Chapter 3

Mathematical Formulation for Response of Structures Isolated with Curved Sliding Surface

3.1 INTRODUCTION

As discussed in previous chapters sliding isolation is an effective technique of base isolation among all devices, as it has an isolation, energy dissipation and restoring mechanism in one unit. In sliding devices the structure slides along the isolator surface, reducing the energy that is transmitted to the structure. The friction force at the sliding layer provides energy dissipation when isolator force (inertia force + restoring force) is less than friction force. When isolator force exceeds friction force sliding initiates and continues till it becomes less than friction force. This process will remain continue until the excitation acts on the structure. As a result the nature of response is of stick-slip nature. The structure behaves as fixed base structure during stick phase.

From above discussion it is clear that the sliding type isolation system is highly nonlinear in its dynamic behavior and suitable modeling and analysis techniques have to be used to model their behavior. In this chapter the necessary mathematical formulation and its solution techniques are proposed to evaluate response of these structures. The analysis of such structures involves intricacies due to the frictional element present in the system. Since the isolator damping is usually different from that of the structure, the structure behavior is non-classically damped. During motion the system behaves in two distinct ways in non-sliding and sliding phases that make the system highly non-linear. As a result, the analysis procedure is quite complicated and involves several influencing parameters. In the subsequent sections, the procedure for the analysis of MDOF system isolated by curved sliding surface of general geometrical profile has been developed. (Pranesh M. 2000) The geometry of the curved sliding surface can be represented by a parametric equation and hence the formulation is capable of analyzing structure isolated by any sliding-type isolator.

Although the real structures are multiple degree-of-freedom (MDOF) systems, most of the response of a structure is contributed from the first few modes of vibration. In fact many structures have major contribution of response from the first mode itself. This is especially true for the base isolated structure, as the first mode is virtually a rigid body mode. The response in any mode is equivalent to the response of an equivalent SDOF system. Due to these reasons, and also due to simplicity of a SDOF system, it is convenient to first study the behavior of SDOF system. Such a study will give very useful information in simplest possible way and provides a basis for the further study of real structures. In present chapter mathematical formulation for behavior of structure isolated with curved sliding surfaces like Friction Pendulum System (FPS) and Variable Frequency Pendulum Isolator (VFPI) has been developed.

3.2 ANALYSIS OF STRUCTURE ISOLATED WITH CURVED SLIDING SURFACE

As discussed above the sliding is of stick-slip nature. When the structure isolated by sliding type of isolator is subjected to ground excitation the non-sliding and sliding phases depend upon characteristics of ground motion. During non-sliding phase the structure behaves as fixed base structure and classically damped structure, and the modal analysis procedures for such structures are quite well established (Clough and Penzien 1993; Shabana 1994; Chopra 2003). But during the sliding phases, the dynamic properties of the system are entirely modified, as the isolator degree-of-freedom (DOF) also becomes active. As the structural damping and the isolator damping differs from each other the behavior of the structure is non-classically damped in sliding phase. The analysis of such structures with a classical damping approximation may induce significant errors (Duncan and Taylor 1979). So, this requires a complex-modal analysis. Further, due to the alternate change in phases between non-sliding and sliding, the overall behavior of the structure is highly non-linear and the methods will be different in the two phases. However, the classically damped nature of the superstructure can be utilized in the sliding phase also to advantage by using the perturbation methods. Alternatively, the coupled equations of motion can be directly solved by numerical methods. But this does not provide proper insight into the structural behavior and may involve high computational effort for structures with many DOFs. So, a procedure involving classically damped analysis for non-sliding phase and complex modal analysis for sliding phase has been used in this work as an effective technique to analyze structures isolated by sliding-type isolation systems. (Foss 1958; Hurty and Rubienstein 1967; Clough and Mojtahedi 1976)

The modal analysis procedures for structures with non-classical damping have been proposed by many researchers (Foss 1958; Hurty and Rubienstein 1967; Clough and Mojtahedi 1976; Duncan and Taylor 1979; Traill-Nash 1981; Borino and Muscolino 1986; Velestos and Ventura 1986; Xu and Igusa 1991). Methods to combine the maximum modal responses have also been proposed (Igusa et. al. 1984; Sinha and Igusa 1995a, 1995b). Subsequently these methods have been applied to base-isolated structures (Tsai and Kelly 1988, 1993). Singh and Malushte (1990) have applied these methods to simple sliding type structures. Tsai et. al. (2003) proposed finite element formulation. Lu et. al. (2006) generalized the mathematical formulation of a sliding surface with variable curvature using a polynomial function to define the geometry of the sliding surface. The application of these methods to structures isolated by sliding type of isolators with non-linear restoring force has been developed by Pranesh M. (2000).

One of the basic objectives of using a modal analysis is to uncouple the equations of motion and solve for only few modes to get reasonably accurate results. The uncoupling is possible only if the natural modes of vibration are orthogonal. The normal modes are no longer orthogonal due to the non-classical nature of damping. However, the complex modes that differ in both magnitude and phase are still orthogonal and can be used to uncouple the equations of motion. Once the equations of motion are uncoupled, step-by-step integration procedures can be developed to solve them.

To analyze the isolated structure by complex modal analysis in the sliding phase one can model the structure in two ways. In the first case the stiffness and damping of the isolator can be directly included in the structure matrices and the resulting equations can be solved by complex modal analysis. For example, a multi-storey shear building can be modeled as a lumped mass model with the isolators at the base as shown in Figure 3.1. This requires the complex eigen solution of the whole structure including all the DOFs. Alternatively, the classical damping nature of the superstructure can be taken advantage of and use a mode-synthesis approach. In this model, the undamped normal modes of the superstructure are assumed to remain unaltered and the model can be represented with all the modal masse attached to the base mass. The resulting model is shown in Figure 3.2. In this model only a few superstructure modes having major participation in the response only are considered for the complex eigen solution. If n superstructure modes are considered in the analysis, the complex eigen value problem has a size of $n + 1$. This can greatly reduce the computational effort.

The mode-synthesis approach for a conventional FPS is relatively less tedious as the modal properties are independent of the response quantities due to the linear restoring force characteristics. But in case of VFPI the analysis is far more complicated due to non-linear restoring force characteristics. As a result in VFPI analysis the stiffness matrix changes at each instant during sliding phase due to non-linearity of the isolator restoring force. So, at any given instant the stiffness of the isolator has to be redefined and taken in the stiffness matrix and this in turn requires the determination if the modal properties at each time step.

3.3 MATHEMATICAL FORMULATION FOR CONSTANT COEFFICIENT OF FRICTION

The accurate modeling of a sliding-type bearing that reflects its real behavior is quite complicated. This is due to (1) highly nonlinear behavior of the bearings and (2) dependence of the coefficient of friction on various parameters like bearing pressure and velocity of sliding. Further there is a possibility of uplift and hence loss of point of contact at the sliding surface under extreme conditions. This possibility has to be carefully evaluated especially when the vertical ground motions are also considered in the analysis. Generally there is a very weak correlation between the horizontal and vertical ground motion and the structure is quite strong in the vertical direction. The gravity load of a structure is sufficient to counter balance the excitation force of vertical ground motion. As a result, uplift does not occur under design level of earthquake (Wang et. al. 1995). Further the structure is supported on a group of bearings at large distances from the centre of mass and hence the effect of overturning moment can be neglected (Mokha et. al. 1996)

One of the basic objectives of base isolation is to reduce the structure response to a level that keeps the structure in the elastic range. This reduces the damage that otherwise occurs in the conventional structures. So the present formulations assume that the structure remains in the elastic range. However due to the non-linearity in the isolation system the analysis methods used for linear systems cannot be used. In spite of the difficulties in modeling the behavior of sliding-type isolators, their dynamic behavior can be qualitatively and quantitatively evaluated using simplified assumptions. The following assumptions are made in present formulation.

1. The coefficient of friction remains same under both static and dynamic conditions and friction is modeled by Coulomb's friction. This assumption is valid if the breakaway friction is not significantly large than steady state friction. Generally after the first spike in the frictional force, a steady-state frictional force is observed (ASTM-G115, 1993).
2. The overturning effect due to uneven distribution of normal force is neglected and the isolator is assumed to be always in contact with the sliding surface.
3. The slider of the isolator is assumed to have a point contact with the sliding surface. This can be realized in practice by designing the slider accordingly.
4. All the isolators at the base are connected rigidly with each other so that there is no relative movement between any two isolators. The effect of all the isolators can then be modeled by a single isolator at this level.
5. The structure is subjected to horizontal ground motion only and the effect of vertical acceleration due to rising of the structure is neglected.

Consider an N-storey shear structure isolated by a curved sliding surface of any defined geometrical configuration subjected to a horizontal ground excitation of $\ddot{x}_g$. The conceptual model of the isolator and the structure is shown in Figure 3.3.

To elaborate the conceptual modeling of the isolator, let us consider the structure as rigid body sliding in the isolator. During sliding the isolator provides two forces: (1) the restoring force provided by the weight of the structure sliding on the curved surface, which always acts towards the lowest point on the isolator and (2) the frictional force, which acts opposite to the relative sliding velocity. The effect of the isolator restoring force and sliding friction can be represented by means of a spring and friction damper respectively. The spring may be linear or non-liner, depending on the nature of the restoring force offered by the isolator, which in turn depends on the geometry of the sliding surface.

The FBD of forces acting at the point of contact with the sliding surface are shown in Figure 2.7. Let us neglect the friction for the time being and consider a rigid mass m sliding freely on a frictionless curved surface of predefined geometry, $y = f(x)$ When the mass is located at the point of contact the lateral force required to keep the block in equilibrium at the given position is given by (Pranesh M., 2000)

$$f_R = mg \tan\theta \tag{3.1}$$

This force can be termed as the restoring force as it is responsible for bringing the mass back to the zero displacement position when released. In other words, the curved sliding surface produces restoring force similar to a spring. The motion of the block on smooth curved sliding surface is analogous to the motion of a simple pendulum with mass m where, the normal reaction offered by the smooth surface is analogous to the tension in the string of the simple pendulum. The equation of motion of a freely oscillation simple pendulum of constant length is given by

$$\ddot{\theta} + \frac{g}{R}\sin\theta = 0 \tag{3.2}$$

where θ is the angle of oscillation, R is the length of the pendulum and over dot indicates derivative with respect to time. If the curved motion is approximated to the horizontal displacement x, (for small θ) the equation of motion reduces to

$$\ddot{x} + \omega_b^2 x = 0 \tag{3.3}$$

In the above equation the constant oscillation frequency of the pendulum ω_b is given by $\sqrt{g/R}$ and the restoring force (equal to $\omega_b^2 x$) is directly proportional to the displacement x. If the sliding surface is spherical, the motion of mass will be identical to that of a simple pendulum.

Now, if the sliding surface geometry is not spherical the motion of the block can be assumed to be equivalent to that of a pendulum with varying radius of curvature at each point on the sliding surface resulting in variable oscillation frequency. Therefore the restoring force given by Equation (3.1) can be expressed as

$$f_R = mg\frac{dy}{dx} \tag{3.4}$$

where dy/dx is the slope of the sliding surface at a point. This restoring force mechanism is mathematically represented by an equivalent non-linear spring, wherein the spring force can be expressed as the product of the equivalent spring stiffness and the deformation

$$f_R = k(x)x \tag{3.5}$$

where $k(x)$ is the non-linear spring stiffness and x is the displacement of the mass. Now the equation of motion of a freely oscillating block on any smooth sliding surface can be written as

$$m\ddot{x} + k(x)x = 0 \tag{3.6}$$

Comparing Equation (3.6) with Equation (3.3) the equation of motion of a freely oscillating mass on a smooth curved sliding surface can be written as

$$m\ddot{x} + m\omega_b^2(x)x = 0 \tag{3.7}$$

Here, $\omega_b(x)$ is the instantaneous isolator frequency, and depends solely on geometry of the sliding surface. The restoring force offered by horizontal surface (PF system) is zero and hence its frequency is zero. However it has to be noted that this definition of isolator frequency incorporated only the effect of restoring force and does not consider that of the frictional force. Although the frictional force acting tangential to the sliding surface also contributes to the stiffness and affects the damped frequency of the isolator, the frictional force is considered as a driving force in the present formulation and hence is taken on the RHS of the equation of motion during sliding phase. The effect of frictional damping on the isolator frequency is neglected.

The motion of the structure can be in either of the two phases, non-sliding phase or sliding phase. When there is no relative movement between the base and the ground, the structure is said to be in a non-sliding phase. In this phase the structure behaves like a fixed base structure. Whenever, the static frictional force is overcome, there is a relative sliding between the ground and the mass and now the structure is said to be in sliding phase. The total motion consists of alternating non-sliding and sliding phases. The foregoing text describes the mathematical formulation in these two phases.

3.3.1 Non-sliding Phase

In this phase the structure behaves similar to a fixed-base structure as the structure moves along with the ground and hence the normal modal analysis can be carried out. The equations of motion in this phase are as below

$$\mathbf{M_0}\ddot{\mathbf{x}}_\mathbf{0} + \mathbf{C_0}\dot{\mathbf{x}}_\mathbf{0} + \mathbf{K_0}\mathbf{x_0} = -\mathbf{M_0}\mathbf{r_0}\ddot{x}_g \tag{3.8}$$

and

$$x_b = \text{constant};\ \dot{x}_b = \ddot{x}_b = 0 \tag{3.9}$$

where, $\mathbf{M_0}$, $\mathbf{C_0}$ and $\mathbf{K_0}$ are the mass, damping stiffness matrices of the fixed-base structure respectively, $\mathbf{x_0}$ is the vector of displacements of the DOFs of the superstructure relative to the base (excluding the DOF of base mass), x_b is the displacement of the base mass relative to the ground, $\mathbf{r_0}$ is the influence coefficient vector and dot indicates derivative with respect to time. Since the base mass does not move relative to the ground, the velocity and acceleration of the base relative to the ground are zero. However the sliding displacement may be non-zero as the structure can enter a non-sliding phase at any instant of its motion. But this value remains constant during a given non-sliding phase. Equation (3.8) is standard equation of motion for a MDOF structure and can be solved by standard modal analysis procedures (Clough and Penziene 1993). Modal analysis can be carried out using the co-ordinate transformation.

$$\mathbf{x}_0 = \mathbf{\Phi}_0 \mathbf{y} \tag{3.10}$$

In the above equation $\mathbf{\Phi}_0$ is the mode-shape matrix of the superstructure DOFs and **y** is a vector of modal co-ordinates. Now the undamped modal properties of the structure can be got by,

$$\left[\mathbf{K}_0 - \mathbf{M}_0 \omega_i^2\right] \phi_{0i} = \mathbf{0} \tag{3.11}$$

This eigen value problem can be solved to get the frequencies ω_i and the associated mode shapes. Using the modal decomposition and orthogonality conditions of normal modes, Equation (3.8) can now be uncoupled to get N set of equations in terms of the modal co-ordinates in the following form,

$$\ddot{y}_m + 2\zeta_m \omega_m \dot{y}_m + \omega_m^2 y_m = -\gamma_m \ddot{x}_g \qquad m = 1,2,3,\ldots,N \tag{3.12}$$

Here, the suffix m indicates m^{th} mode, ζ_m is the modal damping ratio, y_m is the modal co-ordinate and γ_m is the modal participation factor. The participation factor for m^{th} mode is given by

$$\gamma_m = \frac{\phi_{0m}^T \mathbf{M}_0 \mathbf{r}_0}{\phi_{0m}^T \mathbf{M}_0 \phi_{0m}} \tag{3.13}$$

where, ϕ_{0m} is the m^{th} mode shape vector. Each equation in Equation (3.12) corresponding to different i is equivalent to equation of motion for a SDOF structure with natural frequency ω_i and damping ratio ζ_i and can be solved using the available methods.

3.3.2 Beginning of Sliding Phase

The structure will remain in a non-sliding phase until the absolute value of sum of the components of total inertia force and the restoring force acting tangential to the sliding surface at the isolator level do not exceed the absolute value of total frictional force acting along the sliding surface. Referring to Figure 2.7, the condition for the beginning of the sliding phase can be expressed as,

$$\left| \left\{ \sum_{i=1}^{N} m_i (\ddot{x}_i + \ddot{x}_g) + m_b \ddot{x}_g \right\} \cos\theta + m_t g \sin\theta \right| \geq m_t \mu g \cos\theta \tag{3.14}$$

where the suffix i indicate the i^{th} DOF, m_t is the total mass of the structure including the base mass m_b, μ is the coefficient of sliding friction and g is the acceleration due to gravity. Dividing the Equation (3.14) by $\cos\theta$.

$$\left|\left\{\sum_{i=1}^{N} m_i(\ddot{x}_i + \ddot{x}_g) + m_t g \tan\theta\right\}\right| \geq m_t \mu g \tag{3.15}$$

Now $\tan\theta$ is the slope of the sliding surface which can be expressed as,

$$\tan\theta = \frac{dy}{dx_b} \tag{3.16}$$

where, y is the vertical rigid body displacement of the structure at the isolator level due to the induced rising of the structure along the curved sliding surface. So Equation (3.15) may be written as,

$$\left|\left\{\sum_{i=1}^{N} m_i(\ddot{x}_i + \ddot{x}_g) + m_b\ddot{x}_g\right\} + m_t g \frac{dy}{dx_b}\right| \geq m_t \mu g \tag{3.17}$$

As seen earlier, the term $m_t g\, dy/dx_b$ is the restoring force offered by the structure of mass m_t for a given sliding displacement x_b. Using the definition of the frequency of the isolator as defined in Equations (3.4) through (3.7), the restoring force can be expressed as a product of the total mass of the structure and square of instantaneous isolator frequency. The condition for beginning of sliding finally reduces to,

$$\left|\left\{\sum_{i=1}^{N} m_i(\ddot{x}_i + \ddot{x}_g) + m_b\ddot{x}_g\right\} + m_t \omega_b^2(x_b) x_b\right| \geq m_t \mu g \tag{3.18}$$

3.3.3 Sliding Phase

Once the inequality (3.18) is satisfied the structure enters a sliding phase and the DOF corresponding to the base is also activated. The equations of motion now are given by,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -\mathbf{M}\mathbf{r}\ddot{x}_g - \mathbf{r}\mu_f \tag{3.19}$$

where, **M, C, K** are the modified mass, damping and stiffness matrices of order N+1, **r** is the modified influence coefficient vector and μ_f is the frictional force as given below.

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_0 & \mathbf{M}_0\mathbf{r}_0 \\ [\mathbf{M}_0\mathbf{r}_0]^T & m_t \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_0 & \mathbf{0} \\ \mathbf{0} & c_b \end{bmatrix},$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_0 & \mathbf{0} \\ \mathbf{0} & k_b \end{bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} \mathbf{x}_0 \\ x_b \end{Bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and}$$

$$\mu_f = m_t \mu g \,\mathrm{sgn}(\dot{x}_b) \tag{3.20}$$

In the above equation, $\mathrm{sgn}(\dot{x}_b)$ is the signum function that decides the direction of frictional force in a given sliding phase. The value of this function depends on the direction of sliding velocity and takes a value of +1 for positive sliding and -1 for negative sliding velocity. The equations of motion are coupled by the acceleration terms. The equations of motion can be directly solved by standard techniques in order to obtain the time history of responses. However, this approach is computationally expensive and does not provide any insight into the structure behavior. For practical solution, a modal formulation analogous to that of fixed-base structure has been presented below.

In sliding phase, all natural modes of the structure are activated, with the fundamental frequency corresponding to sliding mode of the isolator. Since the structure behaves essentially like a rigid body in this mode of motion, it can be assumed that the eigenvectors of other natural frequencies of the structure are not modified significantly from those of the corresponding fixed-base structure (non-sliding phase). Under this assumption, mode synthesis approach can be adopted using modal properties of the fixed-base structure to also represent the modal properties of the sliding structure. The modal properties of the sliding structure can be expressed as a linear combination of modal properties of the fixed-base structure as given below.

$$\mathbf{x} = \mathbf{\Phi Y} \tag{3.21}$$

where,

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_0 & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \text{ and } \mathbf{Y} = \begin{Bmatrix} \mathbf{y} \\ x_b \end{Bmatrix} \tag{3.22}$$

Substituting Equations (3.21) in Equations (3.19) and pre multiplying by $\mathbf{\Phi}^{\mathrm{T}}$ we get,

$$(\mathbf{\Phi}^{\mathrm{T}}\mathbf{M\Phi})\ddot{\mathbf{Y}} + (\mathbf{\Phi}^{\mathrm{T}}\mathbf{C\Phi})\dot{\mathbf{Y}} + (\mathbf{\Phi}^{\mathrm{T}}\mathbf{K\Phi})\mathbf{Y} = -(\mathbf{\Phi}^{\mathrm{T}}\mathbf{Mr})\ddot{x}_g - (\mathbf{\Phi}^{\mathrm{T}}\mathbf{r})\mu_f \tag{3.23}$$

Carrying out matrix multiplication and using the modal properties of the superstructure the Equation (3.23) can be written as,

$$\widetilde{\mathbf{M}}\ddot{\mathbf{Y}} + \widetilde{\mathbf{C}}\dot{\mathbf{Y}} + \widetilde{\mathbf{K}}\mathbf{Y} = -\mathbf{L}_g\ddot{x}_g - \mathbf{L}_f\mu_f \tag{3.24}$$

where,

$$\widetilde{\mathbf{M}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{M\Phi} = \begin{bmatrix} M_1 & & & & \gamma_1 M_1 \\ & M_2 & \mathbf{0} & & \gamma_2 M_2 \\ & & \cdot & & \cdot \\ & \mathbf{0} & & \cdot & \cdot \\ & & & M_n & \gamma_n M_n \\ \gamma_1 M_1 & \gamma_2 M_2 & \cdot & \gamma_n M_n & m_t \end{bmatrix}$$

$$\widetilde{\mathbf{C}}=\mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi}=\begin{bmatrix} 2M_1\xi_1\omega_1 & & & & \\ & 2M_2\xi_2\omega_2 & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & 2M_n\xi_n\omega_n & \\ & & & & 2m_t\xi_b\omega_b \end{bmatrix}$$

$$\widetilde{\mathbf{K}}=\mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi}=\begin{bmatrix} M_1\omega_1^2 & & & & \\ & M_2\omega_2^2 & & \mathbf{0} & \\ & & \ddots & & \\ & \mathbf{0} & & M_n\omega_n^2 & \\ & & & & m_t\omega_b^2 \end{bmatrix}$$

$$\mathbf{L}_{\mathrm{g}}=\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{r}=\begin{Bmatrix} \gamma_1 M_1 \\ \gamma_2 M_2 \\ \cdot \\ \cdot \\ \gamma_n M_n \\ m_t \end{Bmatrix} \text{ and } \mathbf{L}_{\mathrm{f}}=\mathbf{\Phi}^{\mathrm{T}}\mathbf{r}=\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

In the above matrices n is the number of superstructure modes considered for analysis and M_i is the i^{th} modal mass given by,

$$M_i=\phi_{0i}^T\mathbf{M}_0\phi_{0i} \tag{3.25}$$

So, the number of equations to be solved thus reduces to n + 1. With the modal damping in the superstructure different from the isolator damping, the system is non-classically damped. So the complex eigen value problem needs to be solved for a size of n + 1 only. This can be done by a state-vector approach (Malushte and Singh 1989, Singh and Malushte 1990, Singh and Suarez 1992, Singh et.al. 1992, Sinha 1993).

Let us define state-vector as,

$$\mathbf{u}=\begin{Bmatrix} \mathbf{Y} \\ \dot{\mathbf{Y}} \end{Bmatrix} \tag{3.26}$$

So that we may write Equations (3.24) in the form,

$$\dot{\mathbf{u}}=\mathbf{A}\mathbf{u}-\mathbf{R}_{\mathrm{g}}\ddot{x}_g-\mathbf{R}_{\mathrm{f}}\mu_f \tag{3.27}$$

with,

$$\mathbf{A}=\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{K}} & -\widetilde{\mathbf{M}}^{-1}\widetilde{\mathbf{C}} \end{bmatrix},\ \mathbf{R}_{\mathrm{g}}=\begin{Bmatrix} \mathbf{0} \\ \widetilde{\mathbf{M}}^{-1}\mathbf{L}_{\mathrm{g}} \end{Bmatrix} \text{ and } \mathbf{R}_{\mathrm{f}}=\begin{Bmatrix} \mathbf{0} \\ \widetilde{\mathbf{M}}^{-1}\mathbf{L}_{\mathrm{f}} \end{Bmatrix}$$

The homogeneous form of Equation (3.27) is,

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u} \tag{3.28}$$

This is an eigen value problem of order 2(n + 1). As a matrix **A** is unsymmetrical it has two associated eigenvalue problems of order 2(n + 1) each. The right and left eigenvalue problems can be represented respectively as,

$$|\mathbf{A}-\lambda_m\mathbf{I}|\psi_{\mathbf{R}_m} = \mathbf{0} \text{ and } \psi_{\mathbf{L}_m}^{\mathbf{T}}|\mathbf{A}-\lambda_m\mathbf{I}| = \mathbf{0} \tag{3.29}$$

where λ_m are the complex eigen values and ψ_{Rm} and ψ_{Lm} are the associated right and left eigenvectors respectively. If the structure is under-critically damped the 2(n +1) eigenvalues and associated eigenvectors occur in complex conjugate pairs. The 2(n+1) order eigenvalue problem, in this case, results in only (n+1) independent eigenvalues. This result is expected from physical considerations as the calculation of eigenvalues in Equation (3.29) is a procedure analogous to the calculation of natural frequencies for classically damped structure and should therefore only give (n+1) independent eigenvalues. The eigenvalues and eigenvectors can be rearranged such that,

$$\lambda_{n+m} = \bar{\lambda}_m \qquad m = 1,2,\ldots,n+1 \tag{3.30}$$

$$\psi_{\mathbf{L}_{n+m}} = \bar{\psi}_{\mathbf{L}_m} \text{ and } \psi_{\mathbf{R}_{n+m}} = \bar{\psi}_{\mathbf{R}_m} \qquad m = 1,2,\ldots,n+1 \tag{3.31}$$

where the over bar represents the complex conjugate of a quantity. For comparison with the case of classical damping, it is possible to represent the eigenvalues in terms of i^{th} natural frequency and damping ratio as,

$$\lambda_m = \omega_m(-\xi_m + i\sqrt{1-\xi_m^2}) \text{ and } \bar{\lambda}_m = \omega_m(-\xi_m - i\sqrt{1-\xi_m^2}) \tag{3.32}$$

The natural frequencies and damping ratios of the structure can be determined from the complex eigenvalues by,

$$\omega_m = |\lambda_m|;\ \xi_m = \frac{\mathrm{Re}(\lambda_m)}{|\lambda_m|} \qquad m = 1,2,\ldots,n+1 \tag{3.33}$$

Now the Equations (3.27) can be written in the modal form by the transformation,

$$\mathbf{u} = \boldsymbol{\psi}_{\mathbf{R}}\mathbf{z} \tag{3.34}$$

where **z** is the vector of modal co-ordinates and $\boldsymbol{\psi}_{\mathbf{R}}$ is the matrix of right eigen vectors. Substituting Equation (3.34) in Equation (3.27) and pre multiplying by the m^{th} left eigenvector we get,

$$\psi_{\mathbf{L}_m}^{\mathbf{T}}\boldsymbol{\psi}_{\mathbf{R}}\dot{\mathbf{z}} = \psi_{\mathbf{L}_m}^{\mathbf{T}}\mathbf{A}\boldsymbol{\psi}_{\mathbf{R}}\mathbf{z} - \psi_{\mathbf{L}_m}^{\mathbf{T}}\mathbf{R}_{\mathbf{g}}\ddot{x}_g - \psi_{\mathbf{L}_m}^{\mathbf{T}}\mathbf{R}_{\mathbf{f}}\mu_f \qquad m = 1,2,\ldots,2(n+1) \tag{3.35}$$

The left and right eigen vectors are orthogonal to each other and to matrix **A**. so we have,

$$\psi_{\mathbf{L}_m}^{\mathbf{T}}\psi_{\mathbf{R}_j} = \psi_{\mathbf{L}_m}^{\mathbf{T}}\mathbf{A}\psi_{\mathbf{R}_j} = 0 \text{ with } m \neq j \qquad m = 1,2,\ldots,2(n+1) \tag{3.36}$$

These orthogonal conditions of the complex modes uncouple the Equations (3.35). Rearranging the terms we can write,

$$\dot{z}_m - \lambda_m z_m = -P_{g_m} \ddot{x}_g - P_{f_m} \mu_f \qquad m = 1,2,\ldots,2(n+1) \quad (3.37)$$

where P_{gm} and P_{fm} are the complex participation factors respectively for the ground motion and the frictional force. These participation factors are given by,

$$P_{g_m} = \frac{\psi_{\mathbf{L}_m}^{\mathbf{T}} \mathbf{R}_{\mathbf{g}}}{\psi_{\mathbf{L}_m}^{\mathbf{T}} \psi_{\mathbf{R}_m}^{\mathbf{T}}} \text{ and } P_{f_m} = \frac{\psi_{\mathbf{L}_m}^{\mathbf{T}} \mathbf{R}_{\mathbf{f}}}{\psi_{\mathbf{L}_m}^{\mathbf{T}} \psi_{\mathbf{R}_m}^{\mathbf{T}}} \qquad m = 1,2,\ldots,2(n+1) \quad (3.38)$$

Once the participation factors are known, Equations (3.37) can be solved by step-by-step integration procedure developed in the subsequent sections. As the right and left eigenvectors are in complex conjugate pairs, the participation factors also occur in complex conjugate pairs. This means the solution of Equations (3.37) leads to modal co-ordinates that are also complex conjugate pairs. So, when the back transformation is carried out using Equation (3.34) to get the state vector $\mathbf{u}$, it will be a real vector. This is obvious as the vector $\mathbf{u}$ consists of the modal coordinates of the superstructure which are real quantities. Knowing the state vector consisting of the modal coordinates, the structural deformations can be determined using the normal transformation given by Equation (3.10).

3.3.4 Direction of Sliding

The direction of sliding depends on the signum function that in turn depends on the forces acting on the structure at the end of a non-sliding phase. Once the inequality Equation 3.18 is satisfied, the structure starts sliding in a direction opposite to the direction of the sum of the total inertia force and the restoring force at the isolator level. So, we have

$$\operatorname{sgn}(\dot{x}_b) = -\frac{\left[\sum_{i=1}^{n} m_i(\ddot{x}_i + \ddot{x}_b + \ddot{x}_g)\right] + m_b(\ddot{x}_b + \ddot{x}_g) + m_t\omega_b^2 x_b}{\left|\left[\sum_{i=1}^{n} m_i(\ddot{x}_i + \ddot{x}_b + \ddot{x}_g)\right] + m_b(\ddot{x}_b + \ddot{x}_g) + m_t\omega_b^2 x_b\right|} \quad (3.39)$$

The value of signum function remains the same in a given sliding phase. The end of a sliding phase is governed by the condition that the sliding velocity of the base is equal to zero,

$$\dot{x}_b = 0 \quad (3.40)$$

Once the sliding velocity is zero, the structure may enter a non-sliding phase or reverse its direction of sliding or has a momentary abrupt stop and continue in the same direction. To decide the correct state, the solution process should continue with the equations of non-sliding phase wherein the relative acceleration at the isolator level is forced to zero and further check the validity of the inequality Equation 3.18. If this inequality is satisfied at the same instant of time when the sliding velocity is zero, it shows that there is an abrupt stop at that instant and then the Equation (3.39) decides the further direction of sliding.

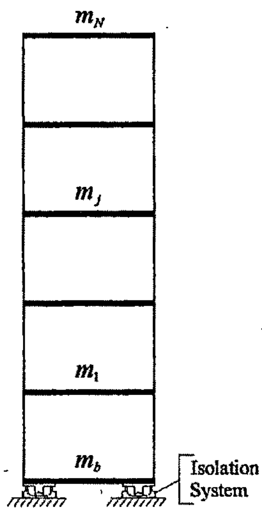

Figure 3.1: Lumped mass model with isolator

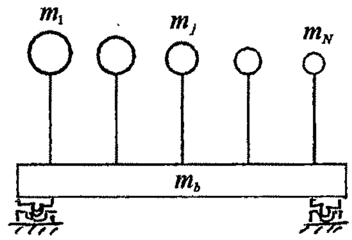

Figure 3.2: Modal masses attached to base mass

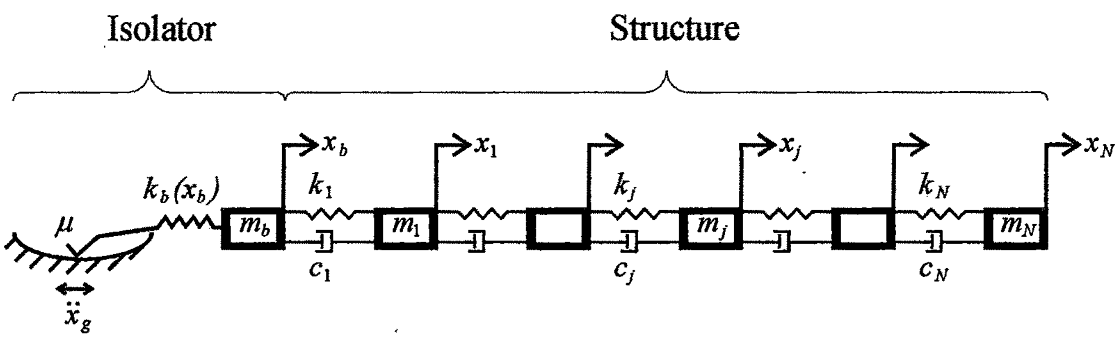

Figure 3.3: Analytical model of multi storied structure with base isolator