$Schwarz$ function $w(z) \in A$ such that

$$\left\{ \frac{(s-t)zL'(z)}{L(sz)-L(tz)} \right\} = \phi(w(z)) \quad (z \in U, s \neq t) \tag{2.3.3}$$

If $p_1(z)$ is analytic and has positive real part in U and $p_1(0) = 1$, then

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + ... \quad (z \in U) \tag{2.3.4}$$

From (2.3.4) we obtain

$$w(z) = \frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + ... \tag{2.3.5}$$

Let

$$p(z) = \frac{(s-t)zL'(z)}{L(sz)-L(tz)} = 1 + b_1 z + b_2 z^2 + ... \quad (z \in U) \tag{2.3.6}$$

$$\frac{(s-t)z\left[1 + 2g_2a_2z + 3g_3a_3z^2 + ...\right]}{(s-t)z + g_2a_2(s^2-t^2)z^2 + g_3a_3(s^3-t^3)z^3} = 1 + b_1z + b_2z^2 + ...$$

which gives

$$b_1 = (2-s-t)g_2a_2 \quad and$$

$$b_2 = (s+t)(s+t-2)g_2^2a_2^2 + (3-s^2-st-t^2)g_3a_3 \tag{2.3.7}$$

Since $\phi(z)$ is univalent and $p \prec \phi$, therefore using (2.3.5), we obtain:

$$p(z) = \phi(w(z)) = 1 + \frac{B_1c_1}{2} + \left\{ \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4} c_1^2 B_2 \right\} z^2 + ... \tag{2.3.8}$$

Now from (2.3.6), (2.3.7) and (2.3.8), we have

$$(2-s-t)g_2a_2 = \frac{B_1c_1}{2},$$

$$\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{1}{4}c_1^2B_2 = (s+t)(s+t-2)g_2^2a_2^2+(3-s^2-st-t^2)g_3a_3$$

Therefore we have

$$a_3-\mu a_2^2 = \frac{B_1}{2g_3(3-s^2-st-t^2)}\left\{c_2-\nu c_1^2\right\},$$

$$(s+t\neq 2, s\neq t) \tag{2.3.9}$$

where

$$\nu = \frac{1}{2}\left\{1-\frac{B_2}{B_1}-\left(\frac{s+t}{2-s-t}\right)B_1+\left(\frac{\mu g_3B_1(3-s^2-st-t^2)}{g_2^2(2-s-t)^2}\right)\right\},$$

$$(s+t\neq 2, s\neq t)$$

Our result now follows by an application of Lemma 2.1.2. The result is sharp for the function defined by

$$\left\{\frac{(s-t)zL'(z)}{L(sz)-L(tz)}\right\} = \phi(z), \qquad s\neq t \tag{2.3.10}$$

and

$$\left\{\frac{(s-t)zL'(z)}{L(sz)-L(tz)}\right\} = \phi(z^2), \qquad s\neq t \tag{2.3.11}$$

If we take parameters s and t to be real numbers then by using Lemma 2.1.3 we obtain following result:

Corollary 2.3.2.

If the function $f(z)$ *given by* (2.1.1) *belongs to* $S^g(\phi, s, t)$, *for* *real parameters s and t such that* $s+t \neq 2$ *and* $s \neq t$*,then*

$$|a_3 - \mu a_2^2|$$

$$\leq \frac{1}{g_3|3-s^2-st-t^2|} \begin{cases} \left| B_2 + B_1^2 \left(\frac{s+t}{2-s-t} \right) \right. \\ \left. \quad -\mu B_1^2 \left(\frac{g_3(3-s^2-st-t^2)}{g_2^2(2-s-t)^2} \right) \right|, & \mu \leq \eta_1 \\ B_1, & \eta_1 \leq \mu \leq \eta_2 \\ \left| \mu B_1^2 \left(\frac{g_3(3-s^2-st-t^2)}{g_2^2(2-s-t)^2} \right) \right. \\ \left. \quad -B_1^2 \left(\frac{s+t}{2-s-t} \right) - B_2 \right|, & \mu \geq \eta_2 \end{cases}$$

where

$$\eta_1 = \frac{g_2^2(2-s-t)^2}{g_3 B_1(3-s^2-st-t^2)} \left[-1 + \frac{B_2}{B_1} + B_1 \left(\frac{s+t}{2-s-t} \right) \right]$$

$$\eta_2 = \frac{g_2^2(2-s-t)^2}{g_3 B_1(3-s^2-st-t^2)} \left[1 + \frac{B_2}{B_1} + B_1 \left(\frac{s+t}{2-s-t} \right) \right]$$

The result is sharp.

Example 2.3.3.

Let $(-1 \leq B < A \leq 1)$*. If* $f(z)$ *given by* (2.1.1) *belongs to* $S^g[A,B,s,t]$*, for real parameters s and t such that* $s+t \neq 2$ *and* $s \neq t$*, then*

$$|a_3-\mu a_2^2|$$

$$\leq \frac{(A-B)}{2g_3|3-s^2-st-t^2|}\begin{cases} \left|-B+\frac{(A-B)(s+t)}{(2-s-t)} -\frac{\mu(A-B)g_3(3-s^2-st-t^2)}{g_2^2(2-s-t)^2}\right|, & \mu\leq\eta_1^* \\ 1, & \eta_1^*\leq\mu\leq\eta_2^* \\ \left|B-\frac{(A-B)(s+t)}{(2-s-t)} +\frac{\mu(A-B)g_3(3-s^2-st-t^2)}{g_2^2(2-s-t)^2}\right|, & \mu\geq\eta_2^* \end{cases}$$

where

$$\eta_1^* = \frac{g_2^2(2-s-t)^2}{g_3(3-s^2-st-t^2)}\left[\left(\frac{s+t}{2-s-t}\right)-\left(\frac{B+1}{A-B}\right)\right]$$

$$\eta_2^* = \frac{g_2^2(2-s-t)^2}{g_3(3-s^2-st-t^2)}\left[\left(\frac{s+t}{2-s-t}\right)-\left(\frac{B-1}{A-B}\right)\right]$$

If $\eta_1\leq\mu\leq\eta_2$, in view of Lemma 2.1.3, Corollary 2.3.2 can be improved.

Theorem 2.3.4.

If the function $f(z)$ given by (2.1.1) belongs to $C^g(\phi, s, t)$, then

$$|a_3-\mu a_2^2| \leq \frac{1}{3g_3|3-s^2-st-t^2|}\max\left\{B_1, \left|B_2+\frac{B_1^2(s+t)}{(2-s-t)} -\frac{3\mu g_3B_1^2(3-s^2-st-t^2)}{4g_2^2(2-s-t)^2}\right|\right\}$$

provided $s+t\neq 2$ and $s\neq t$. The result is sharp.

If we take parameter s and t to be real numbers, then we have following result:

Corollary 2.3.5.

If the function $f(z)$ given by (2.1.1) belongs to $C^g(\phi, s, t)$, for real parameters s and t such that $s+t \neq 2$ and $s \neq t$, then

$$|a_3 - \mu a_2^2| \leq \frac{1}{3g_3|3-s^2-st-t^2|} \begin{cases} \left| B_2 + \frac{B_1^2(s+t)}{(2-s-t)} - \frac{3\mu B_1^2 g_3(3-s^2-st-t^2)}{4g_2^2(2-s-t)^2} \right|, & \mu \leq \eta_1^{**} \\ B_1, & \eta_1^{**} \leq \mu \leq \eta_2^{**} \\ \left| -B_2 - \frac{B_1^2(s+t)}{(2-s-t)} + \frac{3\mu g_3 B_1^2(3-s^2-st-t^2)}{4g_2^2(2-s-t)^2} \right|, & \mu \geq \eta_2^{**} \end{cases}$$

where

$$\eta_1^{**} = \frac{4g_2^2(2-s-t)^2}{3g_3B_1(3-s^2-st-t^2)} \left[-1 + B_1\left(\frac{s+t}{2-s-t}\right) - \left(\frac{B_2}{B_1}\right) \right]$$

$$\eta_2^{**} = \frac{4g_2^2(2-s-t)^2}{3g_3B_1(3-s^2-st-t^2)} \left[1 + B_1\left(\frac{s+t}{2-s-t}\right) - \left(\frac{B_2}{B_1}\right) \right]$$

The result is sharp.

Example 2.3.6.

Let $(-1 \leq B < A \leq 1)$. If $f(z)$ given by (2.1.1) belongs to $C^g[A, B, s, t]$, for real parameters s and t such that $s+t \neq 2, s \neq t$, then

$$|a_3 - \mu a_2^2|$$

$$\leq \frac{A-B}{3g_3|3-s^2-st-t^2|} \begin{cases} \left| -B + \frac{(A-B)(s+t)}{(2-s-t)} \right. \\ \left. - \frac{3\mu g_3 (A-B)(3-s^2-st-t^2)}{4g_2^2(2-s-t)^2} \right|, \ \mu \leq \eta_1' \\ 1, \qquad \eta_1' \leq \mu \leq \eta_2' \\ \left| B - \frac{(A-B)(s+t)}{(2-s-t)} \right. \\ \left. + \frac{3\mu g_3 (A-B)(3-s^2-st-t^2)}{4g_2^2(2-s-t)^2} \right|, \ \mu \geq \eta_2' \end{cases}$$

where

$$\eta_1' = \frac{4g_2^2(2-s-t)^2}{3g_3(3-s^2-st-t^2)} \left[\left(\frac{s+t}{2-s-t} \right) - \left(\frac{B+1}{A-B} \right) \right]$$

$$\eta_2' = \frac{4g_2^2(2-s-t)^2}{3g_3(3-s^2-st-t^2)} \left[\left(\frac{s+t}{2-s-t} \right) - \left(\frac{B-1}{A-B} \right) \right]$$

Since

$$\Omega^\lambda f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n,$$

We have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \qquad (2.3.12)$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)} \qquad (2.3.13)$$

For g_2, g_3 given by (2.3.12) and (2.3.13) respectively, Theorem 2.3.2 reduces to the following:

Theorem 2.3.7.

Let $\lambda < 2$. If $f(z)$ given by (2.1.1) belongs to $S^{\lambda}(\phi, s, t)$ for $s + t \neq 2, s \neq t$, then

$$|a_3 - \mu a_2^2| \leq \frac{(2-\lambda)(3-\lambda)}{6|3 - s^2 - t^2 - st|} max \left\{ B_1, \left| \frac{B_1^2(s+t)}{2-s-t} + B_2 - \frac{3}{2}\frac{\mu(2-\lambda)B_1^2(3-s^2-t^2-st)}{(3-\lambda)(2-s-t)^2} \right| \right\}$$

Corollary 2.3.8

If the function $f(z)$ given by (2.1.1) belongs to $S^{\lambda}(\phi, s, t)$, for real parameters s and t such that $s + t \neq 2$ and $s \neq t$, then

$$|a_3 - \mu a_2^2| \leq \frac{(2-\lambda)(3-\lambda)}{6|3 - s^2 - st - t^2|} \begin{cases} \left| B_2 + B_1^2 \left(\frac{s+t}{2-s-t} \right) - \frac{3}{2}\mu B_1^2 \left(\frac{(3-\lambda)(3-s^2-st-t^2)}{(2-\lambda)(2-s-t)^2} \right) \right|, & \mu \leq \eta_1'' \\ B_1, & \eta_1'' \leq \mu \leq \eta_2'' \\ \left| \frac{3}{2}\mu B_1^2 \left(\frac{(3-\lambda)(3-s^2-st-t^2)}{(2-\lambda)(2-s-t)^2} \right) - B_1^2 \left(\frac{s+t}{2-s-t} \right) - B_2 \right|, & \mu \geq \eta_2'' \end{cases}$$

where

$$\eta_1'' = \frac{2}{3}\frac{(3-\lambda)(2-s-t)^2}{(2-\lambda)B_1(3-s^2-st-t^2)} \left[-1 + \frac{B_2}{B_1} + B_1 \left(\frac{s+t}{2-s-t} \right) \right]$$

$$\eta_2'' = \frac{2}{3}\frac{(3-\lambda)(2-s-t)^2}{(2-\lambda)B_1(3-s^2-st-t^2)} \left[1 + \frac{B_2}{B_1} + B_1 \left(\frac{s+t}{2-s-t} \right) \right]$$

The result is sharp.

Special Case:

For $s = 1$ in aforementioned Theorems 2.3.1, 2.3.4, 2.3.7, Corollaries 2.3.2, 2.3.5, 2.3.8 and Examples 2.3.3, 2.3.6, we arrive at the results obtained by Goyal and Goswami (2009) and for $s = 1, t = -1$ in aforementioned Theorems 2.3.1, 2.3.4, 2.3.7, Corollaries 2.3.2, 2.3.5, 2.3.8 and Examples 2.3.3, 2.3.6, we arrive at the results obtained by Shanmugam et al. (2006).

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CHAPTER 3

SOME STARLIKE AND CONVEXITY PROPERTIES OF SAKAGUCHI CLASSES FOR HYPERGEOMETRIC FUNCTIONS

The main result of the chapter has been published as listed below:

1. "Some starlike and convexity properties of Sakaguchi classes for hypergeometric functions", International Journal of Open Problems in Complex Analysis, 4(1), 2012, 11-19.

This chapter deals with some characterizations for a (Gaussian) hypergeometric function to be in a subclass of Sakaguchi type functions. First we will discuss two lemmas for the two subclasses $S(\alpha, t)$ and $C(\alpha, t)$ of starlike and convex analytic functions defined in the open unit disc U. These subclasses are special cases of subclasses ($s = 1$) defined by (1.2.13) in Chapter 1. Then using these lemmas we shall obtain conditions for (Gaussian) hypergeometric function to be in the subclasses $S(\alpha, t)$ and $C(\alpha, t)$.

3.1. Introduction

Let A be the class of functions defined by (1.2.1) which are analytic and univalent in the open unit disk U.

A function $f(z) \in A$ is said to be in the class $S(\alpha, t)$ if it satisfies equation (1.2.13), where t is a real parameter.

We also denote by $C(\alpha, t)$ the subclass of A consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha, t)$.

We note that $S(\alpha, 0) = S^*(\alpha)$, the class of starlike functions of order α $(0 \leq \alpha < 1)$ and $C(\alpha, 0) = C(\alpha)$, the class of convex functions of order α $(0 \leq \alpha < 1)$ by Silverman(1975).

However, for this chapter, we consider a subclass T of A where T denotes the class consisting of the functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{3.1.1}$$

obviously $T \subset A$. Let $F(a, b; c; z)$ be the (Gaussian) hypergeometric function defined as in Ranville (1971):

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n$$

where $c \neq 0, -1, -2, \ldots.,$ and $(\theta)_n = \theta(\theta+1)\ldots\ldots(\theta+n-1)$, $n \in N = \{1, 2, \ldots..\}$ and $(\theta)_0 = 1$. We note that $F(a, b; c; z)$

converges for $|z| = 1$ and $\text{Re}(c - a - b) > 0$. Also it is related to the gamma function by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

Silverman (1993) gave necessary and sufficient conditions for $zF(a, b; c; z)$ to be in $S^*(\alpha)$ and $C(\alpha)$. For other interesting developments on $zF(a, b; c; z)$ in connections with various subclasses of univalent and p-valent functions, the reader can refer to the works of Carlson and Shaffer (1984), Cho et al.(2002), Goyal et al.(2012), Merkes and Scott (1961), Mostafa (2009), Owa et al. (2007), Ruscheweyh and Singh (1986) etc .

To establish our main results, we shall require results due to Owa et al. (2007) contained in the following Lemma:

Lemma 3.1.1.

(i) If $f(z) \in A$ *satisfies*

$$\sum_{n=2}^{\infty} (|n - u_n| + (1-\alpha)|u_n|)a_n \leq 1 - \alpha, \tag{3.1.2}$$

for some α $(0 \leq \alpha < 1)$, *then* $f(z) \in S(\alpha, t)$.

(ii) If $f(z) \in A$ *satisfies*

$$\sum_{n=2}^{\infty} n(|n - u_n| + (1-\alpha)|u_n|)a_n \leq 1 - \alpha \tag{3.1.3}$$

for some $\alpha(0 \leq \alpha < 1)$*and* $|t| \leq 1$, then $f(z) \in C(\alpha, t)$.

where $(u_n = 1 + t + t^2 + ... + t^{n-1})$

3.2. Main Results

Theorem 3.2.1.

(i) Let $a, b > -1$, $c > 0$, $ab < 0$ *and*

$$1 + \frac{c-a-b-1}{ab} \leq \frac{\alpha(c-a-b-1)(1+|t|)}{ab(1-t)} \tag{3.2.1}$$

Then $zF(a,b;c;z)$ *is in* $S(\alpha, t)$.

(ii) Let $a, b > 0$, $c > a+b+1$, *and*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{ab}{(c-a-b-1)(1-\alpha)} + \frac{1}{1-\alpha} - \frac{\alpha(1+|t|)}{(1-\alpha)(1-t)}\right] \leq 2 \tag{3.2.2}$$

Then

$F_1(a,b;c;z) = z[2 - F(a,b;c;z)]$ is in $S(\alpha, t)$.

Proof (*i*): Since

$$zF(a,b;c;z) = z + \frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n$$

$$= z - \left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n \tag{3.2.3}$$

According to (i) of Lemma 3.1.1, we must show that

$$\sum_{n=2}^{\infty}\left(|n-u_n| + (1-\alpha)|u_n|\right)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left|\frac{c}{ab}\right|(1-\alpha) \tag{3.2.4}$$

Note that left hand side of (3.2.4) diverges if $c < a+b+1$.

Now using results

$$\sum_{n=2}^{\infty}\left(|n-u_n|+(1-\alpha)|u_n|\right) \geq \sum_{n=2}^{\infty}\left(n-\alpha|u_n|\right)$$

and

$$(n-\alpha|u_n|)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)n_2}$$

(3.2.4) reduces to

$$=\sum_{n=2}^{\infty}(n-1)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}+\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$-\alpha\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}|u_n|$$

$$=\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]$$

$$-\alpha\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}|u_{n+1}|-u_1\right]$$

in (3.2.4) we find that

$$\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\right]-\alpha\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}|u_{n+1}|\right]$$

$$\leq(1-\alpha)\left[\left|\frac{c}{ab}\right|+\frac{c}{ab}\right]=0 \qquad (3.2.5)$$

$(u_1=1, ab<0, c>0)$

Since

$$|u_{n+1}|=|1+t+........+t^n|=\left|\frac{1-t^{n+1}}{1-t}\right|\leq\frac{(1+|t|^{n+1})}{1-t}\leq\frac{(1+|t|)}{1-t}$$

as $(|t|^n \leq 1)$, therefore (3.2.5) is valid if

$$\sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n (1)_n} + \frac{c}{ab}\left[\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n}\right]$$

$$-\alpha \frac{c}{ab(1-t)} \left[\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} + (1+|t|) \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n}\right] \leq 0$$

or equivalently,

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \frac{c}{ab} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$-\alpha \frac{c}{ab} \frac{(1+|t|)}{(1-t)} \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \leq 0$$

or

$$1 + \frac{c-a-b-1}{ab} \leq \frac{\alpha(c-a-b-1)(1+|t|)}{ab(1-t)}$$

which is the required inequality (3.2.1).

(ii) Since

$$F_1(a,b;c;z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n \qquad (3.2.6)$$

by (i) of Lemma 3.1.1, we need only to show that

$$\sum_{n=2}^{\infty} (|n-u_n| + (1-\alpha)|u_n|) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq (1-\alpha) \quad (3.2.7)$$

Now left side of (3.2.7)

$$= \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-2}} + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} - \alpha \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} |u_n|$$

Noting that $(\theta)_n = \theta(\theta+1)_{n-1}$ then left side of (3.2.7)

$$= \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} - \alpha \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} |u_{n+1}|$$

$$= \frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right]$$
$$-\alpha\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}|u_{n+1}| - |u_1|\right]$$

But this last expression is bounded above by $(1-\alpha)$ if and only if (3.2.2) holds.

Theorem 3.2.2.

(i) Let $a, b > -1$, $ab < 0$, $c > a+b+2$,

$$ab(a+1)(b+1) + 3ab(c-a-b-2) + (c-a-b-2)_2$$
$$-\alpha\left(\frac{1+|t|^2}{1-t}\right)(c-a-b-2)ab - \frac{\alpha(1+|t|)}{(1-t)}(c-a-b-2)_2 \geq 0 \tag{3.2.8}$$

then $zF(a,b;c;z)$ *is in* $C(\alpha,t)$.

(ii) Let $a, b > 0, c > a+b+2$ *and*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{(a)(b)(a+1)(b+1)}{(c-a-b-2)_2} + \frac{3ab}{c-a-b-1}\right.$$
$$\left.-\alpha\frac{(1+|t|^2)ab}{(1-t)(c-a-b-1)} - \frac{\alpha(1+|t|)}{(1-t)}\right] \leq 2(1-\alpha) \tag{3.2.9}$$

then $F_1(a,b;c;z) = z[2 - F(a,b;c;z)]$ *is in* $C(\alpha,t)$.

Proof (*i*): Since $zF(a,b;c;z)$ has the form (3.2.3), we see from (ii) of Lemma 3.1.1, that our conclusion is equivalent to

$$\sum_{n=2}^{\infty} n(|n-u_n|+(1-\alpha)\,|u_{_n}|)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left|\frac{c}{ab}\right|(1-\alpha) \tag{3.2.10}$$

Note that for $c > a+b+2$, the left side of (3.2.10) converges.

Now left side of (3.2.10)

$$= \sum_{n=2}^{\infty}(n^2-\alpha n\,|u_{_n}|)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$= \sum_{n=2}^{\infty}\left[(n-1)^2+1+2(n-1)\right]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$-\alpha\sum_{n=2}^{\infty}\frac{(n-1)(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\,|u_n| - \alpha\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\,|u_n|$$

$$= \sum_{n=2}^{\infty}(n-1)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} + \sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$+2\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} - \alpha\sum_{n=2}^{\infty}(n-1)\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\,|u_n|$$

$$-\alpha\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\,|u_n|$$

$$= \frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + \frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right]$$

$$+3\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} - \alpha\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\left|\frac{1-t^{n+2}}{1-t}\right|$$

$$-\alpha\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\left|\frac{1-t^{n+1}}{1-t}\right|-|u_1|\right]$$

$$=\frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}+\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]$$
$$+3\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}-\frac{\alpha}{1-t}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}(1+|t|^{n+2})$$
$$-\frac{\alpha c}{ab}\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\frac{(1+|t|^{n+1})}{(1-t)}-|u_1|\right]$$

This last expression is bounded above by $\left|\frac{c}{ab}\right|(1-\alpha)$ if and only if (3.2.8) holds.

(ii) In view of (ii) of Lemma 3.1.1 we need to show that

$$\sum_{n=2}^{\infty}n(|n-u_n|+(1-\alpha)\,|u_n|)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\leq(1-\alpha)\quad(3.2.11)$$

Now left side of (3.2.11)

$$=\sum_{n=2}^{\infty}(n^2-\alpha n\,|u_n|)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$=\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}-\alpha\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}\,|u_{n+2}|$$

$$=\sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}}+3\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$-\alpha\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}\,|u_{n+2}|-\alpha\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}\,|u_{n+1}|$$

Now by the same techniques used in the proof of Theorem 3.2.1 (i) above expression is bounded above by $(1-\alpha)$ if and only if (3.2.9) holds.

Putting $t = 0$ in the above results, we obtain the results of Silverman (1993).

Corollary (3.2.3.)

(i) *Let* $a, b > -1$, $c > 0$, $ab < 0$ *and*

$$1 + \frac{c-a-b-1}{ab} \leq \frac{\alpha(c-a-b-1)}{ab}$$

then $zF(a, b; c; z)$ *is in* $S^*(\alpha)$.

(ii) *Let* $a, b > 0$, $c > a+b+1$ *and*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{ab}{(1-\alpha)(c-a-b-1)} + 1\right] \leq 2$$

then $F_1(a, b; c; z) = z[2 - F(a, b; c; z)]$ *is in* $S^*(\alpha)$

Corollary (3.2.4.)

(i) *Let* $a, b > -1$, $ab < 0$, $c > a+b+2$ *and*

$$ab(a+1)(b+1)+(3-\alpha)ab(c-a-b-2)+(1-\alpha)(c-a-b-2)_2 \geq 0$$

then $zF(a, b; c; z)$ *is in* $C(\alpha)$.

(ii) *Let* $a, b > 0, c > a+b+2$ *and*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{(a)(b)(a+1)(b+1)}{(c-a-b-2)_2} + \frac{(3-\alpha)ab}{c-a-b-1}\right] \leq 2(1-\alpha)$$

then $F_1(a, b; c; z) = z[2 - F(a, b; c; z)]$ *is in* $C(\alpha)$.

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