \) and the chain of level \( \ell \)-ideals of \( f(\mu) \) is \( f(\mu_0) \subset f(\mu_1) \subset \ldots \subset f(\mu_n) = \mathcal{L}'_N \);
2. \( \text{Im}(f^{-1}(\nu)) \subset \text{Im}(\nu) \) and the chain of level \( \ell \)-ideals of \( f^{-1}(\nu) \) is \( f^{-1}(\nu_0) \subset f^{-1}(\nu_1) \subset \ldots \subset f^{-1}(\nu_m) = \mathcal{L}_N \).

**Proof:**

Given that \( f : \mathcal{L}_N \to \mathcal{L}'_N \) is an onto \( \ell \)-homomorphism.

Let \( \mu \) and \( \nu \) be two fuzzy \( \ell \)-ideals of \( \mathcal{L}_N \) and \( \mathcal{L}'_N \) with \( \text{Im}(\mu) = \{t_0, t_1, \ldots, t_n\} \) and \( \text{Im}(\nu) = \{s_0, s_1, \ldots, s_m\} \).

To prove that (1) and (2).

For (1)

Since \( (f(\mu))(y) = \sup_{x \in f^{-1}(y)} \mu(x) \), for all \( y \in \mathcal{L}_N \).

Then obviously \( \text{Im}(f(\mu)) \subset \text{Im}(\mu) \) and if for any \( y \in \mathcal{L}_N \), then \( y \in f(\mu_i) \).

\[ \iff \text{there exists } x \in f^{-1}(y) \text{ such that } \mu(x) \geq t_i. \]

\[ \iff \sup_{z \in f^{-1}(y)} \mu(z) \geq t_i. \]

\[ \iff (f(\mu))(y) \geq t_i. \]

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\[ \iff y \in (f(\mu))_{t_i}. \]

Therefore \( f(\mu_{t_i}) = (f(\mu))_{t_i} \), for \( i = 0, 1, 2, \ldots n \) and hence the chain of level \( \ell \)-ideals of \( f(\mu) \) is \( f(\mu_{t_0}) \subset f(\mu_{t_1}) \subset \cdots \subset f(\mu_{t_n}) = \mathcal{L}_N \).

For (2)

Since \( f^{-1}(\nu(x)) = \nu(f(x)) \) for all \( x \in \mathcal{L}_N \) and \( \text{Im} f^{-1}(\nu) = \text{Im}(\nu) \), since \( f \) is onto.

If for all \( x \in \mathcal{L}_N \), then \( x \in f^{-1}(\mu_{s_i}). \)

\[ \iff f(x) \in \mu_{s_i}. \]

\[ \iff \nu(f(x)) \geq s_i. \]

\[ \iff f^{-1}(\nu(x)) \geq s_i. \]

\[ \iff x \in (f^{-1}(\nu))_{s_i}. \]

Therefore \( f^{-1}(\nu_{s_i}) = (f^{-1}(\nu))_{s_i}, \) for all \( i = 0, 1, 2, \ldots m \) and hence the chain of level \( \ell \)-ideals of \( f^{-1}(\nu) \) is \( f^{-1}(\nu_{s_0}) \subset f^{-1}(\nu_{s_1}) \subset \cdots \subset f^{-1}(\nu_{s_m}) = \mathcal{L}_N. \)