CHAPTER 5

COFFICIENT INEQUALITIES OF MULTIVALENT FUNCTIONS AND CHARACTERIZATION AND SUBORDINATION PROPERTIES FOR λ- SPIRALLIKE GENERALIZED SAKAGUCHI TYPE FUNCTIONS

The main results of the chapter have been published as listed below:

1. "Coefficient inequalities for generalized sakaguchi type multivalent functions", International Journal of Mathematical Archive, 3(7), 2012, 2558-2562.

2. "Characterization and subordination properties for λ-spirallike generalized Sakaguchi type functions", Palestine Journal of Mathematics, 3(1), 2014, 70-76.

This chapter is divided in two sections.

In section A of this chapter we shall study two subclasses $S_p(\alpha, s, t)$ and $C_p(\alpha, s, t)$ concerning with generalized Sakaguchi type functions in the open unit disc U. Further by using the coefficient inequalities for the classes $S_p(\alpha, s, t)$ and $C_p(\alpha, s, t)$, two new subclasses $S_p^0(\alpha, s, t)$ and $C_p^0(\alpha, s, t)$ are defined. Some properties of functions belonging to the subclasses $S_p^0(\alpha, s, t)$ and $C_p^0(\alpha, s, t)$ are also discussed.

In Section B we shall introduce and study subclasses $R^\lambda(\alpha, s, t)$ and $T^\lambda(\alpha, s, t)$ of the class of λ- spirallike generalized Sakaguchi type function. Here we shall prove characterization and subordination properties for these subclasses and point out several interesting consequences of our results.

Section A

5.1. Introduction

Let A_p be the class of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad p = 1, 2, 3, 4..... \qquad (5.1.1)$$

that are analytic in the open unit disk U. A function $f(z) \in A_p$ is said to be in the class $S_p(\alpha, s, t)$ if it satisfies

$$\text{Re}\left\{\frac{(s-t)zf'(z)}{f(sz) - f(tz)}\right\} > \alpha \qquad (5.1.2)$$

for some $\alpha(0 \leq \alpha < 1)$, $s, t \in \mathcal{C}, s \neq t$ and for all $z \in U$. For $p = 1$, the class $S_1(\alpha, s, t) = S(\alpha, s, t)$ was introduced by Frasin (2010) defined in (1.2.14) and for $s = 1$, the class $S_p(\alpha, 1, t) = S_p(\alpha, t)$ was introduced and studied by Goyal et al.(2012).

For $p = 1$ and $s = 1$, the class $S_1(\alpha, 1, t) = S(\alpha, 1, t)$ was introduced and studied by Owa et al.(2007), and by taking $t = -1$, the class $S_1(\alpha, 1, -1) = S_s(\alpha)$ was introduced by Sakaguchi (1993) and is called Sakaguchi function of order α, studied by Owa et al. (2005), where $S_s(0) = S_s$ is the class of starlike functions with respect to symmetrical points in U. We also denote by $C_p(\alpha, s, t)$ the subclass of A_p consisting of all functions $f(z)$ such that $zf'(z) \in S_p(\alpha, s, t)$. Also, we note that $S_1(\alpha, 1, 0) = S^*(\alpha)$ and $C_1(\alpha, 1, 0) = C(\alpha)$ which are, respectively, the familiar classes of starlike functions of order $\alpha(0 \leq \alpha < 1)$ and convex functions of order $\alpha(0 \leq \alpha < 1)$.

5.2. Main Results

We first prove the following two theorems which are similar to the results due to Cho et al.(1993) and Owa et al.(2007).

Theorem 5.2.1.

If $f(z) \in A_p$ satisfies

$$\sum_{n=1}^{\infty} [|p + n - u_{p+n}| + (1 - \alpha) |u_{p+n}|] |a_{p+n}| \leq p - \alpha |u_p| \tag{5.2.1}$$

for some $\alpha(0 \leq \alpha < 1)$, then $f(z) \in S_p(\alpha, s, t)$, where

$$u_p = \sum_{j=1}^{p} s^{p-j} t^{j-1} \tag{5.2.2}$$

Proof: To prove Theorem 5.2.1, we show that if $f(z)$ satisfies (5.2.1) then

$$\left| \frac{(s - t)zf'(z)}{f(sz) - f(tz)} - 1 \right| < 1 - \alpha$$

Evidently, since

$$\frac{(s - t)(zf'(z))}{f(sz) - f(tz)} - 1$$

$$= \frac{(s-t)\{pz^p + \sum_{n=1}^{\infty} (p+n)a_{p+n}z^{p+n}\}}{\{(sz)^p + \sum_{n=1}^{\infty} a_{p+n}(sz)^{p+n}\} - \{(tz)^p + \sum_{n=1}^{\infty} a_{p+n}(tz)^{p+n}\}} - 1$$

$$= \frac{(p-u_p)z^p + \sum_{n=1}^{\infty} (p+n-u_{p+n})a_{p+n}z^{p+n}}{u_p z^p + \sum_{n=1}^{\infty} u_{p+n}a_{p+n}z^{p+n}}$$

we see that

$$\left| \frac{(s-t)(zf'(z))}{f(sz)-f(tz)} - 1 \right|$$

$$\leq \frac{|p-u_p| + \sum_{n=1}^{\infty} |p+n-u_{p+n}|\,|a_{p+n}|}{|u_p| - \sum_{n=1}^{\infty} |u_{p+n}|\,|a_{p+n}|}$$

Therefore, if $f(z)$ satisfies (5.2.1), then we have

$$\frac{|p-u_p| + \sum_{n=1}^{\infty} |p+n-u_{p+n}|\,|a_{p+n}|}{|u_p| - \sum_{n=1}^{\infty} |u_{p+n}|\,|a_{p+n}|} \leq 1-\alpha$$

This completes the proof of Theorem 5.2.1.

Theorem 5.2.2.

If $f(z) \in A_p$ satisfies

$$\sum_{n=1}^{\infty} (p+n)\left[|p+n-u_{p+n}| + (1-\alpha)\,|u_{p+n}|\right]|a_{p+n}| \leq p^2 - \alpha p\,|u_p| \tag{5.2.3}$$

for some $\alpha (0 \leq \alpha < 1)$, then $f(z) \in C_p(\alpha, s, t)$.

Proof: Noting that $f(z) \in C_p(\alpha, s, t)$ if and only if

$zf'(z) \in S_p(\alpha, s, t)$, we can prove Theorem 5.2.2.

$$\left| \frac{(s-t)z[zf'(z)]'}{(sz)f'(sz) - (tz)f'(tz)} - 1 \right| < 1 - \alpha$$

Evidently, since

$$\frac{(s-t)(z[zf'(z)]')}{(sz)f'(sz) - (tz)f'(tz)} - 1$$

$$= \frac{(s-t)\{p^2 z^p + \sum\limits_{n=1}^{\infty} (p+n)^2 a_{p+n} z^{p+n}\}}{\mu - r} - 1$$

where

$$\mu = \{p(sz)^p + \sum_{n=1}^{\infty} (p+n) a_{p+n} (sz)^{p+n}\}$$

and

$$r = \{p(tz)^p + \sum_{n=1}^{\infty} (p+n) a_{p+n} (tz)^{p+n}\}$$

$$= \frac{(p^2 - pu_p)z^p + \sum\limits_{n=1}^{\infty} \left\{(p+n)^2 - (p+n)u_{p+n}\right\} a_{p+n} z^{p+n}}{pu_p z^p + \sum\limits_{n=1}^{\infty} (p+n) u_{p+n} a_{p+n} z^{p+n}}$$

we see that

$$\left| \frac{(s-t)(z[f'(z) + zf''(z)])}{(sz)f'(sz) - (tz)f'(tz)} - 1 \right|$$

$$\leq \frac{\left|p^2 - pu_p\right| + \sum\limits_{n=1}^{\infty} \left|(p+n)^2 - (p+n)u_{p+n}\right| \left|a_{p+n}\right|}{\left|pu_p\right| - \sum\limits_{n=1}^{\infty} \left|(p+n)u_{p+n}\right| \left|a_{p+n}\right|}$$

Therefore, if $f(z)$ satisfies (5.2.3), then we have

$$\frac{\left|p^2 - pu_p\right| + \sum\limits_{n=1}^{\infty} \left|(p+n)^2 - (p+n)u_{p+n}\right| |a_{p+n}|}{|pu_p| - \sum\limits_{n=1}^{\infty} |(p+n)u_{p+n}| \, |a_{p+n}|} \leq 1 - \alpha$$

This completes the proof of Theorem 5.2.2.

5.3. Cofficient Inequalities For Subclasses $S_p(\alpha, s, t)$ and $C_p(\alpha, s, t)$

Applying Caratheodry function $\phi(z)$ defined by

$$\phi(z) = 1 + \sum_{n=1}^{\infty} \phi_n z^n \tag{5.3.1}$$

in U, we discuss the coefficient inequalities for functions $f(z)$ in $S_p(\alpha, s, t)$ and $C_p(\alpha, s, t)$.

Theorem 5.3.1.

If $f(z) \in S_p(\alpha, s, t)$, then

$$|a_{p+n}| \leq \frac{\beta\,|u_p|}{|v_{p+n}|} \left\{ 1 + \beta \sum_{j=p+1}^{p+n-1} \frac{|u_j|}{|v_j|} + \beta^2 \sum_{j_2 > j_1}^{p+n-1} \sum_{j_1=p+1}^{p+n-2} \frac{|u_{j_1} u_{j_2}|}{|v_{j_1} v_{j_2}|} + \right.$$

$$\left. ..... + \beta^{n-1} \prod_{j=p+1}^{p+n-1} \frac{|u_j|}{|v_j|} \right\} \tag{5.3.2}$$

where

$$\beta = 2|(p - \alpha u_p)|, \qquad v_n = n u_p - p u_n \tag{5.3.3}$$

for some $\alpha (0 \leq \alpha < 1)$, $s, t \in \mathcal{C}, s \neq t$

Proof: We define the function $\phi(z)$ by

$$\phi(z) = \frac{u_p}{p - \alpha u_p}\left(\frac{(s-t)zf'(z)}{f(sz) - f(tz)} - \alpha\right) = 1 + \sum_{n=1}^{\infty} \phi_n z^n \quad (5.3.4)$$

for $f(z) \in S_p(\alpha, s, t)$. Then $\phi(z)$ is a Caratheodry function and satisfies

$$|\phi_n| \leq 2 \qquad (n \geq 1) \qquad (5.3.5)$$

Since

$$(s-t)zf'(z) = \{f(sz) - f(tz)\}\{\alpha + \frac{(p - \alpha u_p)}{u_p}\phi(z)\}$$

we have

$$(s-t)z^p\left[p + \sum_{n=1}^{\infty}(p+n)a_{p+n}z^n\right]$$

$$= z^p\left[(s^p - t^p) + \sum_{n=1}^{\infty}(s^{p+n} - t^{p+n})a_{p+n}z^n\right]$$

$$\times \left[\frac{p}{u_p} + \frac{(p - \alpha u_p)}{u_p}\sum_{n=1}^{\infty}\phi_n z^n\right]$$

So we have

$$u_p\left[p + \sum_{n=1}^{\infty}(p+n)a_{p+n}z^n\right]$$

$$= \left[u_p + \sum_{n=1}^{\infty} u_{p+n}a_{p+n}z^n\right]\left[p + (p - \alpha u_p)\sum_{n=1}^{\infty}\phi_n z^n\right]$$

So we get

$$a_{p+n} = \frac{(p-\alpha u_p)}{\{(p+n)u_p - pu_{p+n}\}}[u_{p+n-1}a_{p+n-1}\phi_1 + a_{p+n-2}u_{p+n-2}\phi_2 + ...$$

$$..... + u_{p+1}a_{p+1}\phi_{n-1} + u_p\phi_n]$$

$$(5.3.6)$$

From equation (5.3.6), we easily have that

$$|a_{p+1}| = \left|\frac{(p-\alpha u_p)}{\{u_p(p+1) - pu_{p+1}\}}\phi_1 u_p\right|$$

$$\leq 2|(p-\alpha u_p)|\left[\frac{|\ u_p\ |}{|\ u_p(p+1) - pu_{p+1}\ |}\right]$$

$$|a_{p+2}| = \left|\frac{(p-\alpha u_p)}{\{u_p(p+2) - pu_{p+2}\}}[\phi_2 u_p + \phi_1 u_{p+1}a_{p+1}]\right|$$

$$\leq \frac{2|(p-\alpha u_p)|}{|(p+2)u_p - pu_{p+2}|}\ |\ u_p\ |\ \left[1 + 2|(p-\alpha u_p)|\frac{|\ u_{p+1}\ |}{|\ (p+1)u_p - pu_{p+1}\ |}\right]$$

$$|\ a_{p+3}\ | \leq \frac{2|(p-\alpha u_p)|\ |\ u_p\ |}{|\ (p+3)u_p - pu_{p+3}\ |}\left[1 + \frac{2|(p-\alpha u_p)|\ |\ u_{p+1}\ |}{|\ (p+1)u_p - pu_{p+1}\ |}\right.$$

$$\left. + \frac{2|(p-\alpha u_p)|\ |\ u_{p+2}\ |}{|\ (p+2)u_p - pu_{p+2}\ |} + \frac{2^2|(p-\alpha u_p)|^2\ |\ u_{p+1}\ ||\ u_{p+2}\ |}{|\ (p+1)u_p - pu_{p+1}\ ||\ (p+2)u_p - pu_{p+2}\ |}\right]$$

Thus, using mathematical induction, we obtain the inequality

(5.3.2).

Equality in Theorem 5.3.1 hold for the function $f(z)$ defined by

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = 1 + \frac{(1-2\alpha)z}{1-z} \tag{5.3.7}$$

Special Cases:

If we put $s = 1,\ p = 1, \alpha = 0, t = 0$ in Theorem 5.3.1, then we have well known result

$f(z) \in S^* \Longrightarrow | a_n | \leq n$ where S^* is usual starlike class and if we put $s = 1, p = 1, \alpha = 0, t = -1$ in Theorem 5.3.1, then we have the result due to Sakaguchi (1959)

$f(z) \in S_s \Longrightarrow | a_n | \leq 1$, where S_s is Sakaguchi function class.

For functions in class $C_p(\alpha, s, t)$, similarly we have,

Theorem 5.3.2.

If $f(z) \in C_p(\alpha, s, t)$, *then*

$$|a_{p+n}| \leq \frac{\beta p\,|u_p|}{(p+n)|v_{p+n}|}\left\{1 + \beta \sum_{j=p+1}^{p+n-1} \frac{|u_j|}{|v_j|} + \beta^2 \sum_{j_2>j_1}^{p+n-1} \sum_{j_1=p+1}^{p+n-2} \frac{|u_{j_1}u_{j_2}|}{|v_{j_1}v_{j_2}|} + \right.$$

$$\left. \ldots.. + \beta^{n-2} \prod_{j=p+1}^{p+n-1} \frac{|u_j|}{|v_j|}\right\} \tag{5.3.8}$$

Proof: We define the function $\phi(z)$ by

$$\phi(z) = \frac{u_p}{p-\alpha u_p}\left(\frac{(s-t)z[zf'(z)]'}{(sz)f'(sz)-(tz)f'(tz)} - \alpha\right) = 1 + \sum_{n=1}^{\infty} \phi_n z^n \tag{5.3.9}$$

for $zf'(z) \in S_p(\alpha, s, t)$. Then $\phi(z)$ is a Caratheodry function and satisfies (5.3.5). Since

$$(s-t)z[zf'(z)]' = \{(sz)f'(sz)-(tz)f'(tz)\}\{\alpha+\frac{(p-\alpha u_p)}{u_p}\phi(z)\}$$

we have

$$(s-t)z^p\left[p^2+\sum_{n=1}^{\infty}(p+n)^2 a_{p+n}z^n\right]$$

$$= z^p\left[p(s^p-t^p)+\sum_{n=1}^{\infty}(p+n)(s^{p+n}-t^{p+n})a_{p+n}z^n\right]$$
$$\times\left[\frac{p}{u_p}+\frac{(p-\alpha u_p)}{u_p}\sum_{n=1}^{\infty}\phi_n z^n\right]$$

and

$$u_p\left[p^2+\sum_{n=1}^{\infty}(p+n)^2 a_{p+n}z^n\right]$$
$$=\left[pu_p+\sum_{n=1}^{\infty}(p+n)u_{p+n}a_{p+n}z^n\right]\left[p+(p-\alpha u_p)\sum_{n=1}^{\infty}\phi_n z^n\right]$$

So we get

$$a_{p+n}$$
$$=\frac{(p-\alpha u_p)}{(p+n)\{(p+n)u_p-pu_{p+n}\}}[u_{p+n-1}(p+n-1)a_{p+n-1}\phi_1$$
$$+(p+n-2)a_{p+n-2}u_{p+n-2}\phi_2+.....+(p+1)u_{p+1}a_{p+1}\phi_{n-1}+pu_p\phi_n] \tag{5.3.10}$$

From equation (5.3.10), we easily have that

$$|a_{p+1}| = \left|\frac{p(p-\alpha u_p)}{(u_p(p+1)^2-p(p+1)u_{p+1})}\phi_1 u_p\right|$$

$$\leq 2p|(p-\alpha u_p)|\left[\frac{|\, u_p\,|}{(p+1)\mid u_p(p+1)-pu_{p+1}\mid}\right]$$

$$|a_{p+2}|=\left|\frac{(p-\alpha u_p)}{(p+2)(u_p(p+2)-pu_{p+2})}[\phi_2 pu_p+\phi_1(p+1)u_{p+1}a_{p+1}]\right|$$

$$\leq \frac{2|(p-\alpha u_p)|}{|(p+2)(p+2)u_p-pu_{p+2}|}p\mid u_p\mid\left[1+\frac{2|(p-\alpha u_p)||\, u_{p+1}\mid}{\mid(p+1)u_p-pu_{p+1}\mid}\right]$$

$$\mid a_{p+3}\mid\leq \frac{2|(p-\alpha u_p)|p\mid u_p\mid}{(p+3)\mid(p+3)u_p-pu_{p+3}\mid}\left[1+\frac{2|(p-\alpha u_p)|\mid u_{p+1}\mid}{\mid(p+1)u_p-pu_{p+1}\mid}\right.$$

$$\left.+\frac{2|(p-\alpha u_p)|\mid u_{p+2}\mid}{\mid(p+2)u_p-pu_{p+2}\mid}+\frac{2^2|(p-\alpha u_p)|^2\mid u_{p+1}\mid\mid u_{p+2}\mid}{\mid(p+1)u_p-pu_{p+1}\mid\mid(p+2)u_p-pu_{p+2}\mid}\right]$$

Thus, using mathematical induction, we obtain the inequality (5.3.8).

We now define following two new subclasses of functions

$f(z)\in A_p$

$S_p^0(\alpha,s,t)=\{f(z)\in A_p$ such that $f(z)$ satisfies **(5.2.1)**$\}$

$C_p^0(\alpha,s,t)=\{f(z)\in A_p$ such that $f(z)$ satisfies **(5.2.3)**$\}$

and discuss certain geometrical properties associated with these subclasses.

5.4. Distortion Inequalities for Subclasses $S_p^0(\alpha, s, t)$ and $C_p^0(\alpha, s, t)$

For functions $f(z)$ in the classes $S_p^0(\alpha, s, t)$ and $C_p^0(\alpha, s, t)$, we derive

Theorem 5.4.1

If $f(z) \in S_p^0(\alpha, s, t)$, *then*

$$|z|^p - \sum_{n=1}^{j} |a_{p+n}||z|^{p+n} - A_j |z|^{p+j+1} \leq |f(z)|$$

$$\leq |z|^p + \sum_{n=1}^{j} |a_{p+n}||z|^{p+n} + A_j |z|^{p+j+1} \tag{5.4.1}$$

where

$$A_j = \frac{p - \alpha\,|u_p| - \sum\limits_{n=1}^{j} \left[|p+n-u_{p+n}| + (1-\alpha)\,|u_{p+n}|\right] |a_{p+n}|}{p + j + 1 - \alpha\,|u_{p+n}|}$$

$$(j \geq 1) \tag{5.4.2}$$

Proof: From the inequality (5.2.1), we have

$$\sum_{n=j+1}^{\infty} \left[|p+n-u_{p+n}| + (1-\alpha)\,|u_{p+n}|\right] |a_{p+n}|$$

$$\leq p - \alpha\,|u_p| - \sum_{n=1}^{j} \left[|p+n-u_{p+n}| + (1-\alpha)\,|u_{p+n}|\right] |a_{p+n}|$$

On the other hand we know that,

$$|p+n-u_{p+n}| + (1-\alpha)\,|u_{p+n}| \geq p + n - \alpha\,|u_{p+n}|$$

and hence $p+n-\alpha\left|u_{p+n}\right|$ is monotonically increasing with respect to n. Thus we deduce

$$p+j+1-\alpha\left|u_{p+n}\right|\sum_{n=j+1}^{\infty}\left|a_{p+n}\right|$$

$$\leq p-\alpha\left|u_p\right|-\sum_{n=1}^{j}\left[\left|p+n-u_{p+n}\right|+(1-\alpha)\left|u_{p+n}\right|\right]\left|a_{p+n}\right|$$

which implies that

$$\sum_{n=j+1}^{\infty}\left|a_{p+n}\right|\leq A_j \tag{5.4.3}$$

Therefore we have that

$$|f(z)|\leq|z|^p+\sum_{n=1}^{j}|a_{p+n}||z|^{p+n}+A_j|z|^{p+j+1}$$

and

$$|f(z)|\geq|z|^p-\sum_{n=1}^{j}|a_{p+n}||z|^{p+n}-A_j|z|^{p+j+1}$$

This completes the proof .

For functions in the class $C_p^0(\alpha,s,t)$, similarly we have,

Theorem 5.4.2.

If $f(z)\in C_p^0(\alpha,s,t),$ *then*

$$|z|^p-\sum_{n=1}^{j}|a_{p+n}||z|^{p+n}-B_j|z|^{p+j+1}\leq|f(z)|$$

$$\leq|z|^p+\sum_{n=1}^{j}|a_{p+n}||z|^{p+n}+B_j|z|^{p+j+1} \tag{5.4.4}$$

and

$$p|z|^{p-1} - \sum_{n=1}^{j} (p+n)\,|a_{p+n}||z|^{p+n-1} - C_j|z|^{p+j} \leq \left|f'(z)\right|$$

$$\leq p|z|^{p-1} + \sum_{n=1}^{j} (p+n)\,|a_{p+n}||z|^{p+n-1} + C_j|z|^{p+j} \qquad (5.4.5)$$

where

$$B_j = \frac{p^2 - \alpha p\,|u_p| - \sum_{n=1}^{j} (p+n)\left[|p+n-u_{p+n}| + (1-\alpha)\,|u_{p+n}|\right]|a_{p+n}|}{(p+j+1)(p+j+1-\alpha\,|u_{p+n}|)} \qquad (5.4.6)$$

and

$$C_j = \frac{p^2 - \alpha p\,|u_p| - \sum_{n=1}^{j} (p+n)\left[|p+n-u_{p+n}| + (1-\alpha)\,|u_{p+n}|\right]|a_{p+n}|}{(p+j+1-\alpha\,|u_{p+n}|)} \qquad (5.4.7)$$

Section B

5.5. Introduction

Let A be the class of the form defined in (1.2.1) that are analytic in the unit disc U. An analytic function $f(z) \in A$ is said to be in the generalized Sakaguchi class $S(\alpha, s, t)$ defined by (1.2.14). A function $f(z)$ is said to be in the class $S_p(\lambda)$ if it satisfies the condition

$$Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\} > 0, \qquad (|\lambda| < \frac{\pi}{2}) \tag{5.5.2}$$

Špaěck (1932) proved that members of $S_p(\lambda)$ known as λ- spirallike functions are univalent in the unit disc U. Silverman (1989), Singh (2000) and several others have discussed various properties for spirallike functions.

Recently Goyal and Goswami (2012) have introduced and studied subclasses $P^\lambda(\alpha, t)$ and $M^\lambda(\alpha, t)$ of the classes of λ-spirallike functions. A function $f(z) \in A$ is said to be in the class $P^\lambda(\alpha, t)$ if it satisfies

$$Re\left\{\frac{e^{i\lambda}(1-t)zf'(z)}{f(z)-f(tz)}\right\} > \alpha\cos\lambda$$

If $zf^{'}(z) \in P^\lambda(\alpha, t)$ then $f(z) \in M^\lambda(\alpha, t)$.

Now we introduce a subclass $R^\lambda(\alpha, s, t)$ of the class of λ- spirallike generalied Sakaguchi functions as follows:

Definition 5.5.1.

A function $f(z) \in A$ *is said to be in the class* $R^{\lambda}(\alpha, s, t)$ *if it satisfies*

$$Re\left\{\frac{e^{i\lambda}(s-t)zf'(z)}{f(sz)-f(tz)}\right\} > \alpha cos\lambda, \quad (s \neq t, |\lambda| < \frac{\pi}{2}) \quad (5.5.3)$$

for some $\alpha(0 \leq \alpha < 1)$ *and for all* $z \in U$.

obviously $R^0(\alpha, s, t) = S(\alpha, s, t)$, $R^{\lambda}(\alpha, 1, t) = P^{\lambda}(\alpha, t)$ and $R^{\lambda}(0, 1, 0) = S_p(\lambda)$.

We also denote by $T^{\lambda}(\alpha, s, t)$, the subclass of A consisting of all functions $f(z)$ such that $zf'(z) \in R^{\lambda}(\alpha, s, t)$. To prove our main results, we need the following definition and lemma:

Definition 5.5.2. (Wilf (1961))

A sequence $\{b_n\}_1^{\infty}$ *of complex numbers is said to be a subordinating factor sequence, whenever* $f(z)$ *given by (5.5.1) is regular, univalent and convex in* U, *and*

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad in \quad U \quad (5.5.4)$$

Lemma 5.5.3. (Wilf (1961))

The sequence $\{b_n\}_1^{\infty}$ *is a subordinating factor sequence if and* *only if*

$$Re\left[1+2\sum_{n=1}^{\infty} b_n z^n\right] > 0, \quad z \in U \tag{5.5.5}$$

The purpose of the present section is to investigate the characterization and subordination properties for the class of functions $R^{\lambda}(\alpha, s, t)$ and $T^{\lambda}(\alpha, s, t)$. Some interesting consequences of the main results are also discussed.

5.6. Main Results

We first prove the following theorems dealing with characterization properties for the classes $R^{\lambda}(\alpha, s, t)$ and $T^{\lambda}(\alpha, s, t)$.

Theorem 5.6.1

Let $f(z) \in A$ *such that*

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - 1\right| < 1-\gamma,$$

$$(s, t \in \mathcal{C}, s \neq t, 0 \leq \gamma \leq 1, z \in U) \tag{5.6.1}$$

then $f(z) \in R^{\lambda}(\alpha, s, t)$, *provided that*

$$|\lambda| \leq cos^{-1}\left(\frac{1-\gamma}{1-\alpha}\right) \tag{5.6.2}$$

for some $\alpha (0 \leq \alpha < 1)$ *and* $z \in U$.

Proof: Suppose that

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - 1 = (1-\gamma)\omega(z), \text{ where } |\omega(z)| < 1, \text{ for all } z \in U$$

Now

$$Re\left\{e^{i\lambda}\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\} = \cos\lambda + (1-\gamma)Re\left\{e^{i\lambda}\omega(z)\right\}$$

$$\geq \cos\lambda - (1-\gamma)\left|e^{i\lambda}\omega(z)\right|$$

$$\geq \cos\lambda - (1-\gamma) \geq \alpha\cos\lambda$$

provided that $|\lambda| \leq \cos^{-1}\left(\frac{1-\gamma}{1-\alpha}\right)$. This completes the proof of Theorem 5.6.1.

If we set $\gamma = 1-(1-\alpha)\cos\lambda$, where $|\lambda| < \pi/2$, in Theorem 5.6.1, we obtain the following

Corollary 5.6.2

Let $f(z) \in A$ such that

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)} - 1\right| < (1-\alpha)cos\lambda \qquad (5.6.3)$$

then $f(z) \in R^{\lambda}(\alpha, s, t)$ for $|\lambda| < \pi/2$ and $\alpha(0 \leq \alpha < 1)$.

Remark: On putting $s = 1$ in Theorem 5.6.1 we get the known result due to Goyal et al. (2012), and by putting $s = 1, t = 0$, and $\alpha = 0$ in Theorem 5.6.1 we get the result due to Silverman (1989).

Theorem 5.6.3.

If $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty} \left[|n - u_n| \, sec\lambda \; + (1-\alpha) \, |u_n| \right] |a_n| \leq 1 - \alpha \qquad (5.6.4)$$

for some $\alpha (0 \leq \alpha < 1)$, then $f(z) \in R^{\lambda}(\alpha, s, t)$, where

$$|\lambda| < \pi/2, \;\; u_n = \sum_{j=1}^{n} s^{n-j} t^{j-1}$$

such that $s, t \in \mathcal{C}, s \neq t$.

Proof: To prove the Theorem 5.6.3, we show that if $f(z)$ satisfies the inequality (5.6.4) then

$$\left| \frac{(s-t) z f'(z)}{f(sz) - f(tz)} - 1 \right| < (1-\alpha) \text{cos}\lambda$$

Since

$$\frac{(s-t) z f'(z)}{f(sz) - f(tz)} - 1 = \frac{\left[z + \sum\limits_{n=2}^{\infty} n a_n z^n \right]}{\left[z + \sum\limits_{n=2}^{\infty} u_n a_n z^n \right]} - 1$$

$$= \frac{\sum\limits_{n=2}^{\infty} (n - u_n) a_n z^{n-1}}{1 + \sum\limits_{n=2}^{\infty} u_n a_n z^{n-1}}$$

therefore

$$\left| \frac{(s-t) z f'(z)}{f(sz) - f(tz)} - 1 \right| = \frac{\sum\limits_{n=2}^{\infty} |(n - u_n)| \, |a_n|}{1 - \sum\limits_{n=2}^{\infty} |u_n| \, |a_n|}$$

Thus if $f(z)$ satisfies (5.6.4), then we have

$$\left| \frac{(s-t) z f'(z)}{f(sz) - f(tz)} - 1 \right| < (1-\alpha) \text{cos}\lambda$$

This completes the proof of the Theorem 5.6.3 .

Theorem 5.6.4.

If $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty} n \left[|n - u_n| \sec\lambda \ + (1-\alpha) |u_n| \right] |a_n| \leq 1 - \alpha \qquad (5.6.5)$$

for some $\alpha (0 \leq \alpha < 1)$, then $f(z) \in T^{\lambda}(\alpha, s, t)$, where

$$|\lambda| < \pi/2, \ \ u_n = \sum_{j=1}^{n} s^{n-j} t^{j-1}$$

such that $s, t \in \mathcal{C}$, $s \neq t$.

Remark: For $\lambda = 0$, Theorems 5.6.3 and 5.6.4 reduce to the known results due to Owa et al. (2007).

By setting $s = 1$ in Theorem 5.6.1 and 5.6.3 and 5.6.4 we obtain results of Goyal and Goswami (2012).

By setting $t = -1$ in Theorem 5.6.4, we obtain

Corollary 5.6.5.

If $f(z) \in A$ satisfies the following inequality

$$\sum_{n=2}^{\infty} \left[|n - u_n| \sec\lambda \ + (1-\alpha) |u_n| \right] |a_n| \leq 1 - \alpha \qquad (5.6.6)$$

for some $\alpha (0 \leq \alpha < 1)$, where $|\lambda| < \pi/2$,

$$u_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

then $f(z) \in S(\alpha, 1, -1)$.

5.7. Subordination Property

Theorem 5.7.1.

Let $f(z) \in A$ satisfies the inequality (5.6.4), and K denote the familiar class of the convex univalent functions in U. Then for every $g \in K$, we have

$$\frac{|2-s-t|\, sec\lambda + (1-\alpha)\,|s+t|}{2((1-\alpha)+|2-s-t|\, sec\lambda + (1-\alpha)\,|s+t|)}(f*g)(z) \prec g(z) \tag{5.7.1}$$

where

$$z \in U, s \neq t, 0 \leq \alpha < 1 \ \ and \ \ |\lambda| < \pi/2$$

In particular

$$Re\{f(z)\} > -\frac{((1-\alpha)+|2-s-t|sec\lambda + (1-\alpha)|s+t|)}{|2-s-t|sec\lambda + (1-\alpha)|s+t|},$$

$$(z \in U) \tag{5.7.2}$$

The following constant factor

$$\frac{|2-s-t|sec\lambda + (1-\alpha)|s+t|}{2\left((1-\alpha)+|2-s-t|sec\lambda + (1-\alpha)|s+t|\right)} \tag{5.7.3}$$

is the best dominant.

Proof: Let $f(z) \in A$ satisfies the inequality (5.6.4) and suppose that $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in K$. Then

$$\frac{|2-s-t|\, \sec\lambda + (1-\alpha)\,|s+t|}{2((1-\alpha)+|2-s-t|\, \sec\lambda + (1-\alpha)\,|s+t|)}(f*g)(z)$$

$$= \frac{|2-s-t|\sec\lambda + (1-\alpha)\,|s+t|}{2((1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)\,|s+t|)}(z+\sum_{n=0}^{\infty} a_n c_n z^n) \tag{5.7.4}$$

Thus by definition (5.5.2), the assertion of our theorem will hold if the sequence

$$\left\{ \frac{|2-s-t|\sec\lambda + (1-\alpha)\,|s+t|}{2((1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)\,|s+t|)} a_n \right\}_{n=1}^{\infty}$$

is subordinating factor sequence, with $a_1 = 1$. By virtue of Lemma (5.5.3), this will be the case if and only if

$$Re\left\{ 1 + 2\sum_{n=1}^{\infty} \frac{|2-s-t|\sec\lambda + (1-\alpha)\,|s+t|}{2((1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)\,|s+t|)} a_n z^n \right\} > 0 \tag{5.7.5}$$

Now

$$Re\left\{ 1 + \sum_{n=1}^{\infty} \frac{|2-s-t|\sec\lambda + (1-\alpha)\,|s+t|}{(1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)\,|s+t|} a_n z^n \right\}$$

$$= Re\left\{ 1 + \frac{|2-s-t|\sec\lambda + (1-\alpha)\,|s+t|}{(1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)\,|s+t|} z \right.$$

$$\left. + \sum_{n=2}^{\infty} \frac{|2-s-t|\sec\lambda + (1-\alpha)\,|s+t|}{(1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)\,|s+t|} a_n z^n \right\}$$

$$\geq 1 - \sum_{n=1}^{\infty} \frac{|2-s-t|\sec\lambda + (1-\alpha)\,|s+t|}{(1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)\,|s+t|}\; r$$

$$- \sum_{n=2}^{\infty} \frac{|n-u_n|\sec\lambda + (1-\alpha)|u_n|}{((1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)|s+t|} |a_n| r^n$$

$$> 1 - \frac{|2-s-t|\,\sec\lambda + (1-\alpha)\,|s+t|}{(1-\alpha) + |2-s-t|\,\sec\lambda + (1-\alpha)\,|s+t|}\; r$$

$$- \frac{(1-\alpha)}{(1-\alpha) + |2-s-t|\sec\lambda + (1-\alpha)|s+t|}\; r > 0 \quad (|z| \leq r < 1) \tag{5.7.6}$$

Thus (5.7.6) holds true in U. This proves the subordination result (5.7.1). The inequality (5.7.2) follows from (5.7.1) upon setting

$$g(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \in K \tag{5.7.7}$$

To prove sharpness of the constant given by (5.7.3), we consider the function f_0 defined by

$$f_0(z) = z - \frac{(1-\alpha)\sec\lambda}{(|2-s-t|\sec\lambda + (1-\alpha)|s+t|)} z^2$$

where

$$z \in U, s \neq t, 0 \leq \alpha < 1 \text{ and } |\lambda| < \pi/2 \tag{5.7.8}$$

Then by using (5.7.1), we have

$$\frac{|2-s-t|\,\sec\lambda + (1-\alpha)\,|s+t|}{2((1-\alpha) + |2-s-t|\,\sec\lambda + (1-\alpha)\,|s+t|)} f_0(z) \prec \frac{z}{1-z} \tag{5.7.9}$$

It can be easily verified for the function $f_0(z)$ defined by (5.7.8) that

$$\min_{|z|\leq 1} Re\left\{\frac{|2-s-t|\,\text{sec}\lambda+(1-\alpha)\,|s+t|}{2((1-\alpha)+|2-s-t|\,\text{sec}\lambda+(1-\alpha)\,|s+t|)}f_0(z)\right\}$$
$$=-\frac{1}{2} \tag{5.7.10}$$

This shows that constant given by (5.7.3) is the best dominant.

We also consider the following useful consequence of the subordination Theorem (5.7.1). Upon setting $s=1, t=-1$, we get

Corollary 5.7.2.

Let $f(z)\in A$ is in $S(\alpha,1,-1)$ and satisfies the inequality (5.6.4)*. Then for every $g\in K$, we have*

$$\frac{sec\lambda}{((1-\alpha)+2sec\lambda)}(f*g)(z)\prec g(z) \tag{5.7.11}$$

In particular

$$Re\left\{f(z)\right\}>-\frac{((1-\alpha)+2sec\lambda)}{2sec\lambda}\quad (z\in U) \tag{5.7.12}$$

The following constant factor

$$\frac{sec\lambda}{((1-\alpha)+2sec\lambda)} \tag{5.7.13}$$

is the best dominant.

Remark: Putting $s = 1$ in Theorem 5.7.1 we get known results of Goyal and Goswami (2012) and by putting $t = 0, s = 1$ and $\alpha = 0$ in Theorem (5.7.1) we get a known result obtained by Singh (2000).

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