CHAPTER 2

FEKET-SZEGÖ INEQUALITIES FOR GENERALIZED SAKAGUCHI TYPE FUNCTIONS AND THEIR APPLICATIONS TO FRACTIONAL DERIVATIVE OPERATOR

The main results of the chapter have been published /accepted as listed below:

1. "Fekete- Szegö inequalities for generalized Sakaguchi type functions", Proceeding of the World Congress on Engineering, 2012, Vol. I, July 4 - 6, London, U.K., 210-213.

2. "Applications of Fekete-Szegö inequalities for generalized Sakaguchi type functions to fractional derivative operator", (Accepted) Mathematika UTM.

In this chapter, we have investigated certain coefficient inequalities (Fekete-Szegö) and subordination properties for two subclasses of generalized Sakaguchi type functions defined recently by B.A. Frasin (2010). We have obtained sharp upper bounds of $|a_3 - \mu a_2^2|$ for the functions $f(z) = z + a_2 z^2 + a_3 z^3 + ...$ belonging to this subclass of generalized Sakaguchi type functions. Also, application of our results for subclasses of functions defined by convolution with a normalized analytic functions are given. In particular, Fekete-Szegö inequalities of certain classes of functions defined through fractional derivatives are obtained. The applications of Fekete-Szegö inequalities for subclasses of functions defined by convolution with a normalized analytic functions are also given.

2.1. INTRODUCTION

Let A be the class of analytic and univalent functions of the form defined in (1.2.1).

For two functions $f, g \in A$, we say that the function $f(z)$ is subordinate to $g(z)$ in U and write $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$ if there exists an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $(z \in U)$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

A function $f(z) \in A$ is said to be in the class $S(\alpha, s, t)$ if it satisfies equation (1.2.14) defined by Frasin (2010). Also the class $C(\alpha, s, t)$ is the subclass of A consisting of all functions $f(z)$ such that $zf'(z) \in S(\alpha, s, t)$.

In this chapter we shall define the following classes $S(\phi, s, t)$ and $C(\phi, s, t)$, which provide generalization of aforementioned subclasses studied by Frasin(2010).

Definition 2.1.1.

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ *be univalent starlike function with respect to '1' which maps the unit disk* U *onto a region in the right half plane which is symmetric with respect to the* *real axis, and let* $B_1 > 0$. *The function* $f \in A$ *is in the class* $S(\phi, s, t)$ *if*

$$\left\{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\} \prec \phi(z) \qquad s,t \in \mathcal{C}, s \neq t, z \in U \qquad (2.1.1)$$

Again $C(\phi, s, t)$ *denote the subclass of* A *consisting functions* $f(z)$ *such that* $zf'(z) \in S(\phi, s, t)$.

Obviously $S(\phi, 1, 0) \equiv S^*(\phi)$ and $C(\phi, 1, 0) \equiv C(\phi)$, which are the classes introduced and studied by Ma and Minda (1994).

The class $S(\phi, 1, -1) \equiv S^*_S(\phi)$, which is known class studied by Shanmugam et al.(2006).

When

$$\phi(z) = \frac{(1+Az)}{(1+Bz)}, (-1 \leq B < A \leq 1),$$

we denote the subclasses $S(\phi, s, t)$ and $C(\phi, s, t)$ by $S[A, B, s, t]$ and $C[A, B, s, t]$ respectively.

For $s = 1,\ t = 0$ and $\phi(z)$ defined above the subclass $S(\phi, 1, 0)$ reduces to the class $S^*[A, B]$ studied by Janowski(1972).

For $0 \leq \alpha < 1$ let $S(\alpha) := S[1-2\alpha, -1, 1, -1]$, which is a known class studied by Owa et al.(2007).

Also, for $s = 1, t = -1$ and $\phi(z) = \dfrac{1+(1-2\alpha)z}{1-z}$, our class reduces to a known class $S(\alpha, -1)$ studied by Cho et al. (1993, 2005). Further for $s = 1$, our class reduces to a subclass studied by Goyal and Goswami (2009).

In this chapter, we obtain the Fekete-Szegö inequality for the functions in the subclass $S(\alpha, s, t)$. To prove our main results, we need the following Lemmas.

Lemma 2.1.2. (Ma and Minda (1994))

If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ *is a function with positive real part in* U*, then for any complex number* μ

$$|c_2 - \mu c_1^2| \leq 2max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z}{1-z} \quad and \quad p(z) = \frac{1+z^2}{1-z^2}$$

Lemma 2.1.3. (Shanmugam (2006))

If $p(z) = 1 + c_1 z + c_2 z^2 + ...$ *is an analytic function with positive real part in* U*, then for a real number* ν

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \nu \leq 0 \\ 2, & 0 \leq \nu \leq 1 \\ 4\nu - 2, & \nu \geq 1 \end{cases}$$

When $\nu < 0$ *or* $\nu > 1$*, the equality holds if and only if* $p(z)$ *is* $\frac{1+z}{1-z}$ *or one of its rotations. If* $0 < \nu < 1$*, then the equality holds if and only if* $p(z)$ *is* $\frac{1+z^2}{1-z^2}$ *or one of its rotations. If*

$\nu = 0$, the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. if $\nu = 1$, the equality holds if and only if $p(z)$ is the reciprocal of one of its functions such that the equality holds in the case of $\nu = 0$. Also the above upper bound is sharp, and it can be improved as follows when $0 < \nu < 1$:

$$|c_2 - \nu c_1^2| + \nu|c_1|^2 \leq 2 \qquad (0 < \nu \leq 1/2) \text{ and}$$

$$|c_2 - \nu c_1^2| + (1-\nu)|c_1|^2 \leq 2 \qquad (1/2 < \nu < 1)$$

Our main result is contained in the following Theorem:

2.2. MAIN RESULTS

Theorem 2.2.1.

If $f(z)$ is given by (2.1.1) belongs to $S(\phi, s, t)$, then

$$|a_3 - \mu a_2^2| \leq \frac{1}{|3 - s^2 - t^2 - st|} max\left\{B_1, \left|\frac{B_1^2(s+t)}{2-s-t} + B_2 - \frac{\mu B_1^2(3 - s^2 - t^2 - st)}{(2-s-t)^2}\right|\right\}$$

such that $(s, t \in \mathcal{C}, s+t \neq 2, s \neq t)$.

Proof: Let $f \in S(\phi, s, t)$. Then there exists a *Schwarz* function $w(z) \in A$ such that

$$\left\{\frac{(s-t)zf'(z)}{f(sz) - f(tz)}\right\} = \phi(w(z)) \quad (z \in U, s \neq t) \qquad (2.2.1)$$

If $p_1(z)$ is analytic and has positive real part in U and $p_1(0) = 1$, then

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots\ldots\ldots \quad (z \in U) \quad (2.2.2)$$

From (2.2.2) we obtain

$$w(z) = \frac{c_1}{2} z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \ldots\ldots\ldots \qquad (2.2.3)$$

Let

$$p(z) = \frac{(s-t)zf'(z)}{f(sz)-f(tz)} = 1 + b_1 z + b_2 z^2 + \ldots\ldots\ldots\ldots \quad (z \in U) \qquad (2.2.4)$$

which gives

$$b_1 = (2-s-t)a_2 \quad \text{and} \quad b_2 = (s+t)(s+t-2)a_2^2 + (3-s^2-st-t^2)a_3 \qquad (2.2.5)$$

Since $\phi(z)$ is univalent and $p \prec \phi$, therefore using (2.2.3), we obtain:

$$p(z) = \phi(w(z)) = 1 + \frac{B_1 c_1}{2} + \left\{\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4} c_1^2 B_2\right\} z^2 + \ldots \qquad (2.2.6)$$

Now from (2.2.4),(2.2.5) and (2.2.6), we have

$$(2-s-t)a_2 = \frac{B_1 c_1}{2},$$

$$\frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) B_1 + \frac{1}{4} c_1^2 B_2 = (s+t)(s+t-2)a_2^2 + (3-s^2-st-t^2)a_3$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{B_1}{2(3 - s^2 - st - t^2)} \left\{ c_2 - \nu c_1^2 \right\}, \quad (s+t \neq 2, s \neq t) \tag{2.2.7}$$

where

$$\nu = \frac{1}{2}\left\{ 1 - \frac{B_2}{B_1} - \left(\frac{s+t}{2-s-t} \right) B_1 + \left(\frac{\mu B_1 (3 - s^2 - st - t^2)}{(2-s-t)^2} \right) \right\},$$

$$(s+t \neq 2, s \neq t)$$

Our result now follows by an application of Lemma 2.1.2. The result is sharp for the function defined by

$$\left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} = \phi(z), \qquad s \neq t \tag{2.2.8}$$

and

$$\left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} = \phi(z^2), \qquad s \neq t \tag{2.2.9}$$

If we take parameters s and t to be real numbers then by using Lemma 2.1.3 we obtain following result:

Corollary 2.2.2.

If the function $f(z)$ given by (2.1.1) belongs to $S(\phi, s, t)$, for real parameters s and t such that $s + t \neq 2$ and $s \neq t$, then

$$
|a_3 - \mu a_2^2| \leq \frac{1}{|3-s^2-st-t^2|} \begin{cases} \left| B_2 + B_1^2\left(\frac{s+t}{2-s-t}\right) - \mu B_1^2 \left(\frac{3-s^2-st-t^2}{(2-s-t)^2}\right)\right|, & \mu \leq \sigma_1 \\ B_1, & \sigma_1 \leq \mu \leq \sigma_2 \\ \left| \mu B_1^2 \left(\frac{3-s^2-st-t^2}{(2-s-t)^2}\right) - B_1^2\left(\frac{s+t}{2-s-t}\right) - B_2 \right|, & \mu \geq \sigma_2 \end{cases}
$$

where

$$
\sigma_1 = \frac{(2-s-t)^2}{B_1(3-s^2-st-t^2)} \left[-1 + \frac{B_2}{B_1} + B_1\left(\frac{s+t}{2-s-t}\right)\right]
$$

$$
\sigma_2 = \frac{(2-s-t)^2}{B_1(3-s^2-st-t^2)} \left[1 + \frac{B_2}{B_1} + B_1\left(\frac{s+t}{2-s-t}\right)\right]
$$

The result is sharp.

Remark :

To show that these bounds are sharp for real parameters s and t, we define the functions $K_\phi^n (n = 2, 3, ...)$ by

$$
\left\{\frac{(s-t)z(K_\phi^n)'(z)}{K_\phi^n(sz) - K_\phi^n(tz)}\right\} = \phi(z^{n-1}),
$$

$$
K_\phi^n(0) = 0 = (K_\phi^n)'(0) - 1
$$

and the functions F_λ and G_λ $(0 \leq \lambda < 1)$ by

$$
\left\{\frac{(s-t)zF_\lambda{}'(z)}{F_\lambda(sz) - F_\lambda(tz)}\right\} = \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right),
$$

$$F_\lambda(0) = 0 = (F_\lambda)'(0) - 1$$

and

$$\left\{\frac{(s-t)zG_\lambda{}'(z)}{G_\lambda(sz) - G_\lambda(tz)}\right\} = \phi\left(\frac{-z(z+\lambda)}{1+\lambda z}\right)$$

$$G_\lambda(0) = 0 = (G_\lambda)'(0) - 1$$

Obviously the functions $K_\phi^n, F_\lambda, G_\lambda \in S(\phi, s, t)$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then equality holds if and only if f is K_ϕ^2 or one of its rotations. When $\sigma_1 < \mu < \sigma_2$ then equality holds if and only if f is K_ϕ^3 or one of its rotations. If $\mu = \sigma_1$ then equality holds if and only if f is F_λ or one of its rotations . If $\mu = \sigma_2$ then equality holds if and only if f is G_λ or one of its rotations. If $\sigma_1 \leq \mu \leq \sigma_2$, in view of Lemma 2.1.3, Corollary 2.2.2 can be improved.

Corollary 2.2.3.

Let $f(z)$ given by (2.1.1) belongs to $S(\phi, s, t)$, for real parameters s and t such that $s + t \neq 2$ and $s \neq t$ and σ_3 is given by

$$\sigma_3 = \frac{(2-s-t)^2}{(3-s^2-st-t^2)B_1}\left\{\frac{B_2}{B_1} + \frac{B_1(s+t)}{(2-s-t)}\right\}$$

If $\sigma_1 < \mu \leq \sigma_3$, then

$$|a_3-\mu a_2^2| + \frac{1}{B_1^2}\left[(B_1-B_2)\frac{(2-s-t)^2}{(3-s^2-st-t^2)} - B_1^2\frac{(2-s-t)(s+t)}{(3-s^2-st-t^2)} + \mu B_1^2\right]|a_2|^2 \le \frac{B_1}{(3-s^2-st-t^2)}$$

If $\sigma_3 < \mu \le \sigma_2$, *then*

$$|a_3-\mu a_2^2| + \frac{1}{B_1^2}\left[(B_1+B_2)\frac{(2-s-t)^2}{(3-s^2-st-t^2)} + B_1^2\frac{(2-s-t)(s+t)}{(3-s^2-st-t^2)} - \mu B_1^2\right]|a_2|^2 \le \frac{B_1}{(3-s^2-st-t^2)}$$

where σ_1 *and* σ_2 *are same as defined in Corollary* 2.2.2.

Example 2.2.4.

Let $(-1 \le B < A \le 1)$. *If* $f(z)$ *given by* (2.1.1) *belongs to* $S[A,B,s,t]$, *for real parameters* s *and* t, *then*

$$|a_3-\mu a_2^2| \le \frac{(A-B)}{|3-s^2-st-t^2|}\begin{cases} \left|-B+(A-B)\left\{\left(\frac{s+t}{2-s-t}\right) -\mu\left(\frac{3-s^2-st-t^2}{(2-s-t)^2}\right)\right\}\right|, & \mu \le \sigma_1^* \\ 1, & \sigma_1^* \le \mu \le \sigma_2^* \\ \left|B+(A-B)\left\{\mu\left(\frac{3-s^2-st-t^2}{(2-s-t)^2}\right) - \left(\frac{s+t}{2-s-t}\right)\right\}\right|, & \mu \ge \sigma_2^* \end{cases}$$

where

$$\sigma_1^* = \frac{(2-s-t)^2}{(3-s^2-st-t^2)}\left[\left(\frac{s+t}{2-s-t}\right) - \left(\frac{B+1}{A-B}\right)\right]$$

$$\sigma_2^* = \frac{(2-s-t)^2}{(3-s^2-st-t^2)}\left[\left(\frac{s+t}{2-s-t}\right) - \left(\frac{B-1}{A-B}\right)\right]$$

Since $f(z) \in C(\phi, s, t)$ if and only if $zf'(z) \in S(\phi, s, t)$, Theorem 2.2.1 with an obvious change of the parameter μ, leads to the following Corollary:

Corollary 2.2.5.

If the function $f(z)$ given by (2.1.1) *belongs to $C(\phi, s, t)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{3|3-s^2-st-t^2|} max\left\{B_1, \left|B_2 + \frac{B_1^2(s+t)}{(2-s-t)} - \frac{3\mu B_1^2(3-s^2-st-t^2)}{4(2-s-t)^2}\right|\right\}$$

provided $s + t \neq 2$. The result is sharp.

If we take parameter s and t to be real numbers, then we have following result:

Corollary 2.2.6.

If the function $f(z)$ given by (2.1.1) *belongs to $C(\phi, s, t)$, for real parameters s and t such that $s + t \neq 2$ and $s \neq t$, then*

$$|a_3 - \mu a_2^2|$$

$$\leq \frac{1}{3|3-s^2-st-t^2|} \begin{cases} \left| B_2 + \frac{B_1^2(s+t)}{4(2-s-t)} - \frac{3\mu B_1^2(3-s^2-st-t^2)}{4(2-s-t)^2} \right|, & \mu \leq \sigma_1^{**} \\ B_1, & \sigma_1^{**} \leq \mu \leq \sigma_2^{**} \\ \left| -B_2 - \frac{B_1^2(s+t)}{(2-s-t)} + \frac{3\mu B_1^2(3-s^2-st-t^2)}{4(2-s-t)^2} \right|, & \mu \geq \sigma_2^{**} \end{cases}$$

where

$$\sigma_1^{**} = \frac{4(2-s-t)^2}{3B_1(3-s^2-st-t^2)} \left[-1 + B_1 \left(\frac{s+t}{2-s-t} \right) - \left(\frac{B_2}{B_1} \right) \right]$$

$$\sigma_2^{**} = \frac{4(2-s-t)^2}{3B_1(3-s^2-st-t^2)} \left[1 + B_1 \left(\frac{s+t}{2-s-t} \right) - \left(\frac{B_2}{B_1} \right) \right]$$

The result is sharp.

Corollary 2.2.7.

If the function $f(z)$ given by (2.1.1) *belongs to $C(\phi, s, t)$, for real parameters s and t such that $s+t \neq 2$ and $s \neq t$ and σ_3^{**} is given by*

$$\sigma_3^{**} = \frac{(2-s-t)^2}{3(3-s^2-st-t^2)B_1} \left\{ \frac{4B_2}{B_1} + \frac{4B_1(s+t)}{(2-s-t)} \right\}$$

If $\sigma_1^{**} \leq \mu \leq \sigma_3^{**}$, then

$$|a_3 - \mu a_2^2| + \frac{1}{3B_1^2}\left[4(B_1 - B_2)\frac{(2-s-t)^2}{(3-s^2-st-t^2)}\right.$$
$$\left. -B_1^2\frac{(2-s-t)(s+t)}{(3-s^2-st-t^2)} + 3\mu B_1^2\right]|a_2|^2 \leq \frac{B_1}{3(3-s^2-st-t^2)}$$

If $\sigma_3^{**} < \mu \leq \sigma_2^{**}$, *then*

$$|a_3 - \mu a_2^2| + \frac{1}{3B_1^2}\left[4(B_1 + B_2)\frac{(2-s-t)^2}{(3-s^2-st-t^2)}\right.$$
$$\left. +B_1^2\frac{(2-s-t)(s+t)}{(3-s^2-st-t^2)} - 3\mu B_1^2\right]|a_2|^2 \leq \frac{B_1}{3(3-s^2-st-t^2)}$$

where σ_1^{**} and σ_2^{**} are same as defined in Corollary 2.2.6.

Example 2.2.8.

Let $(-1 \leq B < A \leq 1)$. *If* $f(z)$ *given by* (2.1.1) *belongs to* $C[A, B, s, t]$, *for real parameters* s *and* t, *then*

$$|a_3 - \mu a_2^2|$$
$$\leq \frac{A-B}{3|3-s^2-st-t^2|}\begin{cases} \left|-B + \frac{(A-B)(s+t)}{(2-s-t)} - \frac{3\mu(A-B)(3-s^2-st-t^2)}{4(2-s-t)^2}\right|, & \mu \leq \sigma_1{}' \\ 1, & \sigma_1{}' \leq \mu \leq \sigma_2{}' \\ \left|B - \frac{(A-B)(s+t)}{(2-s-t)} + \frac{3\mu(A-B)(3-s^2-st-t^2)}{(2-s-t)^2}\right|, & \mu \geq \sigma_2{}' \end{cases}$$

where

$$\sigma_1{}' = \frac{(2-s-t)^2}{3(3-s^2-st-t^2)}\left[4\left(\frac{s+t}{2-s-t}\right) - 4\left(\frac{B+1}{A-B}\right)\right]$$

$$\sigma_2{}' = \frac{(2-s-t)^2}{3(3-s^2-st-t^2)}\left[4\left(\frac{s+t}{2-s-t}\right)-4\left(\frac{B-1}{A-B}\right)\right]$$

If $\sigma_1^{**} \leq \mu \leq \sigma_2^{**}$, in view of Lemma 2.1.3, Corollary 2.2.6 can be improved.

Remark 2:

For $s = 1$ and $t = -1$ in aforementioned Theorem 2.2.1, Corollaries 2.2.2, 2.2.3, 2.2.5, 2.2.6, 2.2.7 and Example 2.2.4, 2.2.8, we arrive at the results obtained recently by Shanmugham et al. (2006).

2.3. Generalized Classes $S^g(\phi, s, t)$ and $C^g(\phi, s, t)$

Now we define the following class $S^g(\phi, s, t)$ and $T^g(\phi, s, t)$ which are generalizations of the classes $S(\phi, s, t)$ and $C(\phi, s, t)$. For two analytic functions

$f(z) = z + \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=0}^{\infty} g_n z^n$,

their convolution(or Hadamard product) is defined to be the function $(f * g)(z) = z + \sum_{n=0}^{\infty} a_n g_n z^n$.

For a fixed $g(z) \in A$, let $S^g(\phi, s, t)$ be the class of functions $f(z) \in A$ for which $(f * g)(z) \in S(\phi, s, t)$. We also denote by the subclass $T^g(\phi, s, t)$ the subclass of A consisting of all functions $g(z) \in A$ such that $z(f * g)'(z) \in S(\phi, s, t)$ for $f(z) \in A$.

Definition 2.3.1.

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...$ *be univalent starlike function with respect to '1' which maps the unit disk* U *onto a region in the right half plane which is symmetric with respect to the real axis, and let* $B_1 > 0$*. The function* $g \in A$ *is in the class* $S^g(\phi, s, t)$ *for* $f \in A$ *if*

$$\left\{ \frac{(s-t)z(f*g)'(z)}{(f*g)(sz)-(f*g)(tz)} \right\} \prec \phi(z) \qquad s,t \in \mathcal{C}, s \neq t \tag{2.3.1}$$

Again $T^g(\phi, s, t)$ *denotes the subclass of* A *consisting functions* $g(z) \in A$ *such that* $z((f*g)'(z)) \in S^g(\phi, s, t)$ *for* $f(z) \in A$*.*

Obviously $S^g(\phi, 1, t) \equiv S^g(\phi, t)$, which are the classes introduced and studied by Goyal and Goswami (2009).

Definition 2.3.2.

Let $f(z)$ *be analytic in a simply connected region of the z-plane containing origin. The fractional derivative of* $f(z)$ *of order* λ *is defined by*

$$_0D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z (z-\xi)^{-\lambda} f(\xi) d\xi \qquad (0 \leq \lambda < 1) \tag{2.3.2}$$

where the multiplicity of $(z-\xi)^{-\lambda}$ *is removed by requiring that* $log(z-\xi)$ *is real for* $(z-\xi) > 0$*.*

Using Definition 2.3.2, *Owa and Srivastava* (*see* 1978, 1984; *see also*1987, 1989) *introduced a fractional derivative operator* $\Omega^\lambda : A \to A$ *defined by*

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda)z^\lambda \ {}_0D_z^\lambda f(z), \quad (\lambda \neq 2, 3, 4...)$$

The class $S^\lambda(\phi, s, t)$ *consists of the functions* $f(z) \in A$ *for which* $\Omega^\lambda f(z) \in S(\phi, s, t)$. *The class* $S^\lambda(\phi, s, t)$ *is a special case of the class* $S^g(\phi, s, t)$ *when*

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n \quad (z \in U).$$

Now applying Lemma 2.1.2 for the function $(f * g)(z) = z + g_2a_2z^2 + g_3a_3z^3 + ...$, we get following theorem after an obvious change of the parameter μ :

Theorem 2.3.1.

Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n \ (g_n > 0)$. *If* $f(z)$ *is given by* (2.1.1) *belongs to* $S^g(\phi, s, t)$, *then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{g_3|3-s^2-t^2-st|} max\left\{B_1, \left|\frac{B_1^2(s+t)}{2-s-t} + B_2 - \frac{\mu g_3 B_1^2(3-s^2-t^2-st)}{g_2^2(2-s-t)^2}\right|\right\}$$

such that $s, t \in \mathcal{C}, s+t \neq 2$ *and* $s \neq t$

Proof: Let $f * g = L \in S(\phi, s, t)$. Then there exists a