Chapter 3

Fuzzy sub $\ell$-near ring

In this Chapter, the concept of fuzzy sub $\ell$-near ring and level sub $\ell$-near ring of a $\ell$-near ring were introduced and their properties are studied. The characterization theorem of a fuzzy sub $\ell$-near ring is presented through the level sub $\ell$-near ring, further shown that the family of level sub $\ell$-near ring forms a distributive lattice.

To start with,

**Definition 3.1.** Let $\mu$ be a fuzzy subset of a $\ell$-near ring $L_N$. Then $\mu$ is called a fuzzy sub $\ell$-near ring of $L_N$, if it satisfy the following:

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\};$

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\};$

(iii) $\mu(x \lor y) \geq \min\{\mu(x), \mu(y)\};$

(iv) $\mu(x \land y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in L_N$.

**Example 3.1.** Let $L_N = \{(a, b) : a, b \in \mathbb{R}\}$ is a $\ell$-near ring, defined as in Example 2.5. We define a fuzzy subset $\mu$ on $L_N$ by $\mu((a, b)) = \begin{cases} 1 & \text{if } a \geq 0 \\ \frac{a}{b} & \text{if } a \leq 0 \end{cases}$. Then $\mu$ is a fuzzy sub $\ell$-near ring of $L_N$. 

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Note 3.1. Now, let $\mu$ be a fuzzy sub $\ell$-near ring of a $\ell$-near ring $\mathcal{L}_N$ and let $\mu_t = \{x \in \mathcal{L}_N : \mu(x) \geq t\}$ be a level subset of $\mu$. Define the sets $A_t = \{0x : x \in \mu_t\}$ and $A = 0\mathcal{L}_N = \{0n : n \in \mathcal{L}_N\}$.

Proposition 3.1. Let $\mathcal{L}_N$ be a $\ell$-near ring. Then the set $A = 0\mathcal{L}_N = \{0n : n \in \mathcal{L}_N\}$ is a sub $\ell$-near ring.

Proof:
Given that $\mathcal{L}_N$ is a $\ell$-near ring and let $A = 0\mathcal{L}_N = \{0n : n \in \mathcal{L}_N\}$.
To prove that $A$ is a sub $\ell$-near ring.
Let $x, y \in A$. Then $x = 0n$ and $y = 0m$.
$\implies x - y = (0n) - (0m) = 0(n - m)$
$x \lor y = (0n) \lor (0m) = 0(n \lor m)$,
$x \land y = (0n) \land (0m) = 0(n \land m)$
$xy = (0n)(0m) = 0(nm)$
$\implies x - y, x \lor y, x \land y, xy \in \mathcal{L}_N$, since $\mathcal{L}_N$ is a $\ell$-near ring and as $n, m \in \mathcal{L}_N$.
Hence $A$ is a sub $\ell$-near ring.

Proposition 3.2. If $\mu$ is a fuzzy sub $\ell$-near ring of a $\ell$-near ring $\mathcal{L}_N$, then $\mu(x) \leq \mu(0)$, for all $x \in \mathcal{L}_N$.

Proof:
Given that $\mathcal{L}_N$ is a $\ell$-near ring and $\mu$ is a fuzzy sub $\ell$-near ring.
Let $x \in \mathcal{L}_N$ be arbitrary. Then $0 = x - x$, for all $x \in \mathcal{L}_N$.
$\implies \mu(0) = \mu(x - x) \geq \min\{\mu(x), \mu(x)\}$.
$\implies \mu(0) \geq \mu(x)$.
Thus $\mu(x) \leq \mu(0)$, for all $x \in \mathcal{L}_N$.

Proposition 3.3. If $\mu$ is a fuzzy sub $\ell$-near ring of a $\ell$-near ring $\mathcal{L}_N$, then $\mu(x) = \mu(-x)$, for all $x \in \mathcal{L}_N$. 

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Proof:
Given that $\mathcal{L}_N$ is a $\ell$-near ring and $\mu$ is a fuzzy sub $\ell$-near ring.
To prove that $\mu(x) = \mu(-x)$, for all $x \in \mathcal{L}_N$.
Let $x \in \mathcal{L}_N$ be arbitrary. Then $-x = 0 - x$, for all $x \in \mathcal{L}_N$.
$\implies \mu(-x) = \mu(0 - x) \geq \min\{\mu(0), \mu(x)\} = \mu(x)$, by proposition 3.2.
$\implies \mu(-x) \geq \mu(x)$ \hspace{1cm} (1).
Again, $\mu(x) = \mu(-(x)) \geq \mu(-x)$, by (1).
$\implies \mu(x) \geq \mu(-x)$.
Hence $\mu(x) = \mu(-x)$, for all $x \in \mathcal{L}_N$.

**Proposition 3.4.** If $\mu$ is a fuzzy sub $\ell$-near ring of a $\ell$-near ring $\mathcal{L}_N$, then $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in \mathcal{L}_N$.

**Proof:**
Given that $\mathcal{L}_N$ is a $\ell$-near ring and $\mu$ is a fuzzy sub $\ell$-near ring.
To prove that $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$.
Let $x, y \in \mathcal{L}_N$ be arbitrary.
Now, we take $x + y = x - (-y)$.
$\implies \mu(x + y) = \mu(x - (-y))$
$\quad \geq \min\{\mu(x), \mu(-y)\}$
$\quad = \min\{\mu(x), \mu(y)\}$, by proposition 3.3.
Hence $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in \mathcal{L}_N$.

**Proposition 3.5.** Let $\mu$ be a fuzzy sub $\ell$-near ring of a $\ell$-near ring $\mathcal{L}_N$. If $\mu(x - y) = \mu(0)$, then $\mu(x) = \mu(y)$, where $x, y \in \mathcal{L}_N$.

**Proof:**
Given that $\mathcal{L}_N$ is a $\ell$-near ring and $\mu$ is a fuzzy sub $\ell$-near ring.
To prove that if $\mu(x - y) = \mu(0)$, then $\mu(x) = \mu(y)$, where $x, y \in \mathcal{L}_N$.
Let $x, y \in \mathcal{L}_N$ be arbitrary. Then,

$$
\mu(x) = \mu(x + (-y + y)) \\
= \mu((x - y) + y) \\
\geq \min\{\mu(x - y), \mu(y)\}, \text{ by proposition 3.4} \\
= \min\{\mu(0), \mu(y)\}, \text{ by assumption} \\
= \mu(y), \text{ by proposition 3.2}
$$

$$
\mu(x) \geq \mu(y).
$$

Again, $\mu(y) = \mu(x - x + y)$

$$
= \mu(x - (x - y)) \\
\geq \min\{\mu(x), \mu(x - y)\} \\
= \min\{\mu(x), \mu(0)\}, \text{ by assumption} \\
= \mu(x), \text{ by proposition 3.2}
$$

$$
\mu(y) \geq \mu(x).
$$

Hence $\mu(x) = \mu(y)$.

**Proposition 3.6.** Let $\mu$ be a fuzzy sub $\ell$-near ring of a $\ell$-near ring $\mathcal{L}_N$. Then $\mu(x + y) = \mu(y)$ if and only if $\mu(x) = \mu(0)$, for all $x, y \in \mathcal{L}_N$.

**Proof:**

Given that $\mu$ is a fuzzy sub $\ell$-near ring of $\mathcal{L}_N$.

Assume that $\mu(x + y) = \mu(y)$, for all $x, y \in \mathcal{L}_N$.

To prove that $\mu(x) = \mu(0)$.

Let $x, y \in \mathcal{L}_N$ be arbitrary and as $0 \in \mathcal{L}_N$, then $\mu(x + 0) = \mu(0)$.

Hence $\mu(x) = \mu(0)$.

Conversely, assume that $\mu(x) = \mu(0)$.

To prove that $\mu(x + y) = \mu(y)$. 

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By the proposition 3.4, we have

\[ \mu(x + y) \geq \min\{\mu(x), \mu(y)\} \]
\[ = \min\{\mu(0), \mu(y)\}, \text{ by assumption} \]
\[ = \mu(y), \text{ by proposition 3.2} \]
\[ \mu(x + y) \geq \mu(y). \]

Again, \( \mu(y) = \mu((-x + x) + y) \)
\[ = \mu(-x + (x + y)) \]
\[ \geq \min\{\mu(-x), \mu(x + y)\} \]
\[ = \min\{\mu(x), \mu(x + y)\}, \text{ by proposition 3.3} \]
\[ = \min\{\mu(0), \mu(x + y)\}, \text{ by assumption} \]
\[ = \mu(x + y), \text{ since } x + y \in \mathcal{L}_N \text{ and by proposition 3.2} \]
\[ \mu(y) \geq \mu(x + y). \]

Hence \( \mu(x + y) = \mu(y). \)

**Proposition 3.7.** Let \( \mu \) be a fuzzy sub \( \ell \)-near ring of a \( \ell \)-near ring \( \mathcal{L}_N \). If \( x, y \in \mathcal{L}_N \) and \( \mu(x) < \mu(y) \), then \( \mu(x - y) = \mu(x) \).

**Proof:**

Given that \( \mathcal{L}_N \) is a \( \ell \)-near ring and \( \mu \) is a fuzzy sub \( \ell \)-near ring.

To prove that if \( \mu(x) < \mu(y) \), then \( \mu(x - y) = \mu(x) \), for all \( x, y \in \mathcal{L}_N \).

Let \( x, y \in \mathcal{L}_N \) be arbitrary and assume that \( \mu(x) < \mu(y) \).

\[ \implies \mu(x - y) \geq \min\{\mu(x), \mu(y)\} = \mu(x), \text{ by assumption.} \]
\[ \implies \mu(x - y) \geq \mu(x). \quad (1) \]

Again, \( \mu(x) = \mu((x - y) + y) \geq \min\{\mu(x - y), \mu(y)\} = \mu(x - y). \)
\[ \implies \mu(x) \geq \mu(x - y). \quad (2) \]

From (1) and (2), we get \( \mu(x) = \mu(x - y) \).

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Theorem 3.1. Let \( \mu \) and \( \sigma \) be any two fuzzy sub \( \ell \)-near rings of a \( \ell \)-near ring \( \mathcal{L}_N \). If \( x, y \in \mathcal{L}_N, \mu(x) < \sigma(x) \) and \( \mu(y) < \sigma(y) \), then \( \mu(x - y) < \sigma(x - y) \).

Proof:

Given that \( \mu \) and \( \sigma \) are two fuzzy sub \( \ell \)-near rings of a \( \ell \)-near ring \( \mathcal{L}_N \).

Assume that \( \mu(x) < \sigma(x) \) and \( \mu(y) < \sigma(y) \). 

To prove that \( \mu(x - y) < \sigma(x - y) \). 

Case(i)

Let \( \min\{\mu(x), \mu(y)\} = \mu(x) \) and \( \min\{\sigma(x), \sigma(y)\} = \sigma(x) \).

\[ \implies \mu(x) < \mu(y) \text{ and } \sigma(x) < \sigma(y). \]

Then by proposition 3.7, we have \( \mu(x - y) = \mu(x) \) and \( \sigma(x - y) = \sigma(x) \). 

Therefore \( \mu(x - y) < \sigma(x - y) \), by (1) and (2).

Case(ii)

Let \( \min\{\mu(x), \mu(y)\} = \mu(y) \) and \( \min\{\sigma(x), \sigma(y)\} = \sigma(y) \).

\[ \implies \mu(y) < \mu(x) \text{ and } \sigma(y) < \sigma(x) \]

Then by proposition 3.7, we have \( \mu(x - y) = \mu(y) \) and \( \sigma(x - y) = \sigma(y) \). 

Therefore \( \mu(x - y) < \sigma(x - y) \), by (1) and (3).

Case(iii)

Let \( \min\{\mu(x), \mu(y)\} = \mu(x) \) and \( \min\{\sigma(x), \sigma(y)\} = \sigma(y) \).

\[ \implies \mu(x) < \mu(y) \text{ and } \sigma(y) < \sigma(x) \] 

(4)

Then by proposition 3.7, we have \( \mu(x - y) = \mu(x) \) and \( \sigma(x - y) = \sigma(y) \). 

From (4)and (1), we have \( \mu(x) < \mu(y) < \sigma(y) \). 

Therefore \( \mu(x - y) < \sigma(x - y) \), by (5) and (6).

Case(iv)

Let \( \min\{\mu(x), \mu(y)\} = \mu(y) \) and \( \min\{\sigma(x), \sigma(y)\} = \sigma(x) \).

\[ \implies \mu(y) < \mu(x) \text{ and } \sigma(x) < \sigma(y) \] 

(7)

Then by proposition 3.7, we have \( \mu(x - y) = \mu(y) \) and \( \sigma(x - y) = \sigma(x) \). 

From (7)and (1), we have \( \mu(y) < \mu(x) < \sigma(x) \). 

(8)

From (7)and (1), we have \( \mu(y) < \mu(x) < \sigma(x) \). 

(9)
Therefore $\mu(x - y) < \sigma(x - y)$, by (8) and (9).

**Theorem 3.2.** Let $S$ be any nonempty proper subset of a $\ell$-near ring $\mathcal{L}_N$ and let $\mu$ be a fuzzy subset on $\mathcal{L}_N$, defined by $\mu(x) = \begin{cases} 
g & \text{if } x \in S \\
h & \text{if } x \in \mathcal{L}_N - S \end{cases}$ where $g, h \in [0, 1]$ with $g > h$. Then $\mu$ is a fuzzy sub $\ell$-near ring of $\mathcal{L}_N$ if and only if $S$ is a sub $\ell$-near ring of $\mathcal{L}_N$.

**Proof:**
Given that $S$ is a nonempty proper subset of $\mathcal{L}_N$ and let $\mu$ be the fuzzy subset on $\mathcal{L}_N$ defined by $\mu(x) = \begin{cases} 
g & \text{if } x \in S \\
h & \text{if } x \in \mathcal{L}_N - S \end{cases}$ where $g, h \in [0, 1]$ with $g > h$.

Assume that $\mu$ is a fuzzy sub $\ell$-near ring of $\mathcal{L}_N$.

To prove $S$ is a sub $\ell$-near ring of $\mathcal{L}_N$.

Let $x, y \in S$ be arbitrary. Then $\mu(x) = \mu(y) = g$ and $\min\{\mu(x), \mu(y)\} = g$.

$\implies \mu(x - y), \mu(xy), \mu(x \lor y)$ and $\mu(x \land y)$ are greater than or equal to $g$.

But $\mu$ has only two values $g$ and $h$ with $g > h$.

$\implies$ all the values of $\mu(x - y), \mu(xy), \mu(x \lor y)$ and $\mu(x \land y)$ are equal to $g$.

$\implies x - y, xy, x \lor y$ and $x \land y$ all are belongs to $S$.

Hence $S$ is a sub $\ell$-near ring of $\mathcal{L}_N$.

Conversely, assume that $S$ is a sub $\ell$-near ring of $\mathcal{L}_N$.

To prove that $\mu$ is a fuzzy sub $\ell$-near ring of $\mathcal{L}_N$.

It is enough to prove that $\mu$ satisfies the following axioms:

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$;

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$;

(iii) $\mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}$;

(iv) $\mu(x \land y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in \mathcal{L}_N$. 

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Let \( x, y \in \mathcal{L}_N \) be arbitrary.

**Case(i)**

Let \( x, y \in S \). Then \( \mu(x) = g, \mu(y) = g \) and \( \min\{\mu(x), \mu(y)\} = g \).

Since \( S \) is a sub \( \ell \)-near ring, we have \( x - y, xy, x \lor y, x \land y \in S \).

\[
\implies \mu(x - y) = \mu(xy) = \mu(x \lor y) = \mu(x \land y) = g.
\]

\[
\implies \mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \mu(xy) \geq \min\{\mu(x), \mu(y)\},
\]

Thus all the inequalities are satisfied in this case.

**Case(ii)**

Let \( x, y \in \mathcal{L}_N - S \). Then \( \mu(x) = \mu(y) = h \) and \( \min\{\mu(x), \mu(y)\} = h \).

Here \( x - y, xy, x \lor y, x \land y \) may either belong to \( S \) or to \( \mathcal{L}_N - S \), so their images under \( \mu \) will either be \( g \) or \( h \).

If \( \mu(x - y) = g \), then \( \mu(x - y) = g > h = \min\{\mu(x), \mu(y)\} \geq \min\{\mu(x), \mu(y)\} \).

If \( \mu(xy) = h \), then \( \mu(xy) = h = \min\{\mu(x), \mu(y)\} \geq \min\{\mu(x), \mu(y)\} \).

Thus all the inequalities are satisfied in this case.

**Case(iii)**

Let \( x, y \in \mathcal{L}_N - S \). Then \( \mu(x) = g, \mu(y) = h \) and \( \min\{\mu(x), \mu(y)\} = h \).

Here \( x - y, xy, x \lor y \) and \( x \land y \) may either belong to \( S \) or to \( \mathcal{L}_N - S \), so their images under \( \mu \) will either be \( g \) or \( h \). By case(ii) all the inequalities are satisfied. Thus \( \mu \) is a fuzzy sub \( \ell \)-near ring of \( \mathcal{L}_N \).

**Corollary 3.1.** If a nonempty proper subset \( S \) of a \( \ell \)-near ring \( \mathcal{L}_N \) is a sub
\( \ell \)-near ring of \( \mathcal{L}_N \), then \( \lambda_S \) is a fuzzy sub \( \ell \)-near ring,
where \( \lambda_S = \begin{cases} 
1 & \text{if } x \in S \\
0 & \text{if } x \in \mathcal{L}_N - S 
\end{cases} \) is the characteristic function of \( S \).

**Proof:**

It follows from the theorem, by taking \( g = 1 \) and \( h = 0 \).

**Proposition 3.8.** If \( \mu \) and \( \gamma \) are two fuzzy sub \( \ell \)-near rings of \( \mathcal{L}_N \), then \( \mu \cap \gamma \) is a fuzzy sub \( \ell \)-near ring of \( \mathcal{L}_N \).

**Proof:**

Given that \( \mu \) and \( \gamma \) are two fuzzy sub \( \ell \)-near rings of a \( \ell \)-near ring \( \mathcal{L}_N \).

To prove that \( \mu \cap \gamma \) is a fuzzy sub \( \ell \)-near ring of \( \mathcal{L}_N \).

For any \( x, y \in \mathcal{L}_N \), we have \( (\mu \cap \gamma)(x) = \min\{\mu(x), \gamma(x)\} \).

\[
(\mu \cap \gamma)(x - y) = \min\{\mu(x - y), \gamma(x - y)\} \\
\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\gamma(x), \gamma(y)\}\} \\
= \min\{\min\{\mu(x), \gamma(x)\}, \min\{\mu(y), \gamma(y)\}\} \\
= \min\{(\mu \cap \gamma)(x), (\mu \cap \gamma)(y)\}.
\]

Similarly, we can prove

\[
(\mu \cap \gamma)(xy) \geq \min\{(\mu \cap \gamma)(x), (\mu \cap \gamma)(y)\}. \\
(\mu \cap \gamma)(x \lor y) \geq \min\{(\mu \cap \gamma)(x), (\mu \cap \gamma)(y)\}. \\
(\mu \cap \gamma)(x \land y) \geq \min\{(\mu \cap \gamma)(x), (\mu \cap \gamma)(y)\}.
\]

Thus \( \mu \cap \gamma \) is a fuzzy sub \( \ell \)-near rings of a \( \ell \)-near ring \( \mathcal{L}_N \).

**Proposition 3.9.** If \( \mu \) is a fuzzy sub \( \ell \)-near ring of \( \mathcal{L}_N \), then each level subset \( \mu_t, t \in \text{Im}(\mu) \) is a sub \( \ell \)-near ring of \( \mathcal{L}_N \).

**Proof:**

Given that \( \mu \) is a fuzzy sub \( \ell \)-near ring of \( \mathcal{L}_N \).
To prove that each level subset $\mu_t, t \in \text{Im}(\mu)$ is a sub $\ell$-near ring of $\mathcal{L}_N$.
Consider the level subset $\mu_t = \{x \in X : \mu(x) \geq t\}$, where $t \in \text{Im}(\mu)$.
But by proposition 3.1, we have $\mu(x) \leq \mu(0)$, for all $x \in \mathcal{L}_N$.
$\implies \mu(0) \geq t \Rightarrow 0 \in \mu_t$, for all $t$.
$\implies \mu_t \neq \emptyset$.
Let $x, y \in \mu_t$.
$\implies \mu(x) \geq t, \mu(y) \geq t$ and $\min\{\mu(x), \mu(y)\} \geq t$.
$\implies \mu(x - y) \geq t, \mu(xy) \geq t, \mu(x \lor y) \geq t, \mu(x \land y) \geq t$.
$\implies x - y, xy, x \lor y, x \land y \in \mu_t$.
Hence each level subset $\mu_t, t \in \text{Im}(\mu)$ is a sub $\ell$-near ring of $\mathcal{L}_N$.

**Theorem 3.3. Characterization Theorem** A fuzzy subset $\mu$ of a $\ell$-near ring $\mathcal{L}_N$ is a fuzzy sub $\ell$-near ring of $\mathcal{L}_N$ if and only if the level subset $\mu_t, t \in \text{Im}(\mu)$ is a sub $\ell$-near ring of $\mathcal{L}_N$.

**Proof:**
Assume that $\mu$ is a fuzzy sub $\ell$-near ring of $\mathcal{L}_N$.
To prove that the level subset $\mu_t, t \in \text{Im}(\mu)$ is a sub $\ell$-near ring of $\mathcal{L}_N$.
This part follows from the proposition 3.9.
Conversely assume that level subset $\mu_t, t \in \text{Im}(\mu)$ is a sub $\ell$-near ring of $\mathcal{L}_N$.
To prove $\mu$ is a fuzzy sub $\ell$-near ring of $\mathcal{L}_N$.
Let $\min\{\mu(x), \mu(y)\} = r$.
$\implies$ either $\mu(x) = r$ and $\mu(y) \geq \mu(x) = r$ or $\mu(y) = r$ and $\mu(x) \geq \mu(y) = r$.
$\implies \mu(x) \geq r$ and $\mu(y) \geq r$
$\implies x, y \in \mu_r$.
$\implies x - y, xy, x \lor y, x \land y \in \mu_r$, since $\mu_r$ is a sub $\ell$-near ring of $\mathcal{L}_N$.
$\implies \mu(x - y) \geq r, \mu(xy) \geq r, \mu(x \lor y) \geq r, \mu(x \land y) \geq r$.
(1)
Let $\mu(x - y) = r_1$.
(2)
To prove $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$. That is to prove $r_1 \geq r$.
Suppose $r_1 < r$.

From (2) and (3) we have $\mu(x - y) = r_1 < r \Rightarrow \mu(x - y) < r$.

which is a contradiction to (1). Therefore $r_1 \geq r$.

Hence $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$.

Similarly, we can prove $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, $\mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}$; and $\mu(x \land y) \geq \min\{\mu(x), \mu(y)\}$. Thus $\mu$ is a fuzzy sub $\ell$-near ring of $\mathcal{L}_N$.

**Theorem 3.4.** If a nonempty subset $A$ of a $\ell$-near ring $\mathcal{L}_N$ is a sub $\ell$-near ring of $\mathcal{L}_N$, then there exists a fuzzy sub $\ell$-near ring $\mu$ of $\mathcal{L}_N$ such that $\mu_t = A$, for some $t \in [0, 1]$.

**Proof:**

Assume that a nonempty subset $A$ is a sub $\ell$-near ring of a $\ell$-near ring $\mathcal{L}_N$.

To prove that there exists a fuzzy sub $\ell$-near ring $\mu$ of $\mathcal{L}_N$ such that $\mu_t = A$, for some $t \in [0, 1]$.

Let $t \in [0, 1]$ and define a fuzzy subset $\mu$ on $\mathcal{L}_N$ by $\mu(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$

Then clearly $\mu_t = A$.

Now, we prove that $\mu$ is a fuzzy sub $\ell$-near ring.

It is clear that the level subset $\mu_t$ is a sub $\ell$-near ring of $\mathcal{L}_N$.

Hence by the theorem 3.3, $\mu$ is a fuzzy sub $\ell$-near ring of $\mathcal{L}_N$.

**Definition 3.2.** Let $\mu$ be any fuzzy sub $\ell$-near ring of $\mathcal{L}_N$, $t \in [0, 1]$ and $t \leq \mu(0)$. Then sub $\ell$-near ring $\mu_t$ of $\mathcal{L}_N$ is called a level sub $\ell$-near ring of $\mu$.

**Theorem 3.5.** Two level sub $\ell$-near rings $\mu_{t_1}$ and $\mu_{t_2}$ with $t_1 < t_2$ of a fuzzy sub $\ell$-near ring $\mu$ of $\mathcal{L}_N$ are equal if and only if there is no $x \in \mathcal{L}_N$ such that $t_1 \leq \mu(x) < t_2$.

**Proof:**

Given that $\mu_{t_1}$ and $\mu_{t_2}$ are two level sub $\ell$-near ring of a fuzzy sub $\ell$-near ring
\( \mu \) of a \( \ell \)-near ring \( \mathcal{L}_N \). Assume that \( \mu_1 = \mu_2 \) with \( t_1 < t_2 \).

To prove that there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

Suppose that there is an \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

\[ \implies \mu(x) < t_2 \text{ and } \mu(x) \geq t_1. \]

\[ \implies x \notin \mu_1 \text{ and } x \in \mu_1. \]

Which is a contradiction to \( \mu_1 = \mu_2 \).

Therefore there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

Conversely assume that there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

To prove that \( \mu_1 \) and \( \mu_2 \) are equal.

Now, \( \mu_1 = \{x \in \mathcal{L}_N : \mu(x) \geq t_1\} \) and \( \mu_2 = \{x \in \mathcal{L}_N : \mu(x) \geq t_2\} \) with \( t_1 < t_2 \). Then clearly \( \mu_2 \subseteq \mu_1 \). It is enough to prove that \( \mu_1 \subseteq \mu_2 \).

Let \( x \in \mu_1 \). Then \( \mu(x) \geq t_1 \).

Suppose that \( \mu(x) < t_2 \). Then \( t_1 \leq \mu(x) < t_2 \).

Which is a contradiction to our assumption, so \( \mu(x) \geq t_2 \).

Thus \( x \in \mu_2 \) and hence \( \mu_1 \subseteq \mu_2 \). Therefore \( \mu_1 \) and \( \mu_2 \) are equal.

**Remark 3.1.** Let \( \mu \) be any fuzzy sub \( \ell \)-near ring of a \( \ell \)-near ring \( \mathcal{L}_N \). From the theorem 3.3 we have \( \mu_t, t \in \text{Im } \mu \) are level sub \( \ell \)-near rings of \( \mathcal{L}_N \). Then \( \mathcal{F}_\mu = \{\mu_t : t \in \text{Im}(\mu)\} \) is called as the family of level sub \( \ell \)-near rings of \( \mu \).

**Theorem 3.6.** Let \( \mu \) be any fuzzy sub \( \ell \)-near ring of \( \mathcal{L}_N \) and let \( \mu_{t_1}, \mu_{t_2}, \ldots, \mu_{t_n} \) be the collection of level sub \( \ell \)-near rings of \( \mu \), where \( t_1, t_2, t_3, \ldots, t_n \in \text{Im} (\mu) \).

Then

1. \( \mu_{t_1} \cup \mu_{t_2} \cup \cdots \cup \mu_{t_n} = \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_1, t_2, \ldots, t_n\}\} \).

2. \( \mu_{t_1} \cap \mu_{t_2} \cap \cdots \cap \mu_{t_n} = \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_1, t_2, \ldots, t_n\}\} \)

are also level sub \( \ell \)-near rings of \( \mu \).

**Proof:**

Given that \( \mu \) is a fuzzy sub \( \ell \)-near ring of \( \mathcal{L}_N \) and let \( \mu_{t_1}, \mu_{t_2}, \ldots, \mu_{t_n} \) be the
collection of level sub $\ell$-near rings of $\mu$, where $t_1, t_2, t_3, \ldots, t_n \in \text{Im} \ (\mu)$.

To prove that (1) and (2) are level sub $\ell$-near ring.

For(1)

Let $x, y \in \mu_{t_1} \cup \mu_{t_2} \cup \cdots \mu_{t_n}$ be arbitrary and let $\min\{t_1, t_2, \ldots, t_n\} = t_i$, for some $i, 1 \leq i \leq n$. Then $x, y \in \mu_{t_i}$ and since $\mu_{t_i}$ is a sub $\ell$-near ring.

$\Rightarrow x, y \in \mathcal{L}_N$ such that $x - y, xy, x \lor y, x \land y \in \mu_{t_i}$.

$\Rightarrow x, y \in \mathcal{L}_N$ such that $\mu(x - y) \geq t_i, \mu(xy) \geq t_i, \mu(x \lor y) \geq t_i, \mu(x \land y) \geq t_i$.

$\Rightarrow x, y \in \mathcal{L}_N$ such that $\mu(x - y) \geq \min\{t_1, t_2, \ldots, t_n\}, \mu(xy) \geq \min\{t_1, t_2, \ldots, t_n\},$

$\Rightarrow x - y \in \mu_{t_1} \cup \mu_{t_2} \cup \cdots \cup \mu_{t_n}, xy \in \mu_{t_1} \cup \mu_{t_2} \cup \cdots \cup \mu_{t_n}, x \lor y \in \mu_{t_1} \cup \mu_{t_2} \cup \cdots \cup \mu_{t_n},$

$x \land y \in \mu_{t_1} \cup \mu_{t_2} \cup \cdots \cup \mu_{t_n}.$

$\Rightarrow \mu_{t_1} \cup \mu_{t_2} \cup \cdots \cup \mu_{t_n}$ is a level sub $\ell$-near ring of $\mathcal{L}_N$.

Hence $\mu_{t_1} \cup \mu_{t_2} \cup \cdots \cup \mu_{t_n}$ is a level sub $\ell$-near ring of $\mu$.

For(2)

Let $x, y \in \mu_{t_1} \cap \mu_{t_2} \cap \cdots \cap \mu_{t_n}$ be arbitrary and let $\max\{t_1, t_2, \ldots, t_n\} = t_j$, for some $j, 1 \leq j \leq n$. Then $x, y \in \mu_{t_j}$ and since $\mu_{t_j}$ is a sub $\ell$-near ring.

$\Rightarrow x, y \in \mathcal{L}_N$ such that $x - y, xy, x \lor y, x \land y \in \mu_{t_j}$.

$\Rightarrow x, y \in \mathcal{L}_N$ such that $\mu(x - y) \geq t_j, \mu(xy) \geq t_j, \mu(x \lor y) \geq t_j, \mu(x \land y) \geq t_j$.

$\Rightarrow x, y \in \mathcal{L}_N$ such that $\mu(x - y) \geq \max\{t_1, t_2, \ldots, t_n\}, \mu(xy) \geq \max\{t_1, t_2, \ldots, t_n\},$

$\Rightarrow x - y \in \mu_{t_1} \cap \mu_{t_2} \cap \cdots \cap \mu_{t_n}, xy \in \mu_{t_1} \cap \mu_{t_2} \cap \cdots \cap \mu_{t_n}, x \lor y \in \mu_{t_1} \cap \mu_{t_2} \cap \cdots \cap \mu_{t_n},$

$x \land y \in \mu_{t_1} \cap \mu_{t_2} \cap \cdots \cap \mu_{t_n}.$

$\Rightarrow \mu_{t_1} \cap \mu_{t_2} \cap \cdots \cap \mu_{t_n}$ is a level sub $\ell$-near ring of $\mathcal{L}_N$.

Hence $\mu_{t_1} \cap \mu_{t_2} \cap \cdots \cap \mu_{t_n}$ is a level sub $\ell$-near ring of $\mu$.

**Corollary 3.2.** Let $\mu$ be any fuzzy sub $\ell$-near ring of a $\ell$-near ring $\mathcal{L}_N$. If

$\text{Im} \ (\mu) = \{t_1, t_2, t_3, \ldots, t_n\}$ with $t_1 > t_2 > t_3 > \cdots > t_n$, then $\bigcap_{k=1}^{n} \mu_{t_k} = \mathcal{L}_N$

and $\bigcap_{k=1}^{n} \mu_{t_k} = \mu_{t_1}$.
Theorem 3.7. Let \( \mu \) be any fuzzy sub \( \ell \)-near ring of a \( \ell \)-near ring \( \mathcal{L}_N \). If 
\( \text{Im}(\mu) = \{t_1, t_2, t_3, \ldots, t_n\} \) with \( t_1 > t_2 > t_3 > \cdots > t_n \), then we have the 
following chain of level \( \ell \)-sub near rings of \( \mu \).
That is \( \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_n = \mathcal{L}_N \).

Proof:

Given that \( \mu \) is a fuzzy sub \( \ell \)-near ring of \( \mathcal{L}_N \).
To prove that \( \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_n = \mathcal{L}_N \).
Let \( \text{Im}(\mu) = \{t_1, t_2, t_3, \ldots, t_n\} \) with \( t_1 > t_2 > t_3 > \cdots > t_n \). (1)
Let \( x_1 \in \mu_1 \). Then \( x_1 \in \mathcal{L}_N \) such that \( \mu(x_1) \geq t_1 \).
\[ \rightarrow \mu(x_1) \geq t_2, \text{ by (1)}. \]
\[ \rightarrow x_1 \in \mu_2. \]
\[ \rightarrow \mu_1 \subseteq \mu_2. \]
Again \( x_2 \in \mu_2 \). Then \( x_2 \in \mathcal{L}_N \) such that \( \mu(x_2) \geq t_2 \).
\[ \rightarrow \mu(x_2) \geq t_3, \text{ by (1)} \]
\[ \rightarrow x_2 \in \mu_3. \]
\[ \rightarrow \mu_2 \subseteq \mu_3. \]
\[ \rightarrow \mu_1 \subseteq \mu_2 \subseteq \mu_3. \]
Proceeding like this, we get \( \mu_1 \subseteq \mu_2 \subseteq \cdots \subseteq \mu_n = \mathcal{L}_N \).

Theorem 3.8. If \( \mu \) is any fuzzy sub \( \ell \)-near ring of a \( \ell \)-near ring \( \mathcal{L}_N \), then
\( (\mathcal{F}_\mu, \lor, \land) \) is a distributive lattice, where \( \lor \) and \( \land \) are defined by
\( \mu_i \lor \mu_j = \mu_i \cup \mu_j = \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, t_j\}\} \) and
\( \mu_i \land \mu_j = \mu_i \cap \mu_j = \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_i, t_j\}\} \), for all \( t_i, t_j \in \text{Im}(\mu) \).

Proof:

Given that \( \mu \) is a fuzzy sub \( \ell \)-near ring of a \( \ell \)-near ring \( \mathcal{L}_N \).
To prove that \( (\mathcal{F}_\mu, \lor, \land) \) is a distributive lattice.
Idempotency :
\[ \mu_i \lor \mu_i = \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, t_i\}\} \]
\[
\begin{align*}
\mu_i \cap \mu_i & = \{x \in \mathcal{L}_N : \mu(x) \geq t_i\} \\
& = \mu_i. \\
\mu_i \cap \mu_i & = \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_i, t_i\}\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq t_i\} \\
& = \mu_i.
\end{align*}
\]

**Associativity:**

\[
\begin{align*}
(\mu_i \cup \mu_j) \cup \mu_k & = \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, t_j\}\} \cup \{x \in \mathcal{L}_N : \mu(x) \geq t_k\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq \min\{\min\{t_i, t_j\}, t_k\}\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, t_j, t_k\}\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, \min\{t_j, t_k\}\}\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq t_i\} \cup \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_j, t_k\}\} \\
& = \mu_i \cup (\mu_j \cup \mu_k).
\end{align*}
\]

\[
\begin{align*}
(\mu_i \cap \mu_j) \cap \mu_k & = \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_i, t_j\}\} \cap \{x \in \mathcal{L}_N : \mu(x) \geq t_k\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq \max\{\max\{t_i, t_j\}, t_k\}\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_i, t_j, t_k\}\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_i, \max\{t_j, t_k\}\}\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq t_i\} \cap \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_j, t_k\}\} \\
& = \mu_i \cap (\mu_j \cap \mu_k).
\end{align*}
\]

**Commutativity:**

\[
\begin{align*}
\mu_i \cup \mu_j & = \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, t_j\}\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_j, t_i\}\} \\
& = \mu_j \cup \mu_i. \\
\mu_i \cap \mu_j & = \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_i, t_j\}\} \\
& = \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_j, t_i\}\} \\
& = \mu_j \cap \mu_i. 
\end{align*}
\]

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Absorption:

\[
\mu_i \cap (\mu_i \cup \mu_j) = \{x \in \mathcal{L}_N : \mu(x) \geq t_i\} \cap \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, t_j\}\} \\
= \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_i, \min\{t_i, t_j\}\}\} \\
= \{x \in \mathcal{L}_N : \mu(x) \geq t_i\} \\
= \mu_i.
\]

\[
\mu_i \cup (\mu_i \cap \mu_j) = \{x \in \mathcal{L}_N : \mu(x) \geq t_i\} \cup \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_i, t_j\}\} \\
= \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, \max\{t_i, t_j\}\}\} \\
= \{x \in \mathcal{L}_N : \mu(x) \geq t_i\} \\
= \mu_i.
\]

Distributive:

\[
\mu_i \cup (\mu_j \cap \mu_k) = \mu_i \cap \{x \in \mathcal{L}_N : \mu(x) \geq \max\{t_j, t_k\}\} \\
= \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, \max\{t_j, t_k\}\}\} \\
= \{x \in \mathcal{L}_N : \mu(x) \geq (\min\{t_i, t_j\} \cap \min\{t_i, t_k\})\} \\
= \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, t_j\}\} \cap \{x \in \mathcal{L}_N : \mu(x) \geq \min\{t_i, t_k\}\} \\
= (\mu_i \cup \mu_j) \cap (\mu_i \cup \mu_k).
\]

Hence \((\mathcal{F}_\mu, \lor, \land)\) is a distributive lattice.