Chapter 5

Fuzzy $N$-subgroup in $\ell$-near ring

In this chapter, we apply the concept of fuzzy set to $N$-subgroup of a $\ell$-near ring $\mathcal{L}_N$ and introduced the notion of fuzzy $N$-subgroup of a $\ell$-near ring $\mathcal{L}_N$. Further, established the characterization theorem of a fuzzy $N$-subgroup of a $\ell$-near ring and the image of a fuzzy $N$-subgroup of a $\ell$-near ring and the the pre-image of a fuzzy $N$-subgroup of a $\ell$-near ring under the $\ell$-homomorphism are also fuzzy $N$-subgroup. Also, formulate the normalization of fuzzy $N$-subgroup and fuzzy characteristic $N$-subgroup of a $\ell$-near ring.

**Proposition 5.1.** If $\mathcal{L}_N$ is a $\ell$-near ring, then the set $A = 0\mathcal{L}_N = \{0n : n \in \mathcal{L}_N\}$ is a sub $\ell$-near ring.

**Proof:**

Given that $\mathcal{L}_N$ is a $\ell$-near ring and let $A = 0\mathcal{L}_N = \{0n : n \in \mathcal{L}_N\}$.

To prove that $A$ is a sub $\ell$-near ring.

Let $x, y \in A$. Then $x = 0n$ and $y = 0m$.

$x - y = (0n) - (0m) = 0(n - m) \in A$,

$x \lor y = (0n) \lor (0m) = 0(n \lor m) \in A$. 

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\[ x \land y = (0n) \land (0m) = 0(n \land m) \in A \text{ and} \]
\[ xy = (0n)(0m) = 0(nm) \in A \]
\[ \implies x - y, x \lor y, x \land y, xy \in \mathcal{L}_N, \text{ since } \mathcal{L}_N \text{ is a } \ell\text{-near ring and as } n, m \in \mathcal{L}_N. \]
Hence \( A \) is a sub \( \ell\text{-near ring.} \)

**Theorem 5.1.** If \( \mathcal{L}_N \) is a \( \ell\text{-near ring, then} \)

1. \( A \) is a \( \mu_\ell \)-subgroup of \( \mu_\ell \);

2. \( A \) is a left \( A \)-subgroup of \( A \).

**Proof:**

*For (1)*

Let \( a, b \in A \) be arbitrary.
\[ \implies \text{there exists } x, y \in \mu_\ell \text{ such that } a = 0x \text{ and } b = 0y. \]
\[ \implies a - b = (0x) - (0y) = 0(x - y). \]
\[ \implies a - b \in A, \text{ since } x - y \in \mu_\ell. \]
Hence \( (A, +) \) is a subgroup of \( (\mu_\ell, +) \).

Now, we prove that \( (A, \lor, \land) \) is a sublattice of \( (\mu_\ell, \lor, \land) \).

Let \( a, b \in A \).
\[ \implies \text{there exists } x, y \in \mu_\ell \text{ such that } a = 0x \text{ and } b = 0y. \]
\[ \implies a \lor b = (0x) \lor (0y) = 0(x \lor y), \text{ as } x \lor y \in \mu_\ell \text{ and} \]
\[ a \land b = (0x) \land (0y) = 0(x \land y), \text{ as } x \land y \in \mu_\ell. \]
\[ \implies a \lor b, a \land b \in A. \]
Hence \( (A, \lor, \land) \) is a sublattice of \( (\mu_\ell, \lor, \land) \).

Now, let \( x_1 \in \mu_\ell \) and \( a \in A \).
\[ \implies \text{there exist an } x \in \mu_\ell \text{ such that } a = 0x. \]
\[ \implies x_1a = x_1(0x) = (x_10)x = 0x = a \]
\[ \implies \mu_\ell A \subseteq A. \]
Again, \( ax_1 = (0x)x_1 = 0(xx_1). \)
Since $\mu_t$ is a sub $\ell$-near ring of $\mathcal{L}_N$, $xx_1 \in \mu_t$, for some $x, x_1 \in \mu_t$.

$$ax_1 = 0(xx_1) \in A_t.$$  

$$A_t \mu_t \subset A_t.$$  

Hence $A_t$ is a $\mu_t$-subgroup of $\mu_t$.

For (2)

By proposition 5.1 $A$ is a sub $\ell$-near ring and by (1) we have $(A_t, +)$ is a

subgroup of $(A, +)$ and $(A_t, \lor, \land)$ is a sublattice of $(A, \lor, \land)$.

It remains show that $AA_t \subset A_t$.

Let $a \in A, b \in A_t$.

$$\implies \text{there exist } x \in A_t \text{ and } n \in \mathcal{L}_N \text{ such that } a = 0n \text{ and } b = 0x.$$  

$$\implies ab = (0n)(0x) = 0(n0)x = 0(0)x = 0x \in A_t.$$  

$$\implies AA_t \subset A_t.$$  

$$\implies A_t \text{ is a left } A\text{-subgroup of } A.$$  

**Definition 5.1.** Let $(\mathcal{L}_N, +, \cdot, \lor, \land)$ be a $\ell$-near ring. A fuzzy subset $\mu$ of $\mathcal{L}_N$ is called a fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$, if

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$;

(ii) $\mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}$;

(iii) $\mu(x \land y) \geq \min\{\mu(x), \mu(y)\}$;

(iv) $\mu(nx) \geq \mu(x)$ (resp. $\mu(nx) \geq \mu(x)$), for all $x, y, n \in \mathcal{L}_N$.

**Example 5.1.** Let $(\mathcal{L}_N, +, \cdot, \lor, \land)$ be an $\ell$-near ring defined as in example 2.6. We define a fuzzy subset $\mu : \mathcal{L}_N \rightarrow [0, 1] \text{ by } \mu(c) = \mu(d) < \mu(b) < \mu(a)$. Then $\mu$ is a fuzzy left $N$-subgroup.

**Proposition 5.2.** Every fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ is a fuzzy sub $\ell$-near ring.
Proof:
Let \( \mu \) be a fuzzy left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \).
To prove that \( \mu \) is a fuzzy sub \( \ell \)-near ring.
By assumption, we have

(i) \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \);

(ii) \( \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\} \);

(iii) \( \mu(x \land y) \geq \min\{\mu(x), \mu(y)\} \);

(iv) \( \mu(xy) \geq \mu(y) \), (resp. \( \mu(yx) \geq \mu(x) \)), for all \( x, y \in \mathcal{L}_N \).

Then

(i) \( \mu(x - y) \geq \min\{\mu(x), \mu(y)\} \);

(ii) \( \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\} \);

(iii) \( \mu(x \land y) \geq \min\{\mu(x), \mu(y)\} \);

(iv) \( \mu(xy) \geq \min\{\mu(x), \mu(y)\} \), for all \( x, y \in \mathcal{L}_N \).

Hence \( \mu \) is a fuzzy sub \( \ell \)-near ring.

**Proposition 5.3.** If \( \mu \) is a fuzzy left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \), then \( \mu(x) \leq \mu(0) \), for all \( x \in \mathcal{L}_N \).

**Proof:**
Assume that \( \mu \) is a fuzzy left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \).
To prove that \( \mu(x) \leq \mu(0) \), for all \( x \in \mathcal{L}_N \).
By proposition 3.2, we have \( \mu(x) \leq \mu(0) \), since every fuzzy left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \) is a fuzzy sub \( \ell \)-near ring.
Theorem 5.2. Let $H$ be a nonempty proper subset of a $\ell$-near ring $\mathcal{L}_N$ and let $\mu$ be a fuzzy subset in $\mathcal{L}_N$ defined by $\mu(x) = \begin{cases} g & \text{if } x \in H \\ h & \text{if } x \notin H \end{cases}$ where $g, h \in [0, 1]$ with $g > h$. Then $\mu$ is a fuzzy left(resp.right) $N$-subgroup if and only if $H$ is a left(resp.right) $N$-subgroup.

Proof:

Given that $H$ is a nonempty proper subset of $\mathcal{L}_N$ and let $\mu$ be a fuzzy subset in $\mathcal{L}_N$, defined by $\mu(x) = \begin{cases} g & \text{if } x \in H \\ h & \text{if } x \notin H \end{cases}$ with $g > h \in [0, 1]$.

Assume that $\mu$ is a fuzzy left(resp.right) $N$-subgroup.

To prove that $H$ is a left(resp.right) $N$-subgroup.

Let $x, y \in H$ be arbitrary. Then $\mu(x) = \mu(y) = g$ and $\min\{\mu(x), \mu(y)\} = g$.

$\implies \mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$, since $\mu$ is a fuzzy left(resp.right) $N$-subgroup.

$\implies \mu(x - y) \geq g$, $\mu(x \vee y) \geq g$, $\mu(x \wedge y) \geq g$.

$\implies x - y, x \vee y, x \wedge y \in H$.

Since $\mu$ is a fuzzy left(resp.right) $N$-subgroup, we have $\mu(nx) \geq \mu(x) = g$.

Hence $nx \in H$. (resp. $\mu(xn) \geq \mu(x) = g \implies xn \in H$).

Therefore $H$ is a left(resp.right) $N$-subgroup.

Conversely, assume that $H$ is a left(resp.right) $N$-subgroup of $\mathcal{L}_N$.

To prove that $\mu$ is a fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$.

It is enough to prove $\mu$ satisfies the following axioms:

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$;

(ii) $\mu(x \vee y) \geq \min\{\mu(x), \mu(y)\}$;

(iii) $\mu(x \wedge y) \geq \min\{\mu(x), \mu(y)\}$;

(iv) $\mu(nx) \geq \mu(x)$, (resp. $\mu(xn) \geq \mu(x)$), for all $x, y, n \in \mathcal{L}_N$.  

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For (i), (ii), (iii) 
If at least one of \( x \) and \( y \) does not belong to \( H \), then \( \min\{\mu(x), \mu(y)\} = h \).
\[
\implies \mu(x - y) \geq h, \ \mu(x \lor y) \geq h, \ \mu(x \land y) \geq h.
\]
\[
\implies \mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \ \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\};
\]
\[
\mu(x \land y) \geq \min\{\mu(x), \mu(y)\}.
\]
If \( x, y \in H \), then \( \mu(x) = \mu(y) = g \) and \( \min\{\mu(x), \mu(y)\} = g \).
\[
\implies x - y, \ x \lor y, \ x \land y \in H, \text{ since } H \text{ is a left(resp.right) } N\text{-subgroup.}
\]
\[
\implies \mu(x - y) = g, \ \mu(x \lor y) = g, \ \mu(x \land y) = g.
\]
\[
\implies \mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \ \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\};
\]
\[
\mu(x \land y) \geq \min\{\mu(x), \mu(y)\}.
\]
For(iv) 
If \( x \in H \), then \( nx \in H \), since \( H \) is a left \( N \)-subgroup and for all \( n \in \mathcal{L}_N \).
\[
\implies \mu(nx) = g = \mu(x).
\]
\[
\implies \mu(nx) \geq \mu(x).
\]
If \( x \in H \), then \( xn \in H \), since \( H \) is a right \( N \)-subgroup and for all \( n \in \mathcal{L}_N \).
\[
\implies \mu(xn) = g = \mu(x).
\]
\[
\implies \mu(xn) \geq \mu(x).
\]
If \( x \notin H \) then \( nx \notin H \), for all \( n \in \mathcal{L}_N \) and hence \( \mu(nx) \geq \mu(x) \).
If \( x \notin H \) then \( xn \notin H \), for all \( n \in \mathcal{L}_N \) and hence \( \mu(xn) \geq \mu(x) \).
Hence \( \mu \) is a fuzzy left(resp.right) \( N \)-subgroup.

**Corollary 5.1.** Let \( \chi \) be the characteristic function on a left(resp.right) \( N \)-subgroup \( H \) of \( \mathcal{L}_N \). Then \( \chi_H \) is a fuzzy left(resp.right) \( N \)-subgroup.

**Proof:**
It follows from the theorem, when we take \( g = 1 \) and \( h = 0 \).

**Proposition 5.4.** If \( \mu \) is a fuzzy left(resp.right) \( N \)-subgroup, then the level subset \( \mu \) is a left(resp.right) \( N \)-subgroup.

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Proof:
Given that $\mu$ is a fuzzy left(resp.right) $N$- subgroup.
To prove that the level subset $\mu_t$ is a left(resp.right) $N$-subgroup.
Let $x, y \in \mu_t$ be arbitrary. Then $x, y \in \mathcal{L}_N$ such that $\mu(x) \geq t$ and $\mu(y) \geq t$.
Since $\mu$ is a fuzzy left(resp. right) $N$-subgroup, then we have
\[
\mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \quad \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\},
\]
\[
\mu(x \land y) \geq \min\{\mu(x), \mu(y)\} \quad \text{and} \quad \mu(nx) \geq \mu(x), \quad (\text{resp.} \mu(xn) \geq \mu(x) \geq t).
\]
\[
\implies \mu(x - y) \geq t, \mu(x \lor y) \geq t, \mu(x \land y) \geq t \quad \text{and} \quad \mu(nx) \geq t, \quad (\text{resp.} \mu(xn) \geq t).
\]
\[
\implies x - y, \quad x \lor y, \quad x \land y, \quad nx, \quad (\text{resp.} xn) \in \mu_t.
\]
Hence $\mu_t$ is a left(resp.right) $N$-subgroup of $\mathcal{L}_N$.

**Theorem 5.3.** Let $\mu$ be a fuzzy subset of $\mathcal{L}_N$. Then $\mu$ is a fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ if and only if the level subset $\mu_t$ is a left(resp.right) $N$-subgroup where $t \in [0, 1]$.

Proof:
Given that $\mu$ is a fuzzy subset of $\mathcal{L}_N$.
Assume that $\mu$ is a fuzzy left(resp.right) $N$-subgroup.
To prove that the level subset $\mu_t$ is a left(resp.right) $N$-subgroup.
The proof follows from the proposition 5.4.
Conversely, assume that $\mu_t$ is a left(resp.right) $N$-subgroup of $\mathcal{L}_N$.
To prove that $\mu$ is a fuzzy left(resp.right) $N$-subgroup.
It is enough to prove that $\mu$ satisfies the following:

(i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$;

(ii) $\mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}$;

(iii) $\mu(x \land y) \geq \min\{\mu(x), \mu(y)\}$;

(iv) $\mu(nx) \geq \mu(x)$, (resp. $\mu(xn) \geq \mu(x)$), for all $x, y, n \in \mathcal{L}_N$.

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For (i), (ii), (iii)

Let \( \min\{\mu(x), \mu(y)\} = t. \)

\( \implies \) either \( \mu(x) = t \) and \( \mu(y) \geq \mu(x) = t \) or \( \mu(y) = t \) and \( \mu(x) \geq \mu(y) = t. \)

\( \implies \mu(x) \geq t \) and \( \mu(y) \geq t. \)

\( \implies x, y \in \mu_t. \)

\( \implies x - y, x \lor y, x \land y \in \mu_t, \) since \( \mu_t \) is a additive subgroup and sublattice.

\( \implies \mu(x - y) \geq t, \mu(x \lor y) \geq t \) and \( \mu(x \land y) \geq t. \)

\( \implies \mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\} \)

\( \mu(x \land y) \geq \min\{\mu(x), \mu(y)\}. \)

For (iv)

Let \( x \in \mu_t. \)

\( \implies nx \in \mu_t \) (resp. \( xn \in \mu_t \)), for all \( n \in \mathcal{L}_N, \) since \( \mu_t \) is a left(resp.right) \( N \)-subgroup.

\( \implies \mu(nx) \geq t = \mu(x), \) (resp. \( \mu(xn) \geq t = \mu(x)\)).

\( \implies \mu(nx) \geq \mu(x), \) (resp. \( \mu(xn) \geq \mu(x)\)).

**Theorem 5.4.** If \( H \) is a left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N, \) then there exists a fuzzy left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \) such that \( \mu_t = H, \) for some \( t \in (0, 1). \)

**Proof:**

Assume that \( H \) is a left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N. \)

To prove that there exists a fuzzy left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \) such that \( \mu_t = H, \) for some \( t \in [0, 1]. \)

We define the fuzzy subset \( \mu \) on \( \mathcal{L}_N \) by \( \mu(x) = \begin{cases} 
  t & \text{if } x \in H \\
  0 & \text{if } x \notin H 
\end{cases} \)

where \( t \) is a fixed number in \([0, 1]\). Then clearly, \( \mu_t = H. \)

Now, we show that \( \mu \) is a fuzzy left(resp.right) \( N \)-subgroup.

Let \( x, y \in \mathcal{L}_N \) be arbitrary.

If at least one of \( x \) and \( y \) does not belong to \( H, \) then \( \min\{\mu(x), \mu(y)\} = 0. \)
\[\Rightarrow \mu(x - y) \geq 0, \mu(x \lor y) \geq 0, \mu(x \land y) \geq 0.\]
\[\Rightarrow \mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}, \mu(x \land y) \geq \min\{\mu(x), \mu(y)\}.\]

If \(x, y \in H\), then \(\min\{\mu(x), \mu(y)\} = t.\)
\[\Rightarrow \mu(x - y) \geq t, \mu(x \lor y) \geq t, \mu(x \land y) \geq t.\]
\[\Rightarrow \mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}, \mu(x \land y) \geq \min\{\mu(x), \mu(y)\}.\]

If \(x \in H\), then \(nx\) (resp. \(xn\)) \(\in H\), for all \(n \in \mathcal{L}_N.\)
\[\Rightarrow \mu(nx) = t = \mu(x)\) (resp. \(\mu(xn) = t = \mu(x)\)).
\[\Rightarrow \mu(nx) \geq \mu(x)\) (resp. \(\mu(xn) \geq \mu(x)\)).\]

If \(x \notin H\), then \(\mu(x) = 0\) and \(\mu(nx) \notin H.\)
\[\Rightarrow \mu(nx) = 0 = \mu(x)\) (resp. \(\mu(xn) = 0 = \mu(x)\)).
\[\Rightarrow \mu(nx) \geq \mu(x)\) (resp. \(\mu(xn) \geq \mu(x)\)).\]

Hence \(\mu\) is a fuzzy left(resp.right) \(N\)-subgroup of \(\mathcal{L}_N.\)

**Proposition 5.5.** Let \(\mu\) be a fuzzy left(resp.right) \(N\)-subgroup of \(\mathcal{L}_N.\) Then the set \(N_\mu = \{x \in \mathcal{L}_N : \mu(x) = \mu(0)\}\) is a left(resp.right) \(N\)-subgroup of \(\mathcal{L}_N.\)

**Proof:**

Given that \(\mu\) is a fuzzy left(resp.right) \(N\)-subgroup of \(\mathcal{L}_N.\)

Let \(N_\mu = \{x \in \mathcal{L}_N : \mu(x) = \mu(0)\}\).

To prove that \(N_\mu\) is a left(resp.right) \(N\)-subgroup of \(\mathcal{L}_N.\)

Let \(x, y \in N_\mu\) be arbitrary.
\[\Rightarrow \mu(x) = \mu(y) = \mu(0)\) and \(\min\{\mu(x), \mu(y)\} = \mu(0).\]
\[\Rightarrow \mu(x - y) \geq \min\{\mu(x), \mu(y)\}, \mu(x \lor y) \geq \min\{\mu(x), \mu(y)\}, \mu(x \land y) \geq \min\{\mu(x), \mu(y)\}.\]
\(\mu(x \land y) \geq \min\{\mu(x), \mu(y)\}, \mu(nx) \geq \mu(x),\) (resp. \(\mu(xn) \geq \mu(x)\)), since \(\mu\) be a fuzzy left(resp.right) \(N\)-subgroup of \(\mathcal{L}_N.\)
\[\Rightarrow \mu(x - y) \geq \mu(0), \mu(x \lor y) \geq \mu(0), \mu(x \land y) \geq \mu(0), \mu(nx) \geq \mu(0),\)
(resp. \(\mu(xn) \geq \mu(0)\)).
But by proposition 5.3, \( \mu(x - y) = \mu(x \lor y) = \mu(x \land y) = \mu(nx) \leq \mu(0) \) (resp. \( \mu(xn) \leq \mu(0) \)).

\[ \implies \mu(x - y) = \mu(x \lor y) = \mu(x \land y) = \mu(0) \) (resp. \( \mu(xn) = \mu(0) \))

\[ \implies x - y, x \lor y, x \land y \text{ and } nx \text{ (resp. } xn \text{) are belongs to } N_\mu. \]

Hence \( N_\mu \) is a left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \).

**Definition 5.2.** Let \( \mu \) be any fuzzy \( N \)-subgroup of \( \mathcal{L}_N \), \( t \in [0, 1] \) and \( t \leq \mu(0) \). Then left(resp.right) \( N \)-subgroup \( \mu_t \) of \( \mu \) in \( \mathcal{L}_N \) is called a level left(resp.right) \( N \)-subgroup of \( \mu \).

**Proposition 5.6.** Two level left(resp.right) \( N \)-subgroup \( \mu_{t_1} \) and \( \mu_{t_2} \) of \( \mathcal{L}_N \) with \( t_1 < t_2 \) of \( \mu \) are equal if and only if there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

**Proof:**

Assume that \( \mu_{t_1} \) and \( \mu_{t_2} \) are equal with \( t_1 < t_2 \).

To prove that there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

Suppose that there is an \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

\[ \implies \mu(x) \geq t_1 \text{ and } \mu(x) < t_2. \]

\[ \implies x \in \mu_{t_1} \text{ and } x \notin \mu_{t_2}. \]

\[ \implies \mu_{t_1} \neq \mu_{t_2}. \]

Which is a contradiction to \( \mu_{t_1} = \mu_{t_2} \).

Therefore there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

Conversely, assume that there is no \( x \in \mathcal{L}_N \) such that \( t_1 \leq \mu(x) < t_2 \).

To prove that \( \mu_{t_1} \) and \( \mu_{t_2} \) are equal.

Let \( \mu_{t_1} = \{ x \in \mathcal{L}_N : \mu(x) \geq t_1 \} \) and \( \mu_{t_2} = \{ x \in \mathcal{L}_N : \mu(x) \geq t_2 \} \).

Then clearly \( \mu_{t_2} \subseteq \mu_{t_1} \), since \( t_1 < t_2 \). Now, show that \( \mu_{t_1} \subseteq \mu_{t_2} \).

Let \( x \in \mu_{t_1} \). Then \( x \in \mathcal{L}_N \) such that \( \mu(x) \geq t_1 \).

Also \( \mu(x) \geq t_2 \), since \( \mu(x) \not\in t_2 \) and so \( x \in \mu_{t_2} \).

Hence \( \mu_{t_1} \subseteq \mu_{t_2} \). Therefore \( \mu_{t_1} = \mu_{t_2} \).
Proposition 5.7. If $\mu$ and $\nu$ are two fuzzy left(resp.
right) $N$-subgroup of $\mathcal{L}_N$, then $\mu \cap \nu$ is a fuzzy left(resp.
right) $N$-subgroup of $\mathcal{L}_N$.

Proof:
Given that $\mu$ and $\nu$ are two fuzzy left(resp.
right) $N$-subgroup of $\mathcal{L}_N$.
To prove that $\mu \cap \nu$ is a fuzzy left(resp.
right) $N$-subgroup of $\mathcal{L}_N$.
For any $x, y \in \mathcal{L}_N$, $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$. Then
\[
(\mu \cap \nu)(x - y) = \min\{\mu(x - y), \nu(x - y)\} \\
\geq \min\{\min\{\mu(x), \mu(y)\}, \min\{\nu(x), \nu(y)\}\} \\
= \min\{\min\{\mu(x), \nu(x)\}, \min\{\mu(x), \nu(y)\}\} \\
= \min\{(\mu \cap \nu)(x), (\mu \cap \nu)(y)\}.
\]
Similarly, we can prove
\[
(\mu \cap \nu)(x \lor y) \geq \min\{(\mu \cap \nu)(x), (\mu \cap \nu)(y)\}, \\
(\mu \cap \nu)(x \land y) \geq \min\{(\mu \cap \nu)(x), (\mu \cap \nu)(y)\}, \\
(\mu \cap \nu)(nx) \geq \min\{(\mu \cap \nu)(x), (\mu \cap \nu)(nx)\}, \text{if } x, n \in \mathcal{L}_N,
\]
Hence $\mu \cap \nu$ is a fuzzy left(resp.
right) $N$-subgroup of $\mathcal{L}_N$.

Proposition 5.8. Let $\mu$ and $\nu$ be two fuzzy left(resp.
right) $N$-subgroups of $\mathcal{L}_N$. If $\mu \subseteq \nu$ and $\mu(0) = \nu(0)$, then $N_\mu \subseteq N_\nu$.

Proof:
Given that $\mu$ and $\nu$ are two fuzzy left(resp.
right) $N$-subgroups of $\mathcal{L}_N$.
To prove that $N_\mu \subseteq N_\nu$.
Assume that $\mu \subseteq \nu$ and $\mu(0) = \nu(0)$ and let $x \in N_\mu$.
\[
\implies x \in \mathcal{L}_N \text{ such that } \mu(x) = \mu(0). \\
\implies \nu(x) \geq \mu(x) = \mu(0) = \nu(0), \text{ since } \mu \subseteq \nu \\
\implies \nu(x) \geq \nu(0). \\
\implies \nu(x) = \nu(0), \text{ since } \nu(x) \leq \nu(0), \text{ for all } x \in \mathcal{L}_N.
\]
\[ \implies x \in N_\nu. \]

Hence \( N_\mu \subset N_\nu. \)

**Theorem 5.5.** If \( \{\mu_i : i \in \Lambda\} \) is a family of fuzzy left(resp.right) \( N \)-subgroups of \( \mathcal{L}_N \), then \( \bigcap_{i \in \Lambda} \mu_i \) is a fuzzy left(resp.right) \( N \)-subgroups of \( \mathcal{L}_N \), where \( (\bigcap_{i \in \Lambda} \mu_i)(x) = \inf_{i \in \Lambda} \mu_i(x), \) for all \( x \in \mathcal{L}_N \).

**Proof:**

Given that \( \{\mu_i : i \in \Lambda\} \) is a family of fuzzy left(resp.right) \( N \)-subgroups of \( \mathcal{L}_N \). To prove that \( \bigcap_{i \in \Lambda} \mu_i \) is a fuzzy left(resp.right) \( N \)-subgroups of \( \mathcal{L}_N \).

Let \( x, y \in \mathcal{L}_N \) be arbitrary. Then,

\[
(\bigcap_{i \in \Lambda} \mu_i)(x - y) = \inf_{i}(\mu_i(x - y)) \\
\geq \inf_{i}(\min\{\mu_i(x), \mu_i(y)\}) \\
= \min\{\inf_{i} \mu_i(x), \inf_{i} \mu_i(y)\} \\
= \min\{\cap_{i} \mu_i(x), \cap_{i} \mu_i(y)\}.
\]

\[
(\bigcap_{i \in \Lambda} \mu_i)(x \lor y) = \inf_{i}(\mu_i(x \lor y)) \\
\geq \inf_{i}(\min\{\mu_i(x), \mu_i(y)\}) \\
= \min\{\inf_{i} \mu_i(x), \inf_{i} \mu_i(y)\} \\
= \min\{\cap_{i} \mu_i(x), \cap_{i} \mu_i(y)\}.
\]

\[
(\bigcap_{i \in \Lambda} \mu_i)(x \land y) = \inf_{i}(\mu_i(x \land y)) \\
\geq \inf_{i}(\min\{\mu_i(x), \mu_i(y)\}) \\
= \min\{\inf_{i} \mu_i(x), \inf_{i} \mu_i(y)\} \\
= \min\{\cap_{i} \mu_i(x), \cap_{i} \mu_i(y)\}.
\]

If \( x, n \in \mathcal{L}_N \), then we have

\[
(\bigcap_{i \in \Lambda} \mu_i)(nx) = \inf_{i}(\mu_i(nx))
\]
\[
\geq \inf_i (\mu_i(x)) \\
= \bigcap_i \mu_i(x).
\]

(resp. \((\bigcap_{i\in\Lambda} \mu_i)(x_n) = \inf_i (\mu_i(x_n)) \geq \inf_i (\mu_i(x)) = \bigcap_i \mu_i(x).\)

Hence \(\bigcap_{i\in\Lambda} \mu_i\) is a fuzzy left(resp.right) \(N\)-subgroup of \(\mathcal{L}_N\).

**Proposition 5.9.** Let \(f : \mathcal{L}_N \to \mathcal{L'}_N\) be a \(\ell\)-homomorphism from a \(\ell\)-near ring \(\mathcal{L}_N\) to a \(\ell\)-near ring \(\mathcal{L'}_N\). If \(\nu\) is a fuzzy left(resp.right) \(N\)-subgroup of \(\mathcal{L'}_N\), then the preimage of \(\nu\) under \(f\) is a fuzzy left(resp.right) \(N\)-subgroup of \(\mathcal{L}_N\).

**Proof:**

Given that \(f : \mathcal{L}_N \to \mathcal{L'}_N\) is a \(\ell\)-homomorphism and let \(\nu\) is a fuzzy left(resp.right) \(N\)-subgroup of \(\mathcal{L'}_N\).

To prove that the preimage of \(\nu\) under \(f\) is a fuzzy left(resp.right) \(N\)-subgroup of \(\mathcal{L}_N\).

Let \(\mu\) be the preimage of \(\nu\) under \(f\) and it is defined by \(\mu = \nu \circ f\).

\[
\mu(x - y) = (\nu \circ f)(x - y) = \nu(f(x - y)) \\
= \nu(f(x) - f(y)) \\
\geq \min\{\nu(f(x)), \nu(f(y))\} \\
= \min\{\mu(x), \mu(y)\}.
\]

\[
\mu(x \vee y) = (\nu \circ f)(x \vee y) = \nu(f(x \vee y)) \\
= \nu(f(x) \vee f(y)) \\
\geq \min\{\nu(f(x)), \nu(f(y))\} \\
= \min\{\mu(x), \mu(y)\}.
\]

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\[
\mu(x \wedge y) = (\nu \circ f)(x \wedge y) = \nu(f(x) \wedge y) \\
= \nu(f(x) \wedge f(y)) \\
\geq \min\{\nu(f(x)), \nu(f(y))\} \\
= \min\{\mu(x), \mu(y)\}.
\]

\[
\mu(nx) = (\nu \circ f)(nx) = \nu(f(nx)) \\
= \nu(f(n)f(x)) \\
\geq \nu(f(x)) \\
= \mu(x).
\]

(resp. \(\mu(xn) = (\nu \circ f)(xn) = \nu(f(xn))\))

\[
= \nu(f(x)f(n)) \\
\geq \nu(f(x)) \\
= \mu(x).
\]

Hence \(\mu\) is a fuzzy left (resp. right) \(N\)-subgroup of \(\mathcal{L}_N\).

**Theorem 5.6.** If \(f : \mathcal{L}_N \to \mathcal{L}'_N\) is a \(\ell\)-homomorphism from a \(\ell\)-near ring \(\mathcal{L}_N\) to a \(\ell\)-near ring \(\mathcal{L}'_N\), then the \(\ell\)-homomorphic image of a fuzzy left (resp. right) \(N\)-subgroup of \(\mathcal{L}_N\) having the sup property is a fuzzy left (resp. right) \(N\)-subgroup of \(\mathcal{L}_N\).

**Proof:**

Given that \(f : \mathcal{L}_N \to \mathcal{L}'_N\) is a \(\ell\)-homomorphism.

To prove that the \(\ell\)-homomorphic image of a fuzzy left (resp. right) \(N\)-subgroup of \(\mathcal{L}_N\) having the sup property is a fuzzy left (resp. right) \(N\)-subgroup of \(\mathcal{L}_N\).

Let \(\mu\) be a fuzzy left (resp. right) \(N\)-subgroup of \(\mathcal{L}_N\) having the sup property and let \(\nu\) be the image of \(\mu\) under \(f\).

Given that \(f(x), f(y) \in f(\mathcal{L}_N)\) and let \(x_0 \in f^{-1}(f(x))\) and \(y_0 \in f^{-1}(f(y))\)
be such that \( \mu(x_0) = \sup_{t \in f^{-1}(f(x))} \mu(t) \) and \( \mu(y_0) = \sup_{t \in f^{-1}(f(y))} \mu(t) \).

Then,

\[
\nu(f(x) - f(y)) = \sup_{t \in f^{-1}(f(x) - f(y))} \mu(t) \\
= \mu((x_0) - (y_0)) \\
\geq \min\{\mu(x_0), \mu(y_0)\} \\
= \min\{\sup_{t \in f^{-1}(f(x))} \mu(t), \sup_{t \in f^{-1}(f(y))} \mu(t)\} \\
= \min\{\nu(f(x)), \nu(f(y))\}.
\]

\[
\nu(f(x) \lor f(y)) = \sup_{t \in f^{-1}(f(x) \lor f(y))} \mu(t) \\
= \mu((x_0) \lor (y_0)) \\
\geq \min\{\mu(x_0), \mu(y_0)\} \\
= \min\{\sup_{t \in f^{-1}(f(x))} \mu(t), \sup_{t \in f^{-1}(f(y))} \mu(t)\} \\
= \min\{\nu(f(x)), \nu(f(y))\}.
\]

\[
\nu(f(x) \land f(y)) = \sup_{t \in f^{-1}(f(x) \land f(y))} \mu(t) \\
= \mu((x_0) \land (y_0)) \\
\geq \min\{\mu(x_0), \mu(y_0)\} \\
= \min\{\sup_{t \in f^{-1}(f(x))} \mu(t), \sup_{t \in f^{-1}(f(y))} \mu(t)\} \\
= \min\{\nu(f(x)), \nu(f(y))\}.
\]

\[
\nu(f(nx)) = \nu(f(n)f(x)) = \sup_{t \in f^{-1}(f(n)f(x))} \mu(t) \\
= \mu((n_0)(x_0)) \\
\geq \mu(x_0) \\
= \sup_{t \in f^{-1}(f(x))} \mu(t) \\
= \nu(f(x)).
\]

(resp. \( \nu(f(xn)) = \nu(f(x)f(n)) \))

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\[ \begin{align*}
&= \sup_{t \in f^{-1}(f(x) \cap n)} \mu(t) \\
&= \mu((x_0) (n_0)) \\
&\geq \mu(x_0) \\
&= \sup_{t \in f^{-1}(f(x))} \mu(t) = \nu(f(x))
\end{align*} \]

Hence \( \nu \) is a fuzzy left (resp. right) \( N \)-subgroup of \( \mathcal{L}_N \).

**Definition 5.3.** Let \( f : \mathcal{L}_N \to \mathcal{L}'_N \) be a \( \ell \)-homomorphism from a \( \ell \)-near ring \( \mathcal{L}_N \) to a \( \ell \)-near ring \( \mathcal{L}'_N \). Then \( f \) is called a \( \ell \)-endomorphism if \( \mathcal{L}_N = \mathcal{L}'_N \) and \( f \) is called a \( \ell \)-automorphism if \( f \) is an isomorphism and \( \mathcal{L}_N = \mathcal{L}'_N \). For this \( \ell \)-endomorphism, we define a fuzzy set \( \mu \) in \( \mathcal{L}_N \) by \( \mu^f = \mu(f(x)) \), for all \( x \in \mathcal{L}_N \).

**Proposition 5.10.** Let \( f \) be a \( \ell \)-endomorphism of \( \mathcal{L}_N \). If \( \mu \) is a fuzzy left (resp. right) \( N \)-subgroup of \( \mathcal{L}_N \), then \( \mu^f(x) = \mu(f(x)) \) is a fuzzy left (resp. right) \( N \)-subgroup of \( \mathcal{L}_N \), for all \( x \in \mathcal{L}_N \).

**Proof:**

Given that \( f \) is a \( \ell \)-endomorphism of \( \mathcal{L}_N \) and let \( \mu \) is a fuzzy left (resp. right) \( N \)-subgroup of \( \mathcal{L}_N \).

To prove that \( \mu^f(x) = \mu(f(x)) \) is a fuzzy left (resp. right) \( N \)-subgroup of \( \mathcal{L}_N \), for all \( x \in \mathcal{L}_N \).

Let \( x, y \in \mathcal{L}_N \) be arbitrary. Then

\[ \begin{align*}
\mu^f(x - y) &= \mu(f(x - y)) \\
&= \mu(f(x) - f(y)) \\
&\geq \min \{ \mu(f(x)), \mu(f(y)) \} \\
&= \min \{ \mu^f(x), \mu^f(y) \}. \\
\mu^f(x \lor y) &= \mu(f(x \lor y))
\end{align*} \]
\[
\begin{align*}
&= \mu(f(x) \lor f(y)) \\
\geq& \ \min\{\mu(f(x)), \mu(f(y))\} \\
&= \min\{\mu^f(x), \mu^f(y)\}. \\
\mu^f(x \land y) &= \mu(f(x \land y)) \\
&= \mu(f(x) \land f(y)) \\
\geq& \ \min\{\mu(f(x)), \mu(f(y))\} \\
&= \min\{\mu^f(x), \mu^f(y)\}.
\end{align*}
\]

For every \(x, n \in \mathcal{L}_N\), then we have
\[
\begin{align*}
\mu^f(nx) &= \mu(f(n)f(x)) \\
&= \mu(f(x)) \\
&= \mu^f(x). \\
(\text{resp. } \mu^f(nx) &= \mu(f(n)f(x)) \\
&= \mu(f(x)) \\
&= \mu^f(x)).
\end{align*}
\]

Hence \(\mu^f(x) = \mu(f(x))\) is a fuzzy left\(\text{(resp. right)}\) \(N\)-subgroup of \(\mathcal{L}_N\).

**Definition 5.4.** A left\(\text{(resp. right)}\) \(N\)-subgroup \(H\) of \(\mathcal{L}_N\) is called a characteristic left\(\text{(resp. right)}\) \(N\)-subgroup \(H\) of \(\mathcal{L}_N\) if \(f(H) = H\), for all \(f \in \text{Aut}(\mathcal{L}_N)\).

**Definition 5.5.** A fuzzy left\(\text{(resp. right)}\) \(N\)-subgroup of \(\mathcal{L}_N\) is said to be a fuzzy characteristic if \(\mu^f(x) = \mu(x)\), for all \(x \in \mathcal{L}_N\).

**Theorem 5.7.** If \(\{\mu_i : i \in \Lambda\}\) is the collection of fuzzy characteristic left\(\text{(resp. right)}\) \(N\)-subgroup of \(\mathcal{L}_N\), then \(\bigcap_{i \in \Lambda} \mu_i\) is a fuzzy characteristic left\(\text{(resp. right)}\) \(N\)-subgroup of \(\mathcal{L}_N\), where \(\Lambda\) is any index set.
Proof:
Given that \( \{ \mu_i : i \in \Lambda \} \) is the collection of fuzzy characteristic left(resp.right) 
\( N \)-subgroup of \( \mathcal{L}_N \).
To prove \( \bigcap_{i \in \Lambda} \mu_i \) is a fuzzy characteristic left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \).
Let \( \bigcap_{i \in \Lambda} \mu_i = \inf_{i \in \Lambda} \mu_i(x) \), for all \( x \in \mathcal{L}_N \) and by theorem 5.5, we have \( \bigcap_{i \in \Lambda} \mu_i \) is a 
fuzzy left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \).
Now we prove that \( \bigcap_{i \in \Lambda} \mu_i \) is a fuzzy characteristic.
Let \( x \in \mathcal{L}_N \) and \( f \in \text{Aut}(\mathcal{L}_N) \). Then

\[
(\bigcap_{i \in \Lambda} \mu_i)^f(x) = (\bigcap_{i \in \Lambda} \mu_i)(f(x)) \\
= \inf_{i \in \Lambda} (\mu_i(f(x))) \\
= \inf_{i \in \Lambda} (\mu_i^f(x)) \\
= \bigcap_{i \in \Lambda} \mu_i(x) \\
= (\bigcap_{i \in \Lambda} \mu_i)(x).
\]

Hence \( \bigcap_{i \in \Lambda} \mu_i \) is a fuzzy characteristic left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \).

Proposition 5.11. Let \( \mu \) be a fuzzy left(resp.right) \( N \)-subgroup of \( \mathcal{L}_N \). Then 
\( \mu(x) = t \) if and only if \( x \in \mu_t \) and \( x \notin \mu_s \), for all \( s > t \) in \([0,1]\) and \( x \in \mathcal{L}_N \).

Proof:
Assume that \( \mu(x) = t \).
To prove that \( x \in \mu_t \) and \( x \notin \mu_s \), for all \( s > t \) in \([0,1]\).
Since \( t < s \), then we have \( t = \mu(x) < s \), by assumption.
Hence \( x \in \mu_t \) and \( x \notin \mu_s \).
Conversely, assume that \( x \in \mu_t \) and \( x \notin \mu_s \), for all \( s > t \) in \([0,1]\).
To prove that \( \mu(x) = t \).
By assumption, we have \( \mu(x) \geq t \) and \( \mu(x) < s \).
\( \Rightarrow \mu(x) = t \), since \( s > t \).
**Theorem 5.8.** Let $\mu$ be a fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$. Then the following are equivalent:

(i) $\mu$ is a fuzzy characteristic;

(ii) Each level left(resp.right) $N$-subgroup of $\mu$ is characteristic.

**Proof:**

(i) $\implies$ (ii)

By (i), we have $\mu^f(x) = \mu(x)$, for all $x \in \mathcal{L}_N$ and $f \in \text{Aut}(\mathcal{L}_N)$.

Let $t \in \text{Im}(\mu)$, $f \in \text{Aut}(\mathcal{L}_N)$ and $x \in \mu_t$.

$\implies \mu^f(x) = \mu(x) \geq t$

$\implies \mu(f(x)) \geq t$

$\implies f(x) \in \mu_t$

$\implies f(\mu_t) \subseteq \mu_t$. \hspace{1cm} (1)

Now, let $x \in \mu_t$ and $y \in \mathcal{L}_N$ such that $f(y) = x$.

Then $\mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq t$.

$\implies y \in \mu_t$.

But $x = f(y) \in f(\mu_t)$.

$\implies \mu_t \subseteq f(\mu_t)$ \hspace{1cm} (2)

$\implies f(\mu_t) = \mu_t$, by (1) and (2).

(ii) $\implies$ (i)

Let $x \in \mathcal{L}_N$, $f \in \text{Aut}(\mathcal{L}_N)$ and $\mu(x) = t$.

Then by proposition 5.11, we have $x \in \mu_t$ and $x \not\in \mu_s$, for all $s > t$ in $[0,1]$.

$\implies f(x) \in f(\mu_t) = \mu_t$, by (ii).

$\implies f(x) \in \mu_t$

$\implies \mu(f(x)) \geq t$

$\implies \mu^f(x) \geq t$.

Suppose that $s = \mu^f(x)$ and $s > t$.

$\implies \mu(f(x)) = s$
\[ \implies \mu(f(x)) \geq s \]
\[ \implies f(x) \in \mu_s = f(\mu_s) \]
\[ \implies x \in \mu_s, \text{ since } f \text{ is injective.} \]
Which is a contradiction to \( x \notin \mu_s \).
Therefore \( \mu^f(x) = t = \mu(x) \).
Hence \( \mu \) is a fuzzy characteristic.

**Theorem 5.9.** Let \( \mu \) be a fuzzy subset in \( \mathcal{L}_N \) defined by \( \mu(x) = \begin{cases} g & \text{if } x \in H \\ h & \text{if } x \notin H \end{cases} \)
where \( g, h \in [0, 1] \) with \( g > h \). If \( H \) is a characteristic left(resp.right) \( \mathcal{N} \)-subgroup of \( \mathcal{L}_N \), then \( \mu \) is a fuzzy characteristic left(resp.right) \( \mathcal{N} \)-subgroup.

**Proof:**
Given that \( \mu \) is a fuzzy subset in \( \mathcal{L}_N \) defined by \( \mu(x) = \begin{cases} g & \text{if } x \in H \\ h & \text{if } x \notin H \end{cases} \)
where \( g, h \in [0, 1] \) with \( g > h \).
Assume that \( H \) is a characteristic left(resp.right) \( \mathcal{N} \)-subgroup.
To prove that \( \mu \) is a fuzzy characteristic left(resp.right) \( \mathcal{N} \)-subgroup.
By theorem 5.2, if \( H \) is a left(resp.right) \( \mathcal{N} \)-subgroup, then \( \mu \) is a fuzzy left (resp.right) \( \mathcal{N} \)-subgroup.
Now, we prove \( \mu \) is a fuzzy characteristic.
Let \( x \in \mathcal{L}_N \) and \( f \in \text{Aut}(\mathcal{L}_N) \).
If \( x \in H \), then \( \mu(x) = g \) and also \( f(x) \in f(H) = H \), by assumption.
\[ \implies \mu^f(x) = \mu(f(x)) = g = \mu(x) \]
\[ \implies \mu^f(x) = \mu(x). \]
If \( x \notin H \), then \( \mu(x) = h \) and also \( f(x) \notin f(H) = H \).
\[ \implies \mu^f(x) = \mu(f(x)) = h = \mu(x). \]
Hence \( \mu \) is a fuzzy characteristic.

**Definition 5.6.** A fuzzy left(resp.right) \( \mathcal{N} \)-subgroup \( \mu \) of a \( \ell \)-near ring \( \mathcal{L}_N \) is said to be normal if there exists \( x \in \mathcal{L}_N \) such that \( \mu(x) = 1 \).
Remark 5.1. If a fuzzy left(resp.right) $N$-subgroup $\mu$ of $\mathcal{L}_N$ is normal, then $\mu(0) = 1$. Since $\mu$ is normal, there exists $x \in \mathcal{L}_N$ such that $\mu(x) = 1$.

$\implies \mu(0) = \mu(x - x) \geq \min\{\mu(x), \mu(x)\} = \min\{1, 1\} = 1.

\implies \mu(0) \geq 1.

\implies \mu(0) = 1$, since $\mu(0) \geq \mu(1)$, for all $x \in \mathcal{L}_N$.

Proposition 5.12. If $\mu$ and $\nu$ are two normal fuzzy left(resp.right) $N$-subgroups of $\mathcal{L}_N$ satisfying $\mu \subset \nu$, then $N_\mu \subset N_\nu$.

Proof:
Assume that $\mu$ and $\nu$ are normal fuzzy left(resp.right) $N$-subgroups of $\mathcal{L}_N$ and $\mu \subset \nu$. To prove that $N_\mu \subset N_\nu$.

By assumption, we have $\mu(0) = 1$, $\nu(0) = 1$ and so $\mu(0) = \nu(0)$.

Hence $N_\mu \subset N_\nu$, by proposition 5.8.

Theorem 5.10. If $H$ is any left(resp.right) $N$-subgroup of $\mathcal{L}_N$, then the characteristic function of $H$, $\chi_H$ is a normal fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ and $\chi_H = H$.

Proof:
Given that $H$ is a left(resp.right) $N$-subgroup of $\mathcal{L}_N$.

To prove that $\chi_H$ is a normal fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ and $\chi_H = H$.

By theorem 5.2, we have $\chi_H$ is a fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ and $\chi_H = H$. Next we prove $\chi_H$ is normal.

If $x \in H$, then $\chi_H(x) = 1$.

Theorem 5.11. Let $\mu$ be a normal fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ and let $\mu^*$ be a fuzzy subset of $\mathcal{L}_N$ defined by $\mu^*(x) = \mu(x) + 1 - \mu(0)$, for all $x \in \mathcal{L}_N$. Then $\mu^*$ is a normal fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ containing $\mu$. 

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Proof:

Given that $\mu$ is a normal fuzzy left (resp. right) $N$-subgroup of $\mathcal{L}_N$ and let $\mu^*(x) = \mu(x) + 1 - \mu(0)$, for all $x \in \mathcal{L}_N$.

To prove that $\mu^*$ is a normal fuzzy left (resp. right) $N$-subgroup of $\mathcal{L}_N$ containing $\mu$.

Let $x, y \in \mathcal{L}_N$ be arbitrary. Then

$$
\begin{align*}
\min\{\mu^*(x), \mu^*(y)\} &= \min\{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\} \\
&= \min\{\mu(x), \mu(y)\} + 1 - \mu(0) \\
&\leq \mu(x - y) + 1 - \mu(0) \\
&= \mu^*(x - y).
\end{align*}
$$

$$
\begin{align*}
\min\{\mu^*(x), \mu^*(y)\} &= \min\{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\} \\
&= \min\{\mu(x), \mu(y)\} + 1 - \mu(0) \\
&\leq \mu(x \lor y) + 1 - \mu(0) \\
&= \mu^*(x \lor y).
\end{align*}
$$

$$
\begin{align*}
\min\{\mu^*(x), \mu^*(y)\} &= \min\{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\} \\
&= \min\{\mu(x), \mu(y)\} + 1 - \mu(0) \\
&\leq \mu(x \land y) + 1 - \mu(0) \\
&= \mu^*(x \land y).
\end{align*}
$$

and for all $x, n \in \mathcal{L}_N$, we have

$$
\begin{align*}
\mu^*(nx) &= \mu(nx) + 1 - \mu(0) \\
&\geq \mu(x) + 1 - \mu(0) \\
&= \mu^*(x).
\end{align*}
$$

(resp. $\mu^*(xn) = \mu(xn) + 1 - \mu(0) \geq \mu(x) + 1 - \mu(0) = \mu^*(x)$).
Hence $\mu^*$ is a fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$.

Now we prove $\mu^*$ is normal and containing $\mu$.

Since $\mu$ is normal and $\mu(0) = 1$, then $\mu^*(x) = 1$ and for every $x \in \mathcal{L}_N$, $\mu^*(x) \geq \mu(x)$, that is $\mu \subset \mu^*$.

**Corollary 5.2.** A fuzzy left(resp.right) $N$-subgroup $\mu$ of $\mathcal{L}_N$ is normal if and only if $\mu^* = \mu$ and also $(\mu^*)^* = \mu^*$.

**Proof:**

Given that $\mu$ is a fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ and $\mu(0) = 1$.

To prove that $\mu$ is normal if and only if $\mu^* = \mu$.

We have $\mu^*(x) = \mu(x) + 1 - \mu(0) = \mu(x)$, since $\mu(0) = 1$. Hence $\mu^* = \mu$.

Conversely, assume that $\mu^* = \mu$. To prove that $\mu$ is normal.

By theorem we have $\mu^*$ is a normal fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$, then $\mu$ is normal, by assumption.

Also $(\mu^*)^*(x) = \mu^*(x) + 1 - \mu^*(0) = \mu(x) + 1 - \mu(0) = \mu^*(x)$, that is $(\mu^*)^* = \mu^*$.

**Remark:**

1. If $\mu$ is a normal fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ with $\mu(0) = 1$, then $\mu^*(0) = 1$.

2. If $\mu$ is a fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$ satisfying $\mu^*(x) = 0$, for some $x \in \mathcal{L}_N$ then $\mu(x) = 0$.

3. If $\mu$ is a normal fuzzy left(resp.right) $N$-subgroup of $\mathcal{L}_N$, then $(\mu^*)^* = \mu^* = \mu$.