Chapter 0

Introduction

One of the most important developments in applied mathematics over the last few decades has been emerged in the theory of variational inequalities, which constitutes a significant and important extension of the calculus of variations, whose source can be pulled back to the work of Fermat, Newton, Leibnitz, Bernoulli, Euler and Lagrange.

Ever since the introduction of variational inequalities by Stampacchia [59], it has been observed that the linear and nonlinear inequality problems arising in mathematical science and engineering can be studied uniquely in the modeling of variational inequalities. Variational inequalities have achieved a new height from both angle of applications and theory in the past decades. It has been observed that the origin of variational inequality theory is a precious gift to mathematics which is the works of several researchers like Browder [11], Lions and Stampacchia [16], Kinderlehrer and Stampacchia [13], Mosco [51, 52], Chipot [10] etc. to name only few. In fact they applied the theory of variational inequalities to solve obstacle problems, dam problems, plasticity phenomena, viscoplasticity phenomena, confined plasmas, free boundary problems and optimal control problems. In particular, this theory provides us a simple, natural, unified and general frame to study a wide class of unrelated linear and nonlinear problems arising in fluid flow through porous media, elasticity, transportation, economics, operations research, optimization, regional, physical and applied sciences. The ideas and techniques of variational inequalities are being applied in a variety of diverse fields and proved to be productive and innovative.

The theory of variational inequality problems generalizes the theory of com-
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INTRODUCTION

plementarity problems. Historically it is true that there is very less direct contact
among the researchers who work in variational inequalities and complementarity
problems. A wide range of knowledge in classical theory and applied mathematics
is required to research in variational inequalities with some notational differences,
de spite the knowledge in computer application.

In 1961, Dorn [27] has shown that the minimality of a quadratic program-
mapping problem (QPP) associated to a positive definite matrix is zero under certain
conditions. So it can be said that the Dorn’s paper gave the inspiration to con-
sider the complementarity problem as an independent problem. In 1963, Dantzig
and Cottle [21] have generalized the Dorn’s result as follows: the minimality of the
(QPP) associated to a matrix is zero if all the principal minors of it are positive. In
1965, Lemke [45] has solved the matrix game theory to show the application of the
complementarity problem. Ingleton [38] has applied the theory of complementarity
problem in engineering. The development of the complementarity problems is due
to the research work of several authors such as Cottle [18, 19], Eaves [28], Kara-
mardian [41, 42], Mangasarian [47, 48], Saigal [56], More [50, 49], Pang [54] etc. to
name only few.

In [20], Cottle and Yao established some existence results for a nonlinear com-
plementarity problem involving a pseudomonotone mapping over an arbitrary closed
convex cone in a real Hilbert space. They studied some known existence results for a
nonlinear complementarity problem in an infinite dimensional real Hilbert space and
applied to a class of nonlinear complementarity problems to study of the post-critical
equilibrium state of a thin elastic plate subjected to unilateral conditions.

In [37], Heemels, Schumacher and Weiland extended linear complementarity
problem of mathematical programming to rational complementarity problem and
presented that the problem occurs if complementarity conditions are imposed on
input and output variables of linear dynamical input/state/output systems.

The vector variational inequality was first introduced and studied by Giannessi
[31] in the setting of the finite dimensional Euclidean spaces. Since then, a large
number of results for the vector variational inequality and vector complementarity
problems have been obtained. For details, we refer to G. Y. Chen [13], G. Y. Chen
and X. Q. Yang [15], A. Daniilidis and N. Hadjisavvas [25]. In [5], Behera and
Das introduced generalized vector variational inequality problems, generalized vec-

The variational inequality and complementarity problems are defined as follows: Let $X$ be a reflexive real Banach space and $M \neq \emptyset$ be nonempty closed and convex subset of it with $X^*$ dual of $X$. Consider a nonlinear map

$$ A : M \to X^* $$

and

$$ F : M \to \mathbb{R}^n $$

be any map. Let $\langle g, x \rangle$ be the value of $g \in X^*$ at $x \in X$ be denoted by left and right angle.

(a) The Variational Inequality Problem (VIP) is defined as follows:

find $x_0 \in M$ such that for all $x \in M$,

$$ \langle A(x_0), x - x_0 \rangle \geq 0. \quad (VIP) $$

(b) The Complementarity Problem (CP) is defined as follows: find $z_0 \in M$ such that

$$ \langle A(z_0), z_0 \rangle = 0. \quad (CP) $$

(c) The Differential Inequality Problem (DIP) is defined as follows: find $x_0 \in M$
such that for all \( x \in M \),

\[
\langle \nabla F(x_0), x - x_0 \rangle \geq 0. \quad (DIP)
\]

In special cases, \([DIP]\) is the particular case of \([VIP]\) that arises naturally on a closed convex set \( K \) and is equivalent to the following:

\((d)\) The Minimization Problem \((MP)\) is defined as follows: find \( x_0 \in M \) such that for all \( z \in M \),

\[
F(z) \geq F(x_0), \quad (MP)
\]

if the mapping \( F : M \to \mathbb{R} \) is convex and differentiable on \( M \).

Various phenomena which occur in physical and economical sciences are mathematically formulated as nonsmooth variational inequalities or as optimization problems, where some nonsmooth constraints have to be taken into account. The nonsmooth mixed variational inequalities include classical variational inequalities as well as the nonsmooth convex optimization problems:

Clarke \([17]\) studied the generalized gradient of any convex function as follows. Let \( F : M \to \mathbb{R} \) be Lipschitz near a given point \( x \in M \) but not necessarily differentiable at \( x \) and let \( v \) be any other vector in \( X \). Let \( F^o(x; v) \) being used to denote the generalized directional derivative of \( F \) at \( x \) in the direction \( v \), defined as follows:

\[
F^o(x; v) = \limsup_{y \to x, \ t \downarrow 0} \frac{F(y + tv) - F(y)}{t}.
\]

The generalized gradient (or subdifferential) of \( F \) at \( x \), denoted \( \partial F(x) \), is the subset of \( X^* \) given by

\[
\partial F(x) = \{ \xi \in X^* : F^o(x; v) \geq \langle \xi, v \rangle \}.
\]

\((e)\) The Nonsmooth Nonlinear Variational Inequality Problems \((NNVIP)\) is defined as follows:

find \( z_0 \in M \) such that

\[
\langle A(z_0) + \xi, z - z_0 \rangle \geq 0, \quad \text{for all } z \in M, \ \xi \in \partial F(z_0) \quad (NNVIP)
\]
In 2006, Matrino and Xu [53] have defined the variational inequality problem \((\text{VIP})\) in Hilbert space \(\mathbb{H}\) in the following form:

\[
\text{find } z_0 \in M \subset \mathbb{H} \text{ such that } \langle A(z_0), z - z_0 \rangle \geq \gamma \langle \xi(z_0), z - z_0 \rangle \text{ for all } z \in M. \tag{VIP:T,F}
\]

They used this problem to show the optimality condition for the minimization problem

\[
\min_{z \in M} \frac{1}{2} \langle A(z), z \rangle - F(z),
\]

in a Hilbert space \(\mathbb{H}\), where \(A : M \to L(\mathbb{H})\), and \(\xi : \mathbb{H} \to L(\mathbb{H})\) are such the \(F\) is the potential function for \(\gamma \xi\) (i.e., \(F'(z) = \gamma \xi(z)\) for all \(z \in M \subset \mathbb{H}\)).

In 1981, Hanson [35] introduced the historic notion of “invex function” which abbreviates “invariant convex function” and generalizes the convex function. Several researchers establish many interesting results in both variational and complementarity problems inspired by Hanson’s paper. Hanson’s invex function widely used to analyze and formulate the variational inequality problems, complementarity problems and fixed-point theorems.

We recall some results of Hanson [35] and Ben-Israel and Mond [9] in which the bi-function \(\eta(-,-)\) is widely used to many cases of variational inequality problem and complementarity problem as well. In fact this bi-function is a mapping \(\eta : M \times M \to X\) which generalizes the inequality problems such as \((\text{VIP}), (\text{CP})\) and \((\text{DIP})\). The classical problems in generalized variational inequality theory and generalized complementarity theory are defined as follows.

Consider a reflexive real Banach space \(X\) with \(X^*\) as its dual and \(M \neq \emptyset\) is a nonempty subset of \(X\). Also consider a nonlinear mapping \(A : M \to L(X, \mathbb{R}^n) \equiv \mathbb{R}^n\) and \(F : M \to \mathbb{R}^n\) a differentiable mapping, where the differential of \(F\) at \(z \in M\) is denoted by \(\nabla F(z)\).

(f) The Generalized Variational Inequality Problem \((\text{GVIP})\) is defined as follows:

\[
\text{find } z_0 \in M \text{ such that } \langle A(z_0), \eta(z, z_0) \rangle \geq 0 \text{ for all } z \in M. \tag{GVIP}
\]
(g) The Generalized Complementarity Problem (GCP) is defined as follows:

find $z_0 \in M$ such that

$$\langle A(z_0), \eta(z, z_0) \rangle = 0 \quad \text{for all} \ z \in M. \quad (GCP)$$

(h) The Generalized Differential Inequality Problem (GDIP) is defined as follows:

find $z_0 \in M$ such that

$$\langle \nabla F(z_0), \eta(z, z_0) \rangle \geq 0 \quad \text{for all} \ z \in M. \quad (GDIP)$$

(i) The Generalized Differential Complementarity Problem (GDCP) is defined as follows:

find $z_0 \in M$ such that

$$\langle \nabla F(z_0), \eta(z, z_0) \rangle = 0 \quad \text{for all} \ z \in M. \quad (GDCP)$$

The problems (VIP), (CP) are generalized by Behera and Panda [6, 8] in Hausdorff topological vector spaces using a function $\eta(-,-)$ introduced by Hanson [35] as follows.

Consider $X$ a Hausdorff topological vector space and $M \neq \emptyset$ a nonempty convex subset of it, where $A : M \to X^*$, $\eta : M \times M \to X$ and $g : M \times M \to \mathbb{R}$ any three arbitrary maps.

(j) The Generalized Nonlinear Variational Inequality Problem associated with $g$ (GNVIP$_g$) is defined as follows:

find $z_0 \in M$ such that

$$\langle A(z_0), \eta(z, z_0) \rangle + g(z_0, z) \geq 0 \quad \text{for all} \ z \in M. \quad (GNVIP_g)$$

Moreover, in abstract spaces the existence of vector variational inequalities have been investigated by Chadli et.al [12], Yu and Yao [66]. A generalization of vector variational inequality related to the class of $\eta$-connected sets ($\eta$-invex sets) is a vector variational-like inequality which is more general as compare to the class of
convex sets \([58, 65]\).

In \([57]\), Siddiqi, Ansari and Khaliq have studied the following vector \(g\)-variational-type inequality problem in ordered topological vector spaces.

Consider a Hausdorff topological vector space \(X\) and an ordered Hausdorff topological vector space \(Y\). Now consider a nonempty, closed and convex subset \(M\) of \(X\) and \(D\) a compact and convex subset of \(M\). Let \(L(X,Y)\) be the space of linear continuous operators from \(X\) to \(Y\). Let the mapping \(g: M \to M\) and \(A: M \to L(X,Y)\) be any two maps, \(C: M \to Y\) a multivalued mapping. \(C\) is a closed, pointed and convex cone such that \(\text{int } C(x) \neq \emptyset\) for all \(x \in M\).

\((k)\) The \(\text{Vector } g\text{-Variational-type Inequality Problem (VVIP)}_g\) is defined as follows:

\[
\text{find } z_0 \in D \text{ such that } \langle A(z_0), z - g(z_0) \rangle \in -\text{int } C(z_0) \text{ for all } z \in M.
\]

In an ordered topological vector space, Behera and Das \([5]\) have considered the generalized vector variational inequality problem which is defined as follows. Consider a topological vector space \(X\) (or \(H\)-space) and an ordered topological vector space \((Y, P)\) equipped with closed convex pointed cone such that \(\text{int } P \neq \emptyset\). The set of continuous linear functionals from \(X\) to \(Y\) is denoted by \(L(X,Y)\). A pair of left and right angle is used to denote the value of \(g \in L(X,Y)\) at \(z \in X\) such as \(\langle g, z \rangle\). Let a convex set \(M\) of \(X\), with \(0 \in M\) and \(A: M \to L(X,Y)\) be any mapping. Let \(\eta: M \times M \to X\) be any vector valued function.

\((l)\) The \((GVVIP)\) \(\text{Generalized Vector Variational Inequality Problem}\) is defined as follows:

\[
\text{find } z_0 \in M \text{ such that } \langle A(z_0), \eta(z, z_0) \rangle \notin -\text{int } P \text{ for all } z \in M. \quad (GVVIP)
\]

\((m)\) The \((GVCP)\) \(\text{Generalized Vector Complementarity Problem}\) is defined as follows:
Differential Variational Inclusions (DVI)


Let $M \subset \mathbb{R}^n$ be a closed convex set. Let $f: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $F: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow L(X, \mathbb{R}^n) \equiv \mathbb{R}^n$ be any map. The DVIs have the form to:

find $u(t) \in M$ such that

$$\langle F(t, v(t), u(t)), z - u(t) \rangle \geq 0 \quad (DVI)$$

for every $z \in M$ and almost all $t$, where

$$\frac{dv}{dt} = f(t, v(t), u(t)),$$

$$v(t_0) = v_0.$$  

Closely associated with DVIs are the dynamic/differential complementarity problems. If $M$ is closed convex cone, then the above problem reduces to:

find $u(t) \in M$ such that

$$\langle F(t, v(t), u(t)), u(t) \rangle = 0 \quad (DCP)$$

for almost all $t$, where

$$\frac{dv}{dt} = f(t, v(t), u(t)),$$

$$v(t_0) = v_0.$$  

The problem (DCP) is mostly used to solve Mechanical Contact Problems. The
(DVI) does form a special class of differential inclusion as follows.

\[
\begin{align*}
(i) & \quad t \geq 0, x(t) \in M, \\
(ii) & \quad \sup \langle x'(t) - f(x(t)), x(t) - y \rangle \geq 0, \\
(iii) & \quad x(0) = x_0
\end{align*}
\]

when \( M \) is a closed convex subset. Whenever \( x(t) \in \text{int} M \), (ii) reduces to the differential variational inequality problem

\[x'(t) = f(x(t))\]

which is a differential equation.
Layout of Thesis

The entire thesis description is as follows:

In Chapter 1, we recall some known definitions and results of variational inequalities and complementarities on various spaces such as reflexive real Banach spaces, ordered topological vector spaces, $n$-manifolds (that is, Riemannian $n$-manifolds, and $S^n$) and Hilbert spaces.

In Chapter 2, we study the existence theorems of the generalized vector variational inequality problems ($GVVIP$) with nonnegative restriction using the existence of the problems ($GDVVIP$-I), ($GDVVIP$-II) and ($GDVVIP$-III) which are newly defined. The concept of a function of two variables which is invex at its first component and convex at second component is introduced to study their existence theorems of the above problems using a newly defined function $t$-$\eta$-directed map.

In Chapter 3, a generalized variational inequality problem bounded by other two generalized variational inequality problems is studied. We study the existence theorem of generalized $F$-variational inequality problems with the help of generalized $g$-variational inequality problems ($GNVIP$).

In Chapter 4, the function of $T$-$\eta$-invexity of order $\lambda$ and the problem generalized differential dominated variational inequality problem of order $\lambda$ ($GDDVIP;\lambda$) are introduced. The existence of the solution for ($GDDVIP;\lambda$) is studied through a fractional function. The concept of generalized numerical range is introduced and the interrelationship between generalized numerical range and ($GDDVIP;\lambda$) is studied. The problem generalized dominating Hessian nonsmooth quadratic variational inequality problem ($GDHNQVIP;\lambda$) and generalized nonsmooth quadratic variational inequality problem ($GNQVIP$) are introduced. The interrelationship between generalized numerical range and ($GDHNQVIP;\lambda$) is studied.

In Chapter 5, the variable step iterative method for function of $T$-$\eta$-invexity of order $\lambda$ and ($AGDDVIP;\lambda$) is studied in Hilbert space.

In Chapter 6, the generalized variational inequality problems in Riemannian $n$-manifold ($GVIP_n$) is introduced and studied its existence theorem using Lipschitz fixed point theorem. The existence theorems of $x^*$-variational inequality problems,
such as, $(x^*-VIP)$ and $(VIP(x^*, S^n))$ are studied in Hilbert space and $S^n$ respectively. In the proof of these results, Banach contraction theorem is applied. The problems primal nonlinear vector variational inequality problems $(PNVVIP)$ and its corresponding dual nonlinear vector variational inequality problems $(DNVVIP)$ are studied using $P$-pseudomonotone and $P$-quasimonotone operators through the contractible solution set of generalized complementarities in the moving cone which corresponds to $(PNVVIP)$ and $(DNVVIP)$ respectively.

In Chapter 7, the existence theorems of $F$-complementarity problems and its equivalent generalized complementarity problems are studied in reflexive real Banach space using $\phi-\eta$-invex cone. The existence theorems of $x^*$-complementarity problems, such as, $(x^*-CP)$ and $(GCP_n)$ are studied in Hilbert space and $S^n$ respectively. The generalized Banach contraction principle establishes the existence theorem of $(x^*-GCP)$ in Hilbert space.