CHAPTER 6

SUFFICIENT CONDITIONS FOR GENERALIZED SAKAGUCHI TYPE FUNCTION OF ORDER β

The main result of the chapter has been published as listed below:

"**Sufficient conditions for generalized Sakaguchi type functions of order β**", TWMS Journal of Applied and Engineering Mathematics, 4(1), 2014, 127-131.

In this chapter, we obtain some sufficient conditions for generalized Sakaguchi type function of order β, defined on the open unit disk U. Many interesting outcomes of our results are also calculated. We have also discussed some special cases of our results based on the early researches.

6.1. Introduction

Let A_n be the class of the form

$$f(z) = z + a_{n+1}z^{n+1} + ... \qquad (6.1.1)$$

that are analytic in the unit disk U and let $A_1 = A$ defined in (1.2.1). An analytic function $f(z) \in A_n$ is said to be in the generalized Sakaguchi class $S_n(\beta, s, t)$ if it satisfies

$$Re\left\{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\} > \beta, \qquad z \in U \qquad (6.1.2)$$

for some $\beta(0 \leq \beta < 1)$, s and t are real parameters, $s > t$ and for all $z \in U$.

For $n = 1$ the generalized Sakaguchi class $S_n(\beta, s, t)$ reduces to the subclass $S(\beta, s, t)$ defined in equation (2.1.14) with real parameters s and t. For $n = 1, s = 1$, this class is reduced to $S(\beta, t)$ studied by Owa et al. (2005, 2007), Goyal and Goswami (2011) and Cho et al.(1993). The class $S(0, -1)$ was introduced by Sakaguchi (1959). For $s = 1,\ t = 0$, it reduces to the class $S_n(\beta, 0) = S_n^*(\beta)$ studied by Ravichandran et al.(2002). For $s = 1, t = 0,\ \beta = 0,\ n = 1$, we get the known class $S_1(0, 0) = S^*$ studied by Li and Owa (2001). For $s = 1, t = 0,\ \beta = \dfrac{\alpha}{2},\ n = 1$, we get the class $S_1(\frac{\alpha}{2}, 0) = S^*(\frac{\alpha}{2})$ studied by Li and Owa (2001).

In this chapter, we obtain some sufficient conditions for functions $f(z) \in S_n(\beta, s, t)$. To prove our results, we need the following lemma by by Miller and Mocanu (1987):

Lemma 6.1.1.

Let Ω be a set in the complex plane $\mathcal{C}$ and suppose that ϕ is a mapping from $\mathcal{C}^2 \times U$ to $\mathcal{C}$ which satisfies $\phi(ix, y; z) \notin \Omega$ for $z \in U$, and for all real x, y such that $y \leq -n(1+x^2)/2$. If the function $p(z) = 1 + c_n z^n + ...$ is analytic in U and $\phi(p(z), zp'(z); z) \in \Omega$ for all $z \in U$, then $Re(p(z)) > 0$.

6.2. Main Results

Theorem 6.2.1.

If $f(z) \in A_n$ satisfies

$$Re\left[\frac{(s-t)^2 zf'(sz)}{f(sz)-f(tz)}\left\{\frac{\alpha szf''(sz)}{f'(sz)} + \frac{\alpha tzf'(tz)}{f(sz)-f(tz)} + 1\right\}\right]$$
$$> \alpha\beta\left\{s\beta + \frac{n}{2}(s-t) - (s-t)\right\} + \left\{\beta - \frac{n\alpha}{2}\right\}(s-t) \quad (6.2.1)$$

$(z \in U, 0 \leq \alpha \leq 1, 0 \leq \beta < 1$ *and* $t < s)$,

then $f(z) \in S_n(\beta, s, t)$.

Proof: Define $p(z)$ by

$$\left\{\frac{(s-t)zf'(sz)}{f(sz)-f(tz)}\right\} = (1-\beta)p(z) + \beta.$$

Then $p(z) = 1 + c_n z^n + ...$ and is analytic in U.

A computation shows that

$$\frac{szf''(sz)}{f'(sz)} + \frac{tzf'(tz)}{f(sz) - f(tz)}$$

$$= \frac{(s-t)(1-\beta)zp'(z) + s\left[(1-\beta)p(z)+\beta\right]^2 - (s-t)\left[(1-\beta)p(z)+\beta\right]}{(s-t)\left[(1-\beta)p(z)+\beta\right]}$$

and hence

$$\frac{(s-t)^2 zf'(sz)}{f(sz)-f(tz)}\left[\frac{\alpha szf''(sz)}{f'(sz)} + \frac{\alpha tzf'(tz)}{f(sz)-f(tz)} + 1\right]$$

$$= \alpha(s-t)(1-\beta)zp'(z) + \alpha s(1-\beta)^2 p^2(z)$$

$$+(1-\beta)[2s\alpha\beta + (s-t)(1-\alpha)]p(z) + \beta[s\alpha\beta + (s-t)(1-\alpha)]$$

$$= \phi(p(z), zp'(z); z) \ \ \text{(say)} \qquad (6.2.2)$$

where

$$\phi(u, v; z) = \alpha(s-t)(1-\beta)v + \alpha s(1-\beta)^2 u^2$$

$$+(1-\beta)[2s\alpha\beta+(s-t)(1-\alpha)]u+\beta[s\alpha\beta+(s-t)(1-\alpha)] \quad (6.2.3)$$

For all real x and y satisfying $y \leq -n(1+x^2)/2$, we have

$$Re[\phi(ix, y; z)] \leq \alpha(s-t)(1-\beta)y - \alpha s(1-\beta)^2 x^2 + \beta[s\alpha\beta+(s-t)(1-\alpha)]$$

$$\leq \alpha(s-t)(1-\beta)\left\{\frac{-(1+x^2)}{2}\right\} - \alpha s(1-\beta)^2 x^2 + \beta[s\alpha\beta+(s-t)(1-\alpha)]$$

$$= \frac{-\alpha n}{2}(s-t)(1-\beta) - \left\{\frac{\alpha n}{2}(s-t)(1-\beta) + \alpha\beta(1-\beta)^2\right\} x^2$$

$$+\beta[s\alpha\beta + (1-\alpha)(s-t)]$$

$$\leq \frac{-\alpha n}{2}(s-t)(1-\beta) + \beta[s\alpha\beta + (1-\alpha)(s-t)]$$

$$= \alpha\beta \left\{ s\beta + \frac{n}{2}(s-t) - (s-t) \right\} + \left\{ \beta - \frac{n\alpha}{2} \right\} (s-t)$$

Let $\Omega = \left\{ w; Re(w) > \alpha\beta \left\{ \beta + \frac{n}{2}(s-t) - (s-t) \right\} + \left\{ \beta - \frac{n\alpha}{2} \right\} (s-t) \right\}$

Then $\phi(p(z), zp'(z); z) \in \Omega$ and $\phi(ix, y; z) \notin \Omega$ for all real x

and $y \leq -n(1+x^2)/2,\ z \in U$.

By an application of Lemma 6.1.1, the result follows.

On putting $s = 1$, in Theorem 6.2.1, we get the known result

due to Goyal et al. (2011)

Corollary 6.2.2.

If $f(z) \in A$ *satisfies*

$$Re \left[\frac{(1-t)^2 z f'(z)}{f(z) - f(tz)} \left\{ \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha t z f'(tz)}{f(z) - f(tz)} + 1 \right\} \right]$$

$$> \alpha\beta \left\{ \beta + \frac{n}{2}(1-t) - (1-t) \right\} + \left\{ \beta - \frac{n\alpha}{2} \right\} (1-t)$$

$(z \in U, 0 \leq \alpha \leq 1, 0 \leq \beta < 1$ *and* $|t| < 1, t \neq 1)$,

then $f(z) \in S_n(\beta, t)$.

On putting $s = 1, t = -1$, in Theorem 6.2.1, we get,

Corollary 6.2.3.

If $f(z) \in A$ *satisfies*

$$Re \left[\frac{z f'(z)}{f(z) - f(tz)} \left\{ \frac{\alpha z f''(z)}{f'(z)} - \frac{\alpha z f'(-z)}{f(z) - f(-z)} + 1 \right\} \right]$$

$$> \frac{\alpha\beta}{4} \{\beta + n - 2\} + \left\{ \frac{2\beta - n\alpha}{4} \right\}$$

$(z \in U, 0 \leq \alpha < 1, 0 \leq \beta < 1)$,

then $f(z) \in S_n(\beta, -1)$.

On putting $s = 1, t = -1, \beta = 0$, in Theorem 6.2.1, we get,

Corollary 6.2.4.

If $f(z) \in A$ *satisfies*

$$Re\left[\frac{zf'(z)}{f(z)-f(-z)}\left\{\frac{\alpha zf''(z)}{f'(z)} - \frac{\alpha zf'(-z)}{f(z)-f(-z)} + 1\right\}\right] > \frac{-n\alpha}{4}$$

$(z \in U, 0 \leq \alpha < 1)$,

then $f(z) \in S_n(0, -1)$.

On putting $s = 1, t = 0$, in Theorem 6.2.1, we get,

Corollary 6.2.5.

If $f(z) \in A$ *satisfies*

$$Re\left[\frac{zf'(z)}{f(z)}\left\{\frac{\alpha zf''(z)}{f'(z)} + 1\right\}\right] > \alpha\beta\left\{\beta + \frac{n}{2} - 1\right\} + \left\{\beta - \frac{n\alpha}{2}\right\}$$

$(z \in U, 0 \leq \alpha < 1, 0 \leq \beta < 1)$,

then $f(z) \in S_n(\beta, 0) = S_n^*(\beta)$.

On putting $s = 1, t = 0, n = 1, \beta = 0$, in Theorem 6.2.1, we get,

Corollary 6.2.6.

If $f(z) \in A$ *satisfies*

$$Re\left[\frac{zf'(z)}{f(z)}\left\{\frac{\alpha zf''(z)}{f'(z)} + 1\right\}\right] > \frac{-\alpha}{2}$$

$(z \in U, 0 \leq \alpha \leq 1)$,

then $f(z) \in S_1(0,0) = S^*$.

If we take $\beta = \frac{\alpha}{2}$ and $n = 1$, in above relation we get the following result:

Corollary 6.2.7.

If $f(z) \in A$ *satisfies*

$$Re\left[\frac{zf'(z)}{f(z)}\left\{\frac{\alpha z f''(z)}{f'(z)}+1\right\}\right] > \frac{-\alpha^2}{4}(1-\alpha)$$

$(z \in U, 0 \leq \alpha < 2)$,

then $f(z) \in S_1(\frac{\alpha}{2}, 0) = S^*(\frac{\alpha}{2})$.

Theorem 6.2.8.

Let $0 \leq \beta < 1,\ t < s$ *with* $-1 \leq \frac{t}{s} + \beta < 1$,

$$\lambda = (1-\beta)^2\left\{\frac{n}{2}(s-t) + s(1-\beta)\right\}^2,$$

$$\mu = \left\{\frac{n}{2}(s-t)(1-\beta) + \beta|(s-t-s\beta)|\right\}^2,$$

$$\nu = \left\{s(1-\beta)^2 + \beta(s-t-s\beta)\right\}^2$$

and

$$\sigma = \{(1-\beta)(2s\beta + t - s)\}^2 \qquad (6.2.4)$$

satisfies $(\lambda + \mu - \nu + \sigma)\beta^2 < (1-2\beta)\mu$.

Also suppose that r_0 *be the positive real root of the equation*

$$2\lambda(1-\beta)^2r^3+\left\{(1-\beta)^2(2\lambda+\mu-\nu+\sigma)+3\lambda\beta^2\right\}r^2$$

$$+2\beta^2(2\lambda+\mu-\nu+\sigma)r+(\lambda+2\mu-\nu+\sigma)\beta^2-(1-\beta)^2\mu=0 \tag{6.2.5}$$

and

$$\rho^2=\frac{(1-\beta)^2(1+r_0)}{(s-t)^2\left\{(1-\beta)^2r_0+\beta^2\right\}}[\lambda r_0^2+(\lambda+\mu-\nu+\sigma)r_0+\mu] \tag{6.2.6}$$

Now if $f(z)\in A_n$ *satisfies*

$$\left|\left(\frac{(s-t)zf'(sz)}{f(sz)-f(tz)}-1\right)\left(\frac{szf''(sz)}{f'(sz)}+\frac{tzf'(tz)}{f(sz)-f(tz)}\right)\right|\le\rho$$

then $f(z)\in S_n(\beta,s,t)$.

Proof: Define $p(z)$ by

$$\left\{\frac{(s-t)zf'(sz)}{f(sz)-f(tz)}\right\}=(1-\beta)p(z)+\beta.$$

Then $p(z)=1+c_nz^n+.....$ and is analytic in U.

A computation shows that

$$\frac{szf''(sz)}{f'(sz)}+\frac{tzf'(tz)}{f(sz)-f(tz)}$$

$$=\frac{(s-t)(1-\beta)zp'(z)+s[(1-\beta)p(z)+\beta]^2-(s-t)[(1-\beta)p(z)+\beta]}{(s-t)[(1-\beta)p(z)+\beta]}$$

and hence

$$\left(\frac{(s-t)zf'(sz)}{f(sz)-f(tz)}-1\right)\left(\frac{szf''(sz)}{f'(sz)}+\frac{tzf'(tz)}{f(sz)-f(tz)}\right)$$

$$= \frac{(1-\beta)(p(z)-1)}{(s-t)[(1-\beta)p(z)+\beta]}\left\{(s-t)(1-\beta)zp'(z)\right.$$

$$\left.+s[(1-\beta)p(z)+\beta]^2-(s-t)[(1-\beta)p(z)+\beta]\right\}$$

$$= \phi(p(z), zp'(z); z)$$

Then for all real x and y satisfying $y \leq -n(1+x^2)/2$, we have

$$|\phi(ix, y; z)|^2 = \frac{(1-\beta)^2(1+x^2)}{(s-t)^2[(1-\beta)^2x^2+\beta^2]}$$

$$\times \left[(s-t)(1-\beta)y - s(1-\beta)^2x^2 - \beta(s-t-s\beta)\right]^2$$

$$+(1-\beta)^2[2s\beta-(s-t)]^2x^2$$

$$= \frac{(1-\beta)^2(1+r)}{(s-t)^2[(1-\beta)^2r+\beta^2]}$$

$$\times \left[(s-t)(1-\beta)y - s(1-\beta)^2r - \beta(s-t-s\beta)\right]^2$$

$$+(1-\beta)^2[2s\beta-(s-t)]^2r = g(r, y)$$

where $r = x^2 > 0$ and $y \leq -n(1+x^2)/2$

Since

$$\frac{\partial g}{\partial y} = \frac{2(1-\beta)^3(1+r)}{(s-t)[(1-\beta)^2r+\beta^2]}\left\{(s-t)(1-\beta)y\right.$$

$$\left.-\beta(s-t-s\beta) - s(1-\beta)^2r\right\} < 0$$

therefore we have

$$h(r) = g\left[r, -n(1+r)/2\right] \leq g(r, y),$$

where

$$h(r) = \frac{(1-\beta)^2(1+r)}{(s-t)^2[(1-\beta)^2 r + \beta^2]}[\lambda r^2 + (\lambda+\mu-\nu+\sigma)r + \mu] \tag{6.2.7}$$

where λ, μ, ν, and σ are given in (6.2.4).

Now differentiating (6.2.7) and using $h'(r) = 0$, we get

$$2\lambda(1-\beta)^2 r^3 + \left\{(1-\beta)^2(2\lambda+\mu-\nu+\sigma) + 3\lambda\beta^2\right\} r^2$$

$$+2\beta^2(2\lambda+\mu-\nu+\sigma)r + (\lambda+2\mu-\nu+\sigma)\beta^2 - (1-\beta)^2\mu = 0$$

which is a cubic equation in r. Since r_0 is the positive real root of this equation we have $h(r) \geq h(r_0)$ and hence

$$|\phi(ix, y; z)|^2 \geq h(r_0) = \rho^2.$$

Define $\Omega = \{w; |w| < \rho\}$, then $\phi(p(z), zp'(z); z) \in \Omega$ for all real x and $y \leq -n(1+x^2)/2,\ z \in U$. Therefore by an application of Lemma 6.1.1. the result follows.

By taking $s = 1$ in Theorem 6.2.8 we get the known results of Goyal et al. (2011)

Corollary 6.2.9.

Let $0 \leq \beta < 1, |t| \leq 1, t \neq 1$ *with* $-1 \leq t+\beta < 1$,

$$\lambda_1 = (1-\beta)^2 \left\{\frac{n}{2}(1-t) + (1-\beta\right\}^2,$$

$$\mu_1 = \left\{\frac{n}{2}(1-t)(1-\beta) + \beta(1-t-\beta)\right\}^2,$$

$\nu_1 = \left\{(1-\beta)^2 - \beta(1-t-\beta)\right\}^2$ *and* $\sigma_1 = \{(1-\beta)(2\beta+t-1)\}^2$

satisfies $(\lambda_1 + \mu_1 - \nu_1 + \sigma_1)\beta^2 < (1-2\beta)\mu_1$.

Also suppose that r_1 *be the positive real root of the equation*

$$2\lambda_1(1-\beta)^2 r^3 + \left\{(1-\beta)^2(2\lambda_1 + \mu_1 - \nu_1 + \sigma_1) + 3\lambda_1\beta^2\right\} r^2$$

$$+2\beta^2(2\lambda_1+\mu_1-\nu_1+\sigma_1)r + (\lambda_1+2\mu_1-\nu_1+\sigma_1)\beta^2 - (1-\beta)^2\mu_1 = 0$$

and

$$\rho_1^2 = \frac{(1-\beta)^2(1+r_1)}{(1-t)^2\left\{(1-\beta)^2 r_1 + \beta^2\right\}}[\lambda_1 r_1^2 + (\lambda_1+\mu_1-\nu_1+\sigma_1)r_1 + \mu_1]$$

Now if $f(z) \in A_n$ *satisfies*

$$\left|\left(\frac{1-t)zf'(z)}{f(z)-f(tz)} - 1\right)\left(\frac{zf''(z)}{f'(z)} + \frac{tzf'(tz)}{f(z)-f(tz)}\right)\right| \leq \rho_1 \quad (z \in U)$$

then $f(z) \in S_n(\beta, t)$.

If $s = 1, t = -1$, in Theorem 6.2.8 we get

Corollary 6.2.10.

Let $0 \leq \beta < 1$,

$$\lambda_2 = (1-\beta)^2\{n + (1-\beta\}^2,\ \mu_2 = \{n(1-\beta) + \beta(2-\beta)\}^2,$$

$$\nu_2 = \left\{(1-\beta)^2 - \beta(2-\beta)\right\}^2 \text{ and } \sigma_2 = \left\{2(1-\beta)^2\right\}^2$$

satisfies $(\lambda_2 + \mu_2 - \nu_2 + \sigma_2)\beta^2 < (1-2\beta)\mu_2$.

Also suppose that r_2 *be the positive real root of the equation*

$$2\lambda_3(1-\beta)^2 r^3 + \left\{(1-\beta)^2(2\lambda_2 + \mu_2 - \nu_2 + \sigma_2) + 3\lambda_2\beta^2\right\} r^2$$

$$+2\beta^2(2\lambda_3+\mu_3-\nu_3+\sigma_3)r+(\lambda_2+2\mu_2-\nu_2+\sigma_2)\beta^2-(1-\beta)^2\mu_2=0$$

and

$$\rho_2^2=\frac{(1-\beta)^2(1+r_2)}{4\left\{(1-\beta)^2r_2+\beta^2\right\}}[\lambda_1 r_2^2+(\lambda_2+\mu_2-\nu_2+\sigma_2)r_2+\mu_2]$$

Now if $f(z)\in A_n$ *satisfies*

$$\left|\left(\frac{2zf'(z)}{f(z)-f(-z)}-1\right)\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(-z)}{f(z)-f(-z)}\right)\right|\leq\rho_2 \quad (z\in U)$$

then $f(z)\in S_n(\beta,-1)$.

If $s=1, t=-1, \beta=0$ in Theorem 6.2.8 we get

Corollary 6.2.11.

Suppose that r_3 *be the positive real root of the equation*

$$2(n+1)^2r^3+(3n^2+4n+5)r^2-n^2=0$$

and

$$\rho_3^2=\frac{(1+r_3)}{4r_3}\left[(n+1)^2r_3^2+2(n^2+n+2)r_3+n^2)\right]$$

Now if $f(z)\in A_n$ *satisfies*

$$\left|\left(\frac{2zf'(z)}{f(z)-f(-z)}-1\right)\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(-z)}{f(z)-f(-z)}\right)\right|\leq\rho_3 \quad (z\in U)$$

then $f(z)\in S_n(0,-1)$.

By taking $n=1$ in Corollary 6.2.11, we have $r_4=0.266048...$,

thus we have the following result

Corollary 6.2.12

Now if $f(z) \in A$ satisfies

$$\left|\left(\frac{2zf'(z)}{f(z)-f(-z)}-1\right)\left(\frac{zf''(z)}{f'(z)}-\frac{zf'(-z)}{f(z)-f(-z)}\right)\right| \leq \rho_4 \quad (z \in U)$$

where $\rho_4 = 2.0145979...$, *then* $f(z) \in S_n(0,-1)$.

For $s = 1$ and $t = 0$ in Theorem 6.2.2 gives the known results due to Ravichandran et al. (2002) and for $n = 1, \beta = 0, t = 0$, our Theorem 6.2.2 reduces to another known result of Li and Owa. (2002)

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Epilogue

Just to confirm the old saying that there is nothing new under the sun, I offer the following epilogue.

Complex Analysis leads in many different directions,and there are a number of interesting and important facets that we have not even touched upon. This thesis brings together many contributions and aspects of geometric function theory and fractional calculus. We have studied a lot about Sakaguchi classes with real and complex parameters and a lot of characteristics are yet to be defined.

Our further research may include the characterization of Sakaguchi class with Differintegral operator.

Having written this text, I can only hope that this work will be considered useful by many people working with complex functions.The playground for experimentation is endless so many interesting functions are out there waiting to be discovered.