CHAPTER 3

TRANSFORMATIONS ON THE DISCRETE HOLOMETRIC SPACE

In this chapter, we introduce the concept of transformations on the discrete plane. We further investigate those, which preserve certain metric relations. Of principal interest are the discrete transformations which preserve distance, domains, r-sets etc and D-linear transformations. These are discussed in sections 1 and 2. In section 3, certain group theoretic properties are investigated. Section 4 deals with discrete analytic properties of these transformations.

3.1. DISCRETE TRANSFORMATIONS

DEFINITION 3.1.1. A bijective mapping of \( H \) onto itself is called a D-transformation.

NOTATION: D-transformations will in general be denoted by \( T, T_1, T_2, T_3 \) etc.

DEFINITION 3.1.2. A D-transformation \( T \) with the property that for every \( z_1, z_2 \in H \), \( d(z_1, z_2) = d(T(z_1), T(z_2)) \) is called a D-isometry.

+ Some results of this chapter were presented as a paper entitled "Geometry of the discrete plane" in the 50th Session of IMS during February 1985.
DEFINITION 3.1.3. A D-transformation $T$ defined by

$$T(q^m x_0, q^n y_0) = (q^{m+a} x_0, q^{n+b} y_0)$$

where $(q^a x_0, q^b y_0)$ is a fixed point in $H$ is called a D-translation.

EXAMPLES 3.1.4.

(1) $T_1(q^m x_0, q^n y_0) = (q^{-m-1} x_0, q^{n+2} y_0)$ is a D-isometry

(2) $T_2(q^m x_0, q^n y_0) = (q^{m+n} x_0, q^{m-n} y_0)$ is not a D-isometry

(3) $T_3(q^m x_0, q^n y_0) = (q^{m+3} y_0, q^{n+4} y_0)$ is a D-translation

Figures 4 and 5 illustrate the transformations $T_1$ and $T_2$. We shall denote by $z_0$ the origin of the image plane also and by $w_1, w_2, \ldots$, the image of $z_1, z_2, \ldots$.

THEOREM 3.1.5. All D-translations are D-isometries.

PROOF. Let $T : H \longrightarrow H$ be a D-transformation. That is, there exists $(q^a x_0, q^b y_0) \in H$ such that

$$T(q^m x_0, q^n y_0) = (q^{m+a} x_0, q^{n+b} y_0)$$

for every $m, n \in \mathbb{Z}$. Now let $z_1 = (q^{m_1} x_0, q^{n_1} y_0)$ and

$$z_2 = (q^{m_2} x_0, q^{n_2} y_0)$$

be any two points of $H$. Then
Figure 4

\[ T_1(q^mx_0, q^ny_0) = (q^{m-1}x_0, q^{n+1}y_0) \]

\[ z_i \longrightarrow w_i \text{ for } i = 0, 1, 2, \ldots, 8. \]
\[ d(T(z_1), T(z_2)) = d((q^{1+a_1} x_0, q^{1+b_1} y_0), (q^{2+a_2} x_0, q^{2+b_2} y_0)) \]
\[ = |(m_2+a) -(m_1+a)| + |(n_2+b)-(n_1+b)| \]
\[ = |m_2-m_1| + |n_2-n_1| \]
\[ = d(z_1, z_2). \]

Hence \( T \) is a D-isometry.

**THEOREM 3.1.6.** If \( D \) is a domain and \( T:H \rightarrow H \) is a D-isometry, then \( T(D) \) is also a domain.

**PROOF.** Let \( D = \bigcup_{i=1}^{t} B_i \) be a domain. Then the basic sets \( B_i \) and \( B_{i+1} \) are adjacent and for any two points \( z, \xi \) of \( D \), there are points \( z = z_1, z_2, \ldots, \xi = z_n \) in \( D \) such that \( d(z_i, z_{i+1}) = 1 \). Since \( T \) is a D-isometry, the points \( z_1, z_2, \ldots, z_n \) will be mapped onto points \( w_1, w_2, \ldots, w_n \) with \( d(w_i, w_{i+1}) = 1 \) and further the adjacency of basic sets will be preserved. Hence \( T(D) \) is also a domain.
3.2. SOME SPECIAL TYPE OF D-TRANSFORMATIONS

In this section, we shall characterise D-linear transformations and the transformations which preserve the property of being an r-set.

DEFINITION 3.2.1. Let $L_1$ be the D-linear set \( \{z_i\}_{i=1}^t = \{(q_i^{m_i}x_i, q_i^{n_i}y_i)\}_{i=1}^t \) and $L_2$ be the D-linear set, \( \{w_i\}_{i=1}^t = \{(q_i^{\alpha_i}x_i, q_i^{\beta_i}y_i)\}_{i=1}^t \). Consider a D-transformation $T : H \rightarrow H$ taking $L_1$ onto $L_2$. Then

(1) $T$ is called a horizontal reversal (on $L_1$), if \( \{m_i\}_{i=1}^t \) is monotonic increasing (decreasing) implies that \( \{\alpha_i\}_{i=1}^t \) is monotonic decreasing (increasing). It is called a vertical reversal (on $L_1$) if \( \{n_i\}_{i=1}^t \) is monotonic increasing (decreasing) implies that \( \{\beta_i\}_{i=1}^t \) is monotonic decreasing (increasing).

(2) $T$ is called a horizontal enlargement if \( |m_i - m_j| < |\alpha_i - \alpha_j| \), and a vertical enlargement if \( |n_i - n_j| < |\beta_i - \beta_j| \), for every $j > i = 1, 2, \ldots, t$. 
(3) \( T \) is called a horizontal contraction if \( |m_i - m_j| > |\alpha_i - \alpha_j| \), and a vertical contraction if \( |n_i - n_j| > |\beta_i - \beta_j| \) for every \( j > i = 1, 2, \ldots, t \).

**Definition 3.2.2.** A D-transformation \( T : H \rightarrow H \) is called a D-linear transformation if it takes D-linear sets onto D-linear sets and is a reversal, enlargement or contraction, on any D-linear set \( L \), horizontally as well as vertically (not necessarily of the same type).

**Theorem 3.2.3.** A D-transformation \( T : H \rightarrow H \) takes the D-linear set \( L_1 \) onto the D-linear set \( L_2 \) and is a reversal, enlargement or contraction, horizontally as well as vertically (not necessarily of the same type) if and only if \( \alpha_i = m_i + p_i \), \( \beta_i = n_i + s_i \) where \( \{(q^i x_0, q^i y_0)\}_{i=1}^t \) is D-linear.

**Proof.** Let us suppose that \( T : H \rightarrow H \) is a D-transformation taking \( L_1 \) onto \( L_2 \) and is a reversal, enlargement or contraction, horizontally as well as vertically, then we have to prove that \( \alpha_i = m_i + p_i \) and \( \beta_i = n_i + s_i \), where \( (q^i x_0, q^i y_0) \) is D-linear.
Since, $T$ is a horizontal reversal, we have $m_1$s are monotonic increasing (decreasing) implies that $a_1$s are monotonic decreasing (increasing). Suppose that $m_1$s are increasing and $a_1$s are decreasing. Then $m_j = m_1 + \sum_{i=1}^{j-1} a_i$, $a_1 \geq 0$ and $a_j = a_1 + \sum_{i=1}^{j-1} c_i$, $c_i \leq 0$. Further, since $T$ is a D-transformation, we can express $a_j$s and $b_j$s in terms of $m_j$s and $n_j$s as, $a_j = m_j + p_j$ and $b_j = n_j + s_j$ where $p_j$, $s_j \in \mathbb{Z}$.

Therefore, $a_j = m_1 + \sum_{i=1}^{j-1} a_i + p_j$. That is, $a_1 + \sum_{i=1}^{j-1} c_i = m_1 + \sum_{i=1}^{j-1} a_i + p_j$. So for $j > k$ we have

$$p_j - p_k = \sum_{i=k}^{j-1} c_i - \sum_{i=k}^{j-1} a_i$$

(7)$$< 0 \text{ since } c_i \leq 0 \text{ and } a_i > 0.$$ Hence $\{p_j\}_{j=1}^t$ is monotonic decreasing. Now, if $m_1$s are decreasing and $a_1$s are increasing, then $p_j - p_k$ in (7) is greater than zero and consequently $p_j \quad j=1^t$ is monotonic increasing.
Now, if $T$ is a horizontal enlargement, we have as above, an expression (7), where we do not have any condition on the signs of $c_i$'s or $a_i$'s, but then being a horizontal enlargement, we have, $c_i > a_i$ for every $i = k, \ldots, j-1$ and so $\{p_j\}_{j=1}^{t}$ is monotonic increasing. Further, if $T$ is a horizontal contraction, then $\{p_j\}_{j=1}^{t}$ is monotonic decreasing.

Also, when $T$ is a vertical reversal it can be proved along similar lines that $\{s_j\}_{j=1}^{t}$ is either monotonic decreasing or increasing, when $T$ is a vertical enlargement $\{s_j\}_{j=1}^{t}$ is increasing and when $T$ is a vertical contraction, $\{s_j\}_{j=1}^{t}$ is decreasing. Thus, it is proved that $\{p_j\}, \{s_j\}$ are either monotonic increasing or decreasing, not necessarily of the same type. Hence by theorem 2.3.8, it follows that $\{(q_i x_0, q_i y_0)\}_{i=1}^{t}$ is $D$-linear.

Conversely, suppose that $T$ maps the $D$-linear set $L_1 = \{(q_i x_0, q_i y_0)\}_{i=1}^{t}$ onto $L_2 = \{(q_i x_0, q_i y_0)\}_{i=1}^{t}$
and let \( \alpha_i = m_i + p_i \), \( \beta_i = n_i + s_i \) where \( \{(q^i x_0, q^i y_0)\}_{i=1}^t \) is D-linear. Then, required to prove that, \( \{(q^i x_0, q^i y_0)\}_{i=1}^t \) is D-linear and \( T \) is a reversal, enlargement or contraction, horizontally as well as vertically. We have

\[
\{(q^i x_0, q^i y_0)\}_{i=1}^t \text{ and } \{(q^i x_0, q^i y_0)\}_{i=1}^t \text{ are D-linear.}
\]

So, let us suppose that \( m_1, n_1, p_1, q_1 \) all are monotonic increasing. Then

\[
\sum_{i=1}^{t-1} d(w_i, w_{i+1}) = \sum_{i=1}^{t-1} (|\alpha_{i+1} - \alpha_i| + |\beta_{i+1} - \beta_i|)
\]

\[
= \sum_{i=1}^{t-1} (|m_{i+1} + p_{i+1} - m_i - p_i| + |n_{i+1} + s_{i+1} - n_i - s_i|).
\]

\[
= \sum_{i=1}^{t-1} [(m_{i+1} - m_i) + (p_{i+1} - p_i) + (n_{i+1} - n_i) + (s_{i+1} - s_i)].
\]

\[
= [(m_t - m_1) + (p_t - p_1) + (n_t - n_1) + (s_t - s_1)].
\]

\[
= |\alpha_t - \alpha_1| + |\beta_t - \beta_1|.
\]

\[
d(w_1, w_t). \text{ Thus, } \{w_i\}_{i=1}^t \text{ is D-linear.}
\]
Now, let us take $m_i$'s to be decreasing and $p_i$'s to be increasing. So, we have $m_1 > m_2 > m_3 \ldots > m_t$ and $p_1 < p_2 \ldots < p_t$. So from (7), without any restriction on the signs of $c_i$'s but $a_i \leq 0$, $c_1 < a_1$, $c_1+c_2 < a_1+a_2$, \ldots, $c_1+c_2+c_3 + \ldots + c_t < a_1+a_2+a_3 + \ldots + a_t$. Hence, $a_2-a_1 < m_2-m_1 < 0$; $a_3-a_2 = c_1+c_2 < a_1+a_2 = m_3-m_2 < 0$ etc. That is, $a_i$'s are increasing. Similarly, when $n_i$'s are decreasing and $s_i$'s are increasing, then $\beta_i$'s are increasing and so $\{(q^i_x,q^i_y)\}$ is D-linear. These are the only typical cases and for all other cases, the result can be proved on similar lines.

Now, if $m_i$'s are increasing and $p_i$'s are decreasing, then $a_i$'s are decreasing and hence $T$ is a horizontal reversal. If $m_i$'s are decreasing and $p_i$'s are increasing, then $T$ is a horizontal enlargement and if $m_i$'s are increasing and $p_i$'s are decreasing, then $T$ is a horizontal contraction. Similar conditions imposed on $n_i$'s and $s_i$'s will prove that $T$ is a vertical reversal, enlargement or contraction.
Thus, converse part also is proved. Hence the theorem. In the following theorem, we characterise D-linear transformations.

**THEOREM 3.2.4.** A D-transformation \( T : H \rightarrow H \) is a D-linear transformation if and only if

\[
T(q^m x_0, q^n y_0) = (q^\alpha x_0, q^\beta y_0),
\]

where \( \alpha = m + a_m \), \( \beta = n + b_n \) and \( \{a_i\}_{i=-\infty}^{\infty} \), \( \{b_i\}_{i=-\infty}^{\infty} \) are monotonic increasing or decreasing, not necessarily of the same type.

Proof follows from the above theorem, and is omitted.

**NOTE 3.2.5.** Any D-isometry \( T : H \rightarrow H \) carries D-linear sets to D-linear sets, but not necessarily a D-linear transformation. Converse also is not true.

**NOTE 3.2.6.** We shall now consider certain transformations which map \( r \)-sets onto \( r \)-sets. Clearly, D-transformations need not carry \( r \)-sets onto \( r \)-sets. In the study of transformations of this type, since D-transformations are bijective, we need consider only \( r \)-sets of equal radii.
NOTE 3.2.7. A set of points of \( H \) satisfying the conditions of theorem 2.3.8 in this context are called oriented set of points.

THEOREM 3.2.8. A \( D \)-transformation leaves invariant an \( r \)-set with centre at the origin and preserve the centre and orientation of points on it if and only if it is one among the eight transformations belonging to

\[
T^* = \{ T_i \}_{i=1}^8
\]

where \( T_i \) carries \((q^m x_0, q^n y_0)\) to

\[
(q^m x_0, q^n y_0), (q^{-m} x_0, q^n y_0), (q^{-m} x_0, q^{-n} y_0), (q^m x_0, q^{-n} y_0),
(q^n x_0, q^m y_0), (q^{-n} x_0, q^m y_0), (q^{-n} x_0, q^{-m} y_0) \text{ and } (q^n x_0, q^{-m} y_0)
\]

for \( i=1,2, \ldots, 8 \), respectively.

PROOF. Consider the \( r \)-set with origin as centre and radius \( r_1, S_{r_1}(z_0) \). It is clear that every transformation in \( T^* \) leaves invariant the \( r \)-set and preserve the centre. It remains to show that they preserve the orientation of points on the \( r \)-set. We know that the \( 4r_1 \) points on \( S_{r_1}(z_0) \) can be classified into a disjoint union of four sets as,
where \( a = 0, 1, 2, \ldots, r-1 \). It is an easy consequence of the definition that all the \( L_i \)'s are D-linear sets. Each \( T_i \) in \( T^* \) carries a \( L_i \) to some \( L_j \). For example, under \( T_3 \), \( L_1 \leftrightarrow L_3 \) and \( L_2 \leftrightarrow L_4 \). Thus each \( T_i \) preserve orientation.

Conversely, if \( T \) is a D-transformation which leaves invariant \( S_{r_1}(z_0) \), preserving the centre and orientation then \( T \in T^* \). For, since the centre has to be preserved, the transformations should be of the form \((q^m x_0, q^n y_0) \longrightarrow (q^{am} x_0, q^{bn} y_0); a, b \in \mathbb{Z}\). But \( a, b \) have to be either +1 or -1, since the transformations are bijective. Hence by definition of \( S_{r_1}(z_0) \), it is preserved under a D-transformation only if the transformation is one (1) which keeps \( m \) and \( n \) fixed,

(2) which changes the signs of \( m \) and \( n \), or (3) which changes the points as well as signs of \( m \) and \( n \).

That is, the required transformations are in \( T^* \). Hence the theorem.
We shall now discuss two more situations concerning the transformations of r-sets. They are those (1) which take an r-set with centre origin onto an r-set with centre \((q^a x_0, q^b y_0)\); \(a, b \neq 0\), and (2) in which an r-set with centre \(z_1 = (q^m x_0, q^n y_0)\), \(m, n \neq 0\) is mapped onto an r-set with centre \(w_1 = (q^{a_1} x_0, q^{b_1} y_0)\), \(a_1, b_1 \neq 0\). These two cases exhaust all the possibilities because the transformation which takes an \(S_{r_1}(z_1)\) onto \(S_{r_1}(w_1)\) maps \(z_1\) to \(w_1\). The result obtained in this direction is a consequence of the above characterization theorem and are considered in the following corollories.

COROLLORIES 3.2.8.

(1) A \(D\)-transformation takes an r-set with centre at the origin to an r-set with centre \((q^a x_0, q^b y_0)\), \(a, b \neq 0\) and preserve the orientation of points on it, if and only if it is one among the transformations belonging to \(G = \{g_i\}_{i=1}^8\) where \(g_i\) carries \((q^m x_0, q^n y_0)\) to \((q^{m+a} x_0, q^{n+b} y_0)\), \((q^{-m+a} x_0, q^{n+b} y_0)\), \((q^{-m+a} x_0, q^{-n+b} y_0)\), \((q^{m+a} x_0, q^{-n+b} y_0)\), \((q^{-m+a} x_0, q^{-n+b} y_0)\), \((q^{m+a} x_0, q^{-n+b} y_0)\), \((q^{-m+a} x_0, q^{n+b} y_0)\), \((q^{m+a} x_0, q^{n+b} y_0)\).
(q^{m_a} x_0, q^{-n_b} y_0), (q^{n_a} x_0, q^{m_b} y_0), (q^{-n_a} x_0, q^{m_b} y_0),
(q^{n_a} x_0, q^{-m_b} y_0) \text{ and } (q^{n_a} x_0, q^{-m_b} y_0) \text{ for } i = 1, 2, \ldots, 8,
respectively.

(2) An r-set with centre $z_1 = (q^{m_1} x_0, q^{n_1} y_0)$;

$m_1, n_1 \neq 0$ is mapped to an r-set with centre $w_1 = g^*(z_1) =
(q^{m_1} x_0, q^{n_1} y_0), a_1, b_1 \neq 0$ and preserve the orientation of
points of it if and only if $g^*$ is one of the transforma-
tions belonging to

$G = \{g_i^*\}_{i=1}^8$ where $g_i^*$ carries $(q^{m_1} x_0, q^{n_1} y_0)$ to

\[
\begin{align*}
\lambda \cdot (q^{m_1} x_0, q^{n_1} y_0), & \quad (q^{m_1} x_0, q^{n_1} y_0), \\
\lambda \cdot (q^{m_1} x_0, q^{n_1} y_0), & \quad (q^{m_1} x_0, q^{n_1} y_0), \\
\lambda \cdot (q^{m_1} x_0, q^{n_1} y_0), & \quad (q^{m_1} x_0, q^{n_1} y_0), \\
\lambda \cdot (q^{m_1} x_0, q^{n_1} y_0), & \quad (q^{m_1} x_0, q^{n_1} y_0), \\
\lambda \cdot (q^{m_1} x_0, q^{n_1} y_0), & \quad (q^{m_1} x_0, q^{n_1} y_0),
\end{align*}
\]

for $i = 1, 2, \ldots, 8$, respectively.
3.3. GROUP THEORETIC PROPERTIES OF SOME SPECIAL TYPE OF D-TRANSFORMATIONS

DEFINITION 3.3.1. Let $T_1$ and $T_2$ be two D-transformations. Then, we define $T_1 \circ T_2(z) = T_1(T_2(z))$.

THEOREM 3.3.2. $(T^*, \circ)$ is a finite, non commutative, solvable, nilpotent group.

PROOF. $T^*$ consists of transformations leaving invariant an $r$-set with centre origin and preserve the centre and orientation of points on it, and by theorem 3.2.8 these are transformations defined by

$$T_1(z) = z, \quad T_2(z) = (q^{-m}x_0, q^n y_0), \quad T_3(z) = (q^{-m}x_0, q^{-n} y_0),$$

$$T_4(z) = (q^m x_0, q^{-n} y_0), \quad T_5(z) = (q^n x_0, q^m y_0), \quad T_6(z) = (q^{-n} x_0, q^m y_0)$$

$$T_7(z) = (q^{-n} x_0, q^{-m} y_0) \text{ and } T_8(z) = (q^n x_0, q^{-m} y_0) \text{ where}$$

$z = (q^m x_0, q^n y_0)$. The transformations satisfy the composition table given in page 75. The transformations satisfy all the group axioms and hence $(T^*, \circ)$ is a group, which is clearly a finite group. The transformations $T_4$ and $T_5$ give a pair of non commuting elements of $T^*$ and hence the group is non abelian. Further by a result in [16], since the order of
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the group is 8, which is a prime power, it is solvable. Also, the centre of the group is \( c = \{T_1, T_3\} \) and \( T/c \) is abelian. Hence \( T \) is nilpotent. Hence the theorem.

Let us further analyse the properties of the group \((T^*, 0)\). It has the following subgroups.

\[
\begin{align*}
s_1 &= \{T_1\} , \\ s_2 &= \{T_1, T_2\} , \\ s_3 &= \{T_1, T_3\} , \\ s_4 &= \{T_1, T_4\} , \\ s_5 &= \{T_1, T_5\} , \\ s_6 &= \{T_1, T_7\} , \\ s_7 &= \{T_1, T_2, T_3, T_4\} , \\ s_8 &= \{T_1, T_3, T_5, T_7\} , \\ s_9 &= \{T_1, T_3, T_6, T_8\} \\
\end{align*}
\]

Among these subgroups \( s_7, s_8 \) and \( s_9 \) being of index 2, are normal subgroups.

Further, consider the elements \( T_5 \) and \( T_8 \).

\[
\begin{align*}
T_5 \circ T_5 &= T_1 \text{ - the identity of } T^* \quad \text{and} \quad (T_8)^4 = (T_8)^2 \circ (T_8)^2 = T_3 \circ T_3 = T_1. \\
\end{align*}
\]

Also \((T_5 \circ T_8)^2 = T_2^2 = T_1\). Hence \((T^*, 0)\) has the defining relation, "\( A^4 = I; \quad B^2 = (AB)^2 = I \)"

and so \( T^* \) is isomorphic to the octic group.

NOTATION. \( F \) - the set of D-translations.

THEOREM 3.3.3. \((F, 0)\) is an abelian group.
PROOF. Consider any two D-translations, $F_1(q^m x_0, q^n y_0) = (q^{m+a_1} x_0, q^{n+b_1} y_0)$ and $F_2(q^m x_0, q^n y_0) = (q^{m+a_2} x_0, q^{n+b_2} y_0)$ where $(q^{a_1} x_0, q^{b_1} y_0)$ and $(q^{a_2} x_0, q^{b_2} y_0) \in \mathbb{H}$. Now, to prove the result, it is enough if we prove that $F_1 \circ F_2^{-1}$ is also a D-translation. $F_2^{-1}$ the inverse of $F_2$, is defined by $F_2^{-1}(z) = (q^{-m} x_0, q^{-n} y_0)$. Hence $F_1 \circ F_2^{-1}(z) = (q^{m-a_1-a_2} x_0, q^{n-b_1-b_2} y_0)$ is also a D-translation. Further, $(F,o)$ is isomorphic to the additive group of integers and hence is abelian.

3.4. DISCRETE ANALYTIC PROPERTIES OF D-TRANSFORMATIONS

For complex valued functions defined on $\mathbb{H}$, various notions of discrete analyticity are available in [35] and [70]. Consider $f: \mathbb{H} \rightarrow \mathbb{C}$, where $\mathbb{C}$ is the complex plane. Then

\[ f(z) \text{ is } q\text{-analytic at } z = (x,y) \text{ if } \theta_x = \frac{f(z) - f(qx,y)}{(1-q)x} \text{ and } \theta_y = \frac{f(z) - f(x,qy)}{(1-q)iy} \text{ are equal} \]
(2) \( f \) is \( p \)-analytic at \( z \) if

\[
\Delta_x = \frac{f(z) - f(px, y)}{(1-p)x} \quad \text{and} \quad \Delta_y = \frac{f(z) - f(x, py)}{(1-p)y}
\]

are equal, where \( p = q^{-1} \).

(3) \( f \) is bianalytic at \( z \) if it is both \( q \)-analytic and \( p \)-analytic at \( z \).

(4) \( f \) is \( q \)-monodiffric at \( z \) if

\[
\frac{f(q^{-1}x, y) - f(qx, y)}{(q^{-1}-q)x} = \frac{f(x, q^{-1}y) - f(x, qy)}{(q^{-1}-q)y}
\]

The first two discrete analyticity is due to Harman [35] and the other two due to Velukutty [70]. We apply these definitions to the \( D \)-transformations considered in the previous section. Further, by a theorem in [70], the set of bianalytic functions is a proper subset of the set of \( q \)-monodiffric functions.

THEOREM 3.4.1. \( D \)-translations are bianalytic if and only if \( a = b \).
PROOF. Consider the $D$-translation

$$g_1(q^{m}x_0, q^{n}y_0) = (q^{m+a}x_0, q^{n+b}y_0)$$

where $(q^{a}x_0, q^{b}y_0) \in H$.

$$\theta_x g_1 = \frac{(q^{m+a}x_0, q^{n+b}y_0) - (q^{m+a-l}x_0, q^{n+b}y_0)}{(1-q)q^mx_0}$$

$$= \frac{q^{m+a}x_0(1-q)}{(1-q)q^mx_0} = q^a$$

$$\theta_y g_1 = \frac{(q^{m+a}x_0, q^{n+b}y_0) - (q^{m+a}x_0, q^{n+b+1}y_0)}{(1-q)q^ny_0}$$

$$= \frac{q^{n+b}y_0(1-q)}{(1-q)q^ny_0} = q^b$$

Therefore, $g_1$ is $q$-analytic $\iff q^a = q^b \iff a = b$

Now $\bar{\theta}_x g_1 = \frac{(q^{m+a}x_0, q^{n+b}y_0) - (q^{m+a-l}x_0, q^{n+b}y_0)}{(1-q^{-1})q^mx_0}$

$$= \frac{q^{m+a}x_0(1-q^{-1})}{(1-q^{-1})q^mx_0} = q^a$$
So, \( g_1 \) is \( p \)-analytic \( \iff \) \( q^a = q^b \iff a = b \).

Since \( g_1 \) is both \( p \)-analytic and \( q \)-analytic if and only if \( a = b \), the theorem follows.

**THEOREM 3.4.2.** \( g_2 : H \rightarrow H \) defined by 
\[
g_2(q^m x_0, q^n y_0) = (q^{-m+a} x_0, q^{n+b} y_0)
\]
is bianalytic at the points of the form 
\[
(q^{\frac{a-b}{2}} x_0, q^{\frac{a-b}{2}} y_0); \quad \frac{a-b}{2} \in \mathbb{Z}.
\]

**PROOF.** \( g_2(q^m x_0, q^n y_0) = (q^{-m+a} x_0, q^{n+b} y_0) \)

\[
\theta_x g_2 = \frac{(q^{-m+a} x_0, q^{n+b} y_0) - (q^{-m+1+a} x_0, q^{n+b} y_0)}{(1-q) q^m x_0} = q^{-2m+a}
\]

\[
\theta_y g_2 = q^b.
\]

Hence, \( \theta_x = \theta_y \iff q^{-2m+a} = q^b \iff m = \frac{a-b}{2} \).

Thus, \( g_2 \) is \( q \)-analytic at all points of the form 
\[
(q^{\frac{a-b}{2}} x_0, q^{\frac{a-b}{2}} y_0); \quad \frac{a-b}{2} \in \mathbb{Z}.
\]
\textbf{THEOREM 3.4.3.} The D-transformation \( g_3 \) defined by
\[
(1-q) q^m = \frac{(q^{m+a}x_0, q^{n+b}y_0) - (q^{-m-a}x_0, q^{-n-b}y_0)}{(1-q^{-1})q^x_0} = q^{-2m+a}
\]
\( \hat{\theta}_x g_2 = q^b \)

Hence, \( \hat{\theta}_x = \hat{\theta}_y \) if and only if \( m = \frac{a-b}{2} \). Hence \( g_2 \) is
bianalytic at all points of the form \( (q^{m/2} x_0, q^{n} y_0) \).

\textbf{THEOREM 3.4.4.} The D-transformation \( g_4 \) defined by
\[
g_4(q^{m}x_0, q^{n}y_0) = (q^{m+a}x_0, q^{n+b}y_0) \text{ is bianalytic at points of the form}
\]
\[
(\frac{b-a}{2} x_0, q^{\frac{b-a}{2}} y_0) \in \mathbb{Z}.
\]
EXAMPLES 3.4.5.

1. $g_2(q^m x_0, q^n y_0) = (q^{m+4} x_0, q^{n+6} y_0)$ is bianalytic at points $(q^{-1} x_0, q^n y_0), n \in \mathbb{Z}$.

2. $g_3(q^m x_0, q^n y_0) = (q^{m+1} x_0, q^{-n+5} y_0)$ is bianalytic at points $(q^m x_0, q^{m+2} y_0), m \in \mathbb{Z}$.

3. $g_4(q^m x_0, q^n y_0) = (q^{m+2} x_0, q^{-n-8} y_0)$ is bianalytic at points $(q^m x_0, q^{-5} y_0), m \in \mathbb{Z}$.

NOTE 3.4.6.

1. Since bianalytic functions are $q$-monodiffric also, the transformations considered above are $q$-monodiffric in the respective set of points. Also, the discrete analyticity of the $D$-translations do not impose any condition on $m$ and $n$ and hence defines an entire function subject to the only condition that $a = b$.

2. Consider the $q$-analyticity of $g_5(q^m x_0, q^n y_0) = (q^{n+a} x_0, q^{m+b} y_0)$. We have $\theta_x g_5 = i \frac{q^b y_0}{x_0}$ and $\theta_y g_5 = \frac{q^a y_0}{i y_0}$. So $0 = y$ if and only if $q^{2a} x_0^2 = q^{2b} y_0^2$. The condition on $q$ is undesirable from the point of view of the theory considered so far. Similarly, for $g_6, g_7$ and $g_8$. Hence the only transformations, among those mentioned in Cor.3.2.8(1) of interest for discrete analyticity, are $g_1, g_2, g_3$ and $g_4$. 