Chapter 4

Equalizers and Intersection in the Category of Graphs

4.1 Introduction

If \( f : G \to G_1 \) is a morphism in the category of Abelian Groups (OR R-mod, vector spaces etc.) then \( \text{Ker} f = \{ x \in G/ f(x) = 0 \} \) is indeed the equalizer of \( f \) and the zero morphism 0. A generalization of this idea is that of an equalizer of any two morphisms in any arbitrary category \( \mathcal{A} \). This chapter proves the existence of equalizers of any two homomorphisms in the category \( \mathcal{S} \) of graphs by actually constructing the same (up to isomorphism). Dually the coequalizer for morphisms \( f \) and \( g \) in \( \mathcal{S} \) is defined as the equalizer for \( f \) and \( g \) in the dual category \( \mathcal{S}^* \). This is in fact a generalization of a quotient by an equivalence relation. It is clear that \( \mathcal{S} \) has coequalizers if and only if \( \mathcal{S}^* \) has equalizers. This chapter also proves by an example that \( \mathcal{S} \) does not have coequalizers. Finally it proves
that $\mathcal{G}$ has finite intersections also.

4.2 Equalizers

Definition 4.2.1: Let $f, g : X \to Y$ be two given homomorphisms of graphs. Then a homomorphism $h : K \to X$ is said to be an equalizer for $f$ and $g$ if

i) $fh = gh$ and

ii) if $p : Z \to X$ is any graph homomorphism such that $fp = gp$ then there exists a unique homomorphism $q : Z \to K$ such that $hq = p$ [30]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram}
\caption{Figure 4.1:}
\end{figure}

Theorem 4.2.2. The category of graphs $\mathcal{G}$ has equalizers.

Proof. Let $f, g : X \to Y$ be given homomorphisms of graphs. Let $K$ be the graph with vertex set $V(K) = \{ x \in X / f(x) = g(x) \}$ and for all $u, v \in K$, the edge $(u, v) \in E(K)$ if and only if $(u, v)$ belongs to
Let \( i_k = (i_{V(K)}^* i_{E(K)}^*) \) be the inclusion homomorphism.

**Claim:** \( i_k : K \to X \) is an equalizer for \( f \) and \( g \).

Now for all \( x \in K \)

\[
(f i_k)^* (x) = f^* i_k^* (x)
\]

\[
= f^* (x) = g^* (x)
\]

\[
= g^* i_k^* (x) = (g i_k)^* (x)
\]

so that \((f i_k)^* = (g i_k)^* \). Hence by Lemma 1.6 [36].

\( f i_k = g i_k \) which is (i) of Definition 4.2.1

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**Figure 4.2:**

Suppose there exists a homomorphism of graphs \( p : Z \to X \) such that \( f p = g p \). Then for all \( z \in Z \), \( f^* p^*(z) = g^* p^*(z) \) so that by definition of \( K, p^*(z) \in V(K) \). So define a homomorphism \( q : Z \to K \) as follows. \( q^*(z) = p^*(z) \) for all \( z \in Z \).

Moreover if \( (z_1, z_2) \in E(Z) \) then \( (p^*(z_1), p^*(z_2)) \in E(X) \) (since \( p \) is a homomorphism) and hence \( (p^*(z_1), p^*(z_2)) \in E(K) \) (by defn of \( E(K) \)). i.e. \( (q^*(z_1), q^*(z_2)) \in E(K) \) so that \( q : Z \to K \) is a homomorphism of graphs. Also by definition, for all
\[ \exists z \in Z, \]
\[ r_k q'(z) = p'(z) \text{ and so } i_k q = p \]

Further, if there exists \( q_1 : z \to k \) such that \( i_k q_1 = p \). Then
\[ q_1 = i_k q_1 = p = i_k q = q, \]
proving the uniqueness [30].

Hence \( G \) has equalizers.

\[ \blacksquare \]

**Remark 4.2.3:** As in any category the following properties are true in \( G \)

i) If \( h \) is an equalizer for \( f \) and \( g \) then \( h \) is a monomorphism.

ii) Any two equalizers for \( f \) and \( g \) are isomorphic subobjects of \( X \). Hence
we can talk about the equalizer of two given homomorphisms.

iii) \( f = g \) if and only if \( 1_x \) is the equalizer for \( f \) and \( g \).

We have stated earlier that generally when discussing about graphs, we
consider only graphs which are non null (not equal to \( \phi \)). However while
discussing about the category of graphs the null graph \( \phi \) is also considered
as an object with the unique morphism
(\text{also denoted as } \phi) \quad \phi : \phi \to G.

iv) In proof of Theorem 4.2.2.
Suppose \( V(K) = \{ x \in V(X) \mid f(x) = g(x) \} = \phi \). Then \( K \) is the null graph \( \phi \)
which is the equalizer for \( f \) and \( g \). For if there exists a homomorphism of
graphs \( p : Z \to X \) such that \( fp = gp \). Since \( f(x) \neq g(x) \) for any \( x \in V(X) \),
\( fp(x) \neq gp(x) \) for any \( x \in V(Z) \) and hence we must have \( Z = \phi \).
This implies that $p = \phi$. Therefore we have the unique homomorphism $\phi : Z \to K$ such that the above triangle commutes. Thus $K$ is the equalizer for $f$ and $g$.

We have the following interesting proposition and its corollary.

**Proposition 4.2.4.** Let $K \xrightarrow{h} X \xrightarrow{f} Y$ be an equalizer diagram for $f$ and $g$. Then $h$ is an isomorphism if and only if $f = g$.

**Proposition 4.2.5.** Let $f = g$. Then $1_X : X \to X$ is such that $f.1_X = g.1_X$. Therefore by LUMP of equalizers there exists a unique morphism say $\delta : X \to K$ such that $h.\delta = 1_X$. Therefore $h$ is a retraction. Also by Remark 4.2.3(i) $h$ is a monomorphism. Therefore by Proposition 5.1* [30] $h$ is an isomorphism.

**Corollary 4.2.6.** Let $f, g : X \to Y$ be homomorphism of graphs. Then $f = g$ if
and only if $1_X : X \to X$ is an equalizer for $f$ and $g$.

**Proof.** Since $1_X$ is an isomorphism by Proposition 4.2.4, we have $f = g$. Conversely let $f = g$. Then for any $p : Z \to X$, we have $fp = gp$. So we have the unique homomorphism $p : Z \to X$ such that $1_X p = p$. Thus the Universal Mapping Property for equalizers is satisfied so that $1_X$ is an equalizer for $f$ and $g$.

\[\]

**Proposition 4.2.7.** Let $f, g : X \to Y$ be homomorphism of graphs and $h : K \to X$ be the equalizer for $f$ and $g$. If $h$ is also an epimorphism then $h$ is an isomorphism.

![Diagram](image)

**Figure 4.4:**

**Proof.** Given that $h$ is an equalizer for $f$ and $g$. Hence $f$ is a monomorphism. Moreover $fh = gh$ (by definition) and hence $f = g$ (Since $h$ is an epimorphism). Therefore by (iii) in Remark 4.2.3 $1_X$ is an equalizer for $f$ and $g$. This shows
that there exists a Unique homomorphism $\delta : X \rightarrow K$ such that $h\delta = 1_X$.
Therefore $h$ is a retraction.

Thus $h$ is a monomorphism and also a retraction implies that $h$ is an isomorphism [Proposition 5.1* in [30]].

\[\]  

**Definition 4.2.8**: A monomorphism $h : A \rightarrow B$ which fits into an equalizer diagram is called a regular monomorphism. Thus every equalizer is a regular monomorphism.

![Diagram](https://via.placeholder.com/150)

Figure 4.5:

**Remark 4.2.9**: There are categories in which monomorphisms need not be regular monomorphisms. However in $\mathcal{G}$ every coretraction is a regular monomorphism.

**Proof.** Let $h : A \rightarrow B$ be a coretraction in $\mathcal{G}$. Then there exists $K : B \rightarrow A$ such that $kh = 1_A$.

**Claim**: $h$ is the equalizer for $hk$ and $1_B : B \rightarrow B$. 
Now \( (h.k).h = h.(k.h) = h.1_A = h \) and \( 1_B.h = h \) so that condition (i) of definition of an equalizer is satisfied. Suppose there exists \( t : Z \to B \) such that \( hkt = 1_B.t = t \).

Now if \( \bar{z} \in Z \), then \( (kt)(\bar{z}) \in A \).
So define \( \gamma : Z \to A \)
\[
\bar{z} \mapsto (kt)\bar{z} \cdots (1)
\]
Then for \( \bar{z} \in Z \), \( h\gamma(\bar{z}) = h(kt)\bar{z} = t(\bar{z}) \) and \( h\gamma = t \cdots (2) \)

**Claim:** \( \gamma \) is unique.

Suppose there exists \( \delta : Z \to A \) such that \( h\delta = t \cdots (3) \)
Then for all \( \bar{z} \in Z \)
\[
\gamma(\bar{z}) = (kt)\bar{z} \text{ (by } 1) \\
= (kh\delta)(\bar{z}) \text{ by } (3) \\
= 1_A\delta(\bar{z})
\]
which implies that \( \gamma = \delta \).
Thus $h : A \to B \xrightarrow{hk} B$ is the equalizer for $hk, 1_B : B \to B$ so that $h$ is a regular monomorphism.

\section*{4.3 Coequalizers}

\textbf{Definition 4.3.1:} Let $f, g : X \to Y$ be two given homomorphism of graphs. Then a homomorphism $h : Y \to Z$ is said to be a coequalizer for $f$ and $g$ if

i) $hf = hg$ and

ii) if $h_1f = h_1g$ for some homomorphism $h_1 : Y \to Z_1$ then there exists a unique homomorphism $\gamma : Z \to Z_1$ such that $\gamma h = h_1$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{coequalizer_diagram.png}
\caption{Coequalizer Diagram}
\end{figure}

\textbf{Remark 4.3.2:} As for equalizers, we can prove the following statements for coequalizers.

1) If $h$ is a coequalizer for $f$ and $g$ then $h$ is an epimorphism.

2) Any two coequalizer for $f$ and $g$ are isomorphic graphs.

3) If $h$ is a coequalizer for $f$ and $g$ and $h$ is also a monomorphism then $h$ is an isomorphism.
4) Every retraction is a coequalizer.

**Definition 4.3.3:** Let $X, Y$ be arbitrary graphs in $\mathcal{G}$. If coequalizer for every pair of homomorphisms $f, g : X \to Y$ exists then $\mathcal{G}$ is said to have Coequalizers.

**Remark 4.3.4:** We have proved that the category $\mathcal{G}$ has equalizers. However this is not true in the case of coequalizers as the following example shows.

**Example 4.3.5:** Let $X$ and $Y$ be graphs where $V(X) = \{x\}, E(X) = \emptyset$; $V(Y) = \{y_1, y_2\}, E(Y) = \{(y_1, y_2)\}$ and $f, g : X \to Y$ where $f^*(x) = y_1$ and $g^*(x) = y_2$

\[ x \xrightarrow{f} Y \xrightarrow{h} Z \]

**Figure 4.9:**

Suppose $h : Y \to Z$ is a coequalizer for $f$ and $g$ then $hf = hg$. Then $h^*f^*(x) = h^*g^*(x)$

\[(i.e) h^*(y_1) = h^*(y_2) \cdots (1)\]
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But \((y_1, y_2)\) is an edge in \(Y\) implies that \(\bar{h}(y_1, y_2) = (h^*(y_1), h^*(y_2))\) is an edge in \(Z\) which is a contradiction by (1). Therefore \(f, g\) do not have a coequalizer thus proving that \(\mathcal{G}\) does not have coequalizers.

Let us now discuss about the existence of equalizers in some special subcategories of the Category of graphs.

**Example 4.3.6:** Let \(C\) denote the full subcategory of cycles in \(\mathcal{G}\). Then \(C\) does not have equalizers.

**Proof.** Consider the cycles \(C_4\) and \(C_6\) and the homomorphisms
\(f, g : C_4 \to C_6\) given below:

![Figure 4.10](image)

\[f(x_1) = y_1, \quad f(x_2) = y_3, \quad f(x_3) = y_5, \quad f(x_4) = y_3.\]

Similarly \(g : C_4 \to C_6\) is defined as \(g(x_1) = y_1, g(x_2) = y_3, g(x_3) = y_2\) and \(g(x_4) = y_1\). Clearly \(f\) and \(g\) are homomorphism of graphs. Moreover the equalizer of \(f\) and \(g\) is a subgraph of \(C_4\)(in \(\mathcal{G}\)) say \(K\) where \(V(K) = \{x \in V(C_4)/f^*(x) = g^*(x)\}\) i.e \(K\) is the graph \(x_1 \mapsto x_2\) which is not a cycle and hence is not an object in \(C\).
Therefore \( C \) does not have equalizers.

\[ \blacksquare \]

**Example 4.3.7:** Let \( \mathcal{P} \) denote the full subcategory of planar graphs. Then \( \mathcal{P} \) has equalizers.

**Proof.** Let \( G_1 \) and \( G_2 \) be planar graphs and let \( f, g : G_1 \rightarrow G_2 \) be homomorphisms of graphs. Then the equalizer of \( f \) and \( g \) is a subgraph \( K \) of \( G_1 \) where \( V(K) = \{ x \in V(G_1)/f^*(x) = g^*(x) \} \). Since any subgraph of a planar graph is planar, \( K \) is planar and hence is an object in \( \mathcal{P} \). Hence \( \mathcal{P} \) has equalizers.

\[ \blacksquare \]

**Example 4.3.8:** Let \( S_n \) be the star graph \( K_{1,n} \) such that \( a \in S_n \) is adjacent to every other vertex in \( S_n \). such a star graph is denoted as \( S_{n,a} \). A homomorphism \( f : S_{m,a} \rightarrow S_{n,b} \) is said to fix internal nodes if \( f(a) = f(b) \).

Let \( S \) denote the full subcategory of star graphs and \( S_0 \) be the subcategory of \( S \) with the same object as in \( S \) and \( [S_{m,a}, S_{n,b}]_{S_0} = \{ f \in [S_{m,a}, S_{n,b}] / f \) fixes internal nodes\}. Then \( S_0 \) has equalizers.

**Proof.** Let \( f, g : S_{m,a} \rightarrow S_{n,b} \) be homomorphisms which fixes internal nodes.

Let \( K \) be the subgraph of \( S_{m,a} \) where \( V(K) = \{ x \in S_{m,a}/f(x) = g(x) \} \).

By definition \( a \in V(K) \). Moreover if \( u \in V(K) \) and \( u \neq a \), then since \( (u, a) \) is an edge in \( S_{m,a} \), \( (u, a) \) is an edge in \( K \) also. Therefore \( K \) is the subgraph of the
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form $S_{t,a}$ where $0 \leq t \leq m$ and hence $K \in S_0$. Hence $S_0$ has equalizers.

Example 4.3.9: Let $S$ denote the full subcategory of star graphs. Then $S$ does not have equalizers.

**Proof.** Consider the homomorphisms $f, g : S_{m,a} \to S_{n,b}$ where $a \neq b$ and $m, n \geq 2$ defined as follows.

$f(a) = y_1 \neq b$ and $f(u) = b$ for all $u \in S_{m,a}$. Similarly $g(a) = y_2 \neq y_1$ and $g(u) = b$ for all $u \in S_{m,a}$. Then it is easy to verify that

$V(K) = \{x \in S_{m,a}|f(x) = g(x)\}$

$= \{x \in S_{m,a}|x \neq a\}$ since $f(u) = b = g(u)$ for all $u \neq a$. Hence $K$ is the empty graph.

(i.e. $E(K) = \emptyset$) with $m$ vertices ($m \geq 2$) and hence not a star graph. i.e. $K \notin S$ and so $S$ does not have equalizers.

Example 4.3.10: The full subcategory of connected graphs $\mathcal{C}$ does not have equalizers.

**Proof.** We know that star graphs are connected graphs and hence belong to $\mathcal{C}$. Consider the homomorphisms $f, g : S_{m,a} \to S_{n,b}$ as defined in Example 4.3.9. Then equalizer for $f$ and $g$ is the empty subgraph $K$ which is disconnected and hence $K \notin \mathcal{C}$. Therefore $\mathcal{C}$ does not have equalizers.
Example 4.3.11: The full subcategory of complete Bipartite graphs \( K \) does not have equalizers.

**Proof.** Every star graph is a complete bipartite graph of the form \( K_{1,m} \). Hence by previous example it follows that \( K \) does not have equalizers.

Example 4.3.12: Let \( T \) be the full subcategory of Trees. Then \( T \) does not have a equalizer.

**Proof:** We know that a tree is a connected acyclic graph and hence every star graph is a tree. Consider the Example 4.3.9. Then \( S_{m,a} \) and \( S_{n,b} \) are trees and \( f, g \) are homomorphism of trees. Then the equalizer of \( f \) and \( g \) is given by the empty graph \( K \) which is disconnected. Therefore \( K \) is not a tree and hence \( K \notin T \), i.e. \( T \) does not have equalizers.

### 4.4 Intersection

**Definition 4.4.1:** Let \( \{ u_i : A_i \to A \} \) be a family subgraphs of \( A \). A morphism \( u : B \to A \) is called the intersection of the family if

i) for each \( i \in I \) we can write \( u = u_i v_i \) for some morphism \( v_i : B \to A_i \) and

ii) if every morphism \( C \to A \) which factors through each \( u_i \) factors uniquely
through $u$.

![Diagram of graphs](image)

Figure 4.11:

**Remark 4.4.2:** 1) By definition of subgraphs, each $u_i$ is a monomorphism and hence $u'_i$ is injective.

2) The morphism $u$ in the definition is also a monomorphism. For $u\gamma_1 = u\gamma_2$ for $\gamma_1, \gamma_2 : C \to B$, then $u_i\gamma_1 = u_i\gamma_2$. Hence $v_i\gamma_1 = v_i\gamma_2$ (since $u_i$ is a monomorphism).

Take $w_i = v_i\gamma_1 = v_i\gamma_2$ and $w = u_i\gamma_1 = u_i\gamma_2$.

then by uniqueness in the definition $\gamma_1 = \gamma_2$.

So that $u$ is a monomorphism.

3) Since $u = u_i v_i$ is a monomorphism each $v_i$ is a monomorphism.

4) Any two intersections of a given family are isomorphism.

**Proposition 4.4.3.** The category of graphs $\mathcal{G}$ has finite intersections.
Proof. Let \( \{u_i : A_i \to A\}_{i=1}^{n} \) be a finite set of subgraphs of \( A \). Then by the definition of subobjects, each \( u_i : A_i \to A \) is a monomorphism and hence \( u_i^* : V(A_i) \to V(A) \) is injective [36].

Consider the graph \( B \), where

\[
V(B) = \bigcap_{i=1}^{n} u_i^*(V(A_i)) \subseteq V(A).
\]

If \( x_1, x_2 \in V(B) \) then \( x_1, x_2 \in u_i^*(V(A_i)) \) for each \( i = 1 \) to \( n \). So define \( (x_1, x_2) \) is an edge in \( B \) if and only if \( (x_1, x_2) \) is an edge in \( A_i \) for all \( i \).

Consider the graph \( B \), we define a homomorphism

\[
v_i : B \to A_i (i = 1 \text{ to } n)
\]
as follows:

If \( x \in V(B) \) then \( x \in u_i^*(V(A_i)) \) for each \( i \). Hence there is a unique \( y_i \in V(A_i) \) such that \( x = u_i^*(y_i) \) \( (1) \)

Define \( v_i^* : V(B) \to V(A_i) \) by \( v_i^*(x) = y_i \). Clearly \( v_i : B \to A_i \) is a homomorphism.

Let \( u : B \to A \) be defined as \( u = u_i v_i \) \( (2) \)

Then for all \( x \in V(B) \),

\[
u^*(x) = u_i^* v_i^*(x) = u_i^*(y_i) = x \quad \text{by} \quad (1)
\]

Thus \( u^*(x) = x \) for all \( x \in B \) \( \cdots (3) \)

\([i.e. u^* : V(B) \to V(A) \text{ is the inclusion map}]

Claims 1: \( B \) is the intersection of the given family.

Suppose there exist a morphism \( w : C \to A \) such that \( w = u_i w_i (i = 1 \text{ to } n) \) for some morphisms \( w_i : C \to A_i \)
Define $\gamma : C \to A$ as follows. $\gamma^* : V(C) \to V(A)$ Where $\gamma^*(3) = w^*(3)$ for all $3 \in C$.

Since $w$ is a homomorphism, it preserves edges in $C$ and so does $\gamma^*$ and hence $\gamma$ is a homomorphism. Moreover $u^*\gamma^*(3) = u^*(w^*(3)) = w^*(3)$ by (3)

Hence $u\gamma = w$ (Lemma 1.6 in [36])

**Claim 2**: Suppose there exists $\delta : C \to B$ such that $u\delta = w$.

Then for all $3 \in V(B)$,

$$\gamma^*(3) = w^*(3) \text{(by definition)}$$

$$= u^*\delta^*(3) \text{(by assumption)}$$

$$= \delta^*(3) \text{[Since $u^*$ is the inclusion]}$$

and so $\gamma = \delta$ proving the uniqueness of $\gamma$. Thus $\mathcal{G}$ has finite intersections [37].

We now investigate the existence of intersections in some special subcategories of the category of Graphs. Let us first begin with the following Remark.
Remark 4.4.4: If \( u_i : G_i \rightarrow G \) is a monomorphism in \( \mathcal{G} \) then

If \( u_i : V(G_i) \rightarrow V(G) \) is injective. Hence we may identify \( G_i \) with its image in \( G \). Moreover since any two intersections of a given family are isomorphic, we may say without loss of generality assume the intersection of a given family to be the usual intersection of subgraphs [30].

Example 4.4.5: Let \( \mathcal{P} \) be the full subcategory of planar graphs. Then \( \mathcal{P} \) has finite intersections.

Proof. Let \( G \) be a planar graph. Then the intersection of any finite family of subgraphs of \( G \) is also a subgraph of \( G \) and so is planar. Hence \( \mathcal{P} \) has finite intersections.

Example 4.4.6: Let \( \mathcal{C} \) be the full subcategory of Cycles. Then \( \mathcal{C} \) has finite intersections.

Proof. Let \( C_n(n \geq 3) \) be any given cycle. Then the only subgraph of \( C_n \) in \( \mathcal{C} \) is \( C_n \) itself. Therefore the intersection of any finite family of subgraphs of \( C_n \) is \( C_n \) itself and so belongs to \( \mathcal{C} \). ie \( \mathcal{C} \) has finite intersections.
Example 4.4.7: Let $\mathcal{K}$ denote the full subcategory of a complete graphs. Then $\mathcal{K}$ has finite intersections.

**Proof.** Let $\{K_i| i = 1 \text{ to } m\}$ be a finite family of subgraphs of a complete graph $K_n$. Let $K = \bigcap_{i=1}^{m} K_i$ be the intersections of the family. If $u, v \in V(K)$ then $u, v \in V(K_i)$ for all $i = 1$ to $m$ and so $(u, v) \in E(K_i)$ for all $i$ (since each $K_i$ is complete). Therefore $(u, v)$ belongs to $E(K)$. Thus any two vertices in $K$ are adjacent and so $K$ is a complete graph, i.e. $K \in \mathcal{K}$ and so $\mathcal{K}$ has finite intersection.

Example 4.4.8: Let $\mathcal{T}$ be the full subcategory of Trees. Then $\mathcal{T}$ has finite intersections.

**Proof.** Let $\{T_i\}_{i=1}^{m}$ be a family of subgraphs of the tree $T$ in $\mathcal{T}$. Let $T' = \bigcap_{i=1}^{m} T_i$. If $m = 1$ then the result is trivial. Assume that $m \geq 2$. If $T'$ has only one vertex then $T'$ is a tree and hence the result is true. Since $T'$ is a subgraph of $T_i$ for all $i$, $T'$ is acyclic.

Let $u, v \in V(T'), u \neq v$. Then $u, v \in V(T_i)$ for all $i$, and hence there is path from $u$ to $v$ in each $T_i$. Let $u = x_1, x_2, \ldots, x_n = v$ and $v = y_1, y_2, \ldots, y_m = v$ be paths in $T_1$ and $T_2$. If $x_i \neq y_i$ for any $i > 2$, then we get a cycle of the form given below in $\mathcal{T}$ which is a contradiction.
Hence the two paths are identical. In otherwords $T'$ is connected (and acyclic) and therefore a tree. i.e. $T' \in T$ and so $T$ has finite intersections. ■

**Example 4.4.9:** The full subcategory of connected graphs does not have intersections.

**Proof.** Let $\mathcal{C}$ be the full sub category of connected graphs. Let $C$ be the connected graph given below and $C_1, C_2$ its subgraphs.
Then $C_1 \cap C_2$ is the empty graph with vertex set \{ $x_2$, $x_4$ \} which is disconnected. Hence $\mathcal{C}$ does not have intersections.

\textbf{Remark 4.4.10:} We observe that the graph $C$ is not a tree.

\textbf{Example 4.4.11:} Let $S$ be the full subcategory of star graphs. Then $S$ has finite intersections.

\textbf{Proof.} Any star graph will be of the form $S_{m,a}$, $m \geq 0$. Any subgraph of $S_{m,a}$ with only one vertex will be of the form $S_{0,a}(u \in S_{m,a})$ which is a star graph, and any subgraph of $S_{m,a}$ with at least two vertices will be of the form $S_{n,a}$ where $1 \leq n \leq m$. In any case if $\{S_i\}_{i=1}^n$ is a family of subgraphs of $S_{m,a}$ then $\bigcap_{i=1}^n S_i$ being a subgraph of $S_{m,a}$ is a stargraph and so belongs to $S$. Thus $S$ has finite intersections.

\section{4.5 Conclusion}

In this chapter, the existence of equalizers of any two homomorphisms in the category $\mathcal{G}$ of graphs is proved. It is also proved by an example that $\mathcal{G}$ does not have coequalizers and $\mathcal{G}$ has finite intersection.