CHAPTER 2
Preliminaries

2.1 Introduction to fuzzy sets

The revolutionary idea of fuzzy set was introduced by pioneer researcher Zadeh [206]. The concept of fuzzy set generalizes the Boolean property of membership and non-membership of classical set by assigning a graded value of membership to every element of the universe under consideration.

The doctrine of fuzzy sets has been properly defined by M. M. Gupta [42] as: “a body of concepts and techniques aimed at providing a systematic framework for dealing with the vagueness and imprecision inherent in human thought processes.”

Mathematically, a fuzzy set \( A \) over a universal set \( X \) is defined by a membership function \( \mu_A : X \rightarrow [0,1] \) assigning to each individual \( x \in X \) a value \( \mu_A(x) \in [0,1] \), representing its grade of membership in the fuzzy set \( A \). This grade corresponds to the degree to which that individual is similar or compatible with the concept represented by the fuzzy set.

Each fuzzy set \( A \) is completely defined by a membership function describing its fuzziness or fuzzy property. The choice of the membership function is both context and observer dependent so it can have any form. A detailed study and examples of membership functions and other types of fuzzy sets is available in [13,74, 99,133].

2.1.1 Basic characteristics of fuzzy sets

Fuzzy sets can be characterized in more detail by referring to the features used in characterizing the membership functions describing them [24,74,78,218].
Basic characteristics of membership function are described as follows:

**α-cut:** The $\alpha$-cut $^\alpha A$, of fuzzy set $A$, is a crisp subset of $A$, with all elements having membership value at least $\alpha$ i.e. $^\alpha A = \{ x \mid \mu_A(x) \geq \alpha \}$, $\alpha \in [0,1]$.

**Strong α-cut:** The strong $\alpha$-cut $^\alpha^* A$, of fuzzy set $A$, is a crisp subset of $A$, given by the collection of all elements with membership value greater than $\alpha$ i.e. $^\alpha^* A = \{ x \mid \mu_A(x) > \alpha \}$, $\alpha \in [0,1]$.

**Height:** The height $h(A)$, of a fuzzy set $A$ is the largest membership grade attained by any $x \in X$ in $A$ i.e. $h(A) = \sup_{x \in X} \mu_A(x)$.

**Support:** The support of a fuzzy set $A$, is a crisp subset of $A$ with elements having nonzero membership in $A$ i.e. $\text{Supp}(A) = \{ x \in X \mid \mu_A(x) > 0 \}$.

**Core:** The core of a fuzzy set is the set of all elements of $X$ that exhibit a unit level of membership in $A$ denoted as $\text{Core}(A) = \{ x \in X \mid \mu_A(x) = 1 \}$.

**Normality:** A fuzzy set $A$ is normal if its core is non-empty, otherwise it is called subnormal.

**Convexity:** A fuzzy set $A$ is convex if its membership function is such that $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$ for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$ and concave otherwise. Clearly, a fuzzy set is convex if each of its $\alpha$-cut is convex.

**Distance between two fuzzy sets:** If $A$ and $B$ are fuzzy sets defined over $X$, the Minkowski distance between $A$ and $B$, denoted as $d_p(A,B)$ is given by

$$d_p(A,B) = \left( \sum_{x \in X} (\mu_A(x) - \mu_B(x))^p \right)^{1/p}, \quad p \in (1, \infty).$$
For $p = 1$, it becomes the Hamming distance and when $p = 2$, it is said to be the Euclidean distance. Figure 2.1 shows some basic characteristics of fuzzy set $A$.

![Figure 2.1: Height, $\alpha$-cut, core and support of a fuzzy set](image)

**2.1.2 Extension principle for fuzzy sets**

The extension principle is basically a process of constructing a new fuzzy set $Y$ from a given set $X$ using a property $f$ such that the membership grade of an element in $Y$ is determined with the help of membership grade of elements in $X$. Let $X$ and $Y$ be two non-empty sets, $f$ a mapping from $X$ to $Y$ and $A$ be a fuzzy set in $X$. The extension principle states that the image of $A$ under this mapping is a fuzzy set $B = f(A)$ in $Y$ such that, for each $y \in Y$, $\mu_B(y) = \sup_x \mu_A(x)$, subject to $x \in X$ and $y = f(x)$. The importance of this principle is that it extends the applicability of fuzzy sets to all fields [64,72,196].

**2.2 Fuzzy operations**

The generalizations of set operations for fuzzy sets have wide importance. A variety of these generalizations known as fuzzy complements, intersections and unions are presented as follows:
2.2.1 Fuzzy complement

Fuzzy complement over a fuzzy set \( A \) is a unary operation \( c : [0,1] \rightarrow [0,1] \) defined as \( c(A(x)) = cA(x), \forall x \in X \) provided that the following properties are satisfied:

(i) \( c(0) = 1 \) and \( c(1) = 0 \), \hspace{1cm} \text{(boundary condition)}
(ii) For all \( a, b \in [0,1] \), if \( a \leq b \), then \( c(a) \geq c(b) \), \hspace{1cm} \text{(monotonicity)}

Some examples of these complements are:

(i) Yagar class of complements: \( c(a) = (1 - a^\omega)^\frac{1}{\omega}, \ \omega \in (0, \infty) \)
(ii) Sugeno class of complements: \( c(a) = \frac{1-a}{1+\lambda a}, \ \lambda \in (-1, \infty) \)

More examples and other details of fuzzy complements can be found in [71,74].

2.2.2 Fuzzy intersections

Fuzzy intersections or triangular norms (t-norms) were introduced by Berthold Schweizer and Abe Sklar in [160], following some ideas of Karl Menger [98]. Menger in 1942 seeded the concept with the introduction of statistical metric in probabilistic spaces.

A t-norm is a binary operation \( \odot : [0,1]^2 \rightarrow [0,1] \) satisfying the following properties:

(i). \( \odot \) is commutative and associative i.e. for all \( a, b, c \in [0,1] \)
\[
a \odot b = b \odot a, \quad a \odot (b \odot c) = (a \odot b) \odot c, 
\]

(ii). \( \odot \) is non-decreasing in both the arguments i.e.
\[
a \leq b \text{ implies } a \odot c \leq b \odot c, \quad b \leq c \text{ implies } a \odot b \leq a \odot c,
\]
(iii). \( 1 \odot a = a \) and \( 0 \odot a = 0 \), for all \( a \in [0,1] \). (boundary condition)

Some desirable requirements are:

(iv). \( \odot \) is a continuous function (continuity)

(v). \( a \odot a < a \) (sub-idempotent)

(vi). \( a_1 < a_2 \) and \( b_1 < b_2 \) implies \( a_1 \odot b_1 < a_2 \odot b_2 \). (strict monotonicity)

A continuous, sub-idempotent t-norm is called an Archimedean t-norm; if it also satisfies strict monotonicity, it is called strict Archimedean t-norm. Prominent examples of t-norms are as follows:

(i) Standard intersection or Godel t-norm or Zadeh t-norm : \( a \odot b = \min(a,b) \),

(ii) Algebraic product or Goguen t-norm: \( a \odot b = a \cdot b \),

(iii) Bounded difference or Lukasiewicz t-norm: \( a \odot b = \max(0,a+b-1) \),

(iv) Drastic intersection: \( a \odot b = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases} \),

(v) Hamacher product: \( a \odot b = \begin{cases} 0, & \text{if } a = b = 0 \\ \frac{ab}{a+b-ab}, & \text{otherwise} \end{cases} \),

(vi) Nilpotent minimum: \( a \odot b = \begin{cases} \min(a,b), & \text{if } a+b > 1 \\ 0, & \text{otherwise} \end{cases} \)

The graphical presentation of fuzzy intersections (i) to (vi) is shown in figure 2.2. More examples and other details of fuzzy intersections can be found in [71,74].
Figure 2.2: Graphs of fuzzy intersections
Definition 2.2.1. A t-norm \( \odot \) is said to be left continuous if for \( b \in [0,1] \) and sequence \( \{a_n \in [0,1] \mid n \in N = 1, 2, 3, \ldots\} \), \( \sup_{n \in N} (a_n \odot b) = \left( \sup_{n \in N} a_n \right) \odot b \) and right continuous if \( \inf_{n \in N} (a_n \odot b) = \left( \inf_{n \in N} a_n \right) \odot b \).

A t-norm is continuous if and only if it is both left and right continuous. The continuity (left and right) of t-norm imply the interchangeability of the supremum and infimum operators with the t-norm that plays crucial role in resolution of fuzzy relation equations. The three most popular t-norms the minimum t-norm, product t-norm, Lukasiewicz t-norm are all continuous but the drastic intersection is right continuous and not left continuous and the nilpotent minimum is left continuous only.

The product t-norm is the prototypical example of class of strict t-norms. The drastic product is non-continuous, Archimedean t-norm. A well known example of continuous, non-Archimedean t-norm is minimum t-norm but it is not alone candidate for this class, some other examples can be found in [71,195]. A detailed discussion over the algebraic and analytic behavior of t-norms is presented in [71].

2.2.3 Fuzzy union

A t-conorm (or fuzzy union or s-norm) is a binary operation \( \oplus : [0,1]^2 \to [0,1] \) which is associative, commutative, monotone satisfying the boundary condition i.e. \( a \oplus 0 = a \), for all \( a, b, c \in [0,1] \).

Basically, t-conorms are considered as the dual operators of t-norm operation with the same essential characteristics of t-norms except the boundary condition \( a \oplus 0 = a \). Prominent examples of t-conorms are as follows:

(i). Standard union or Zadeh t-conorm: \( a \oplus b = \max(a, b) \)
(ii). Algebraic sum or probabilistic sum: \( a \oplus b = a + b - ab \)

(iii). Bounded sum or Lukasiewicz t-conorm: \( a \oplus b = \min(1, a + b) \)

(iv). Drastic union: \( a \oplus b = \begin{cases} a, & b = 0 \\ b, & a = 0 \\ 1, & \text{otherwise} \end{cases} \)

(v). Nilpotent maximum: \( a \oplus b = \begin{cases} \max(a, b), & \text{if } a + b < 1 \\ 1, & \text{otherwise} \end{cases} \)

(vi). Einstein sum: \( a \oplus b = \frac{a + b}{1 + ab} \)

More examples and other details of fuzzy unions can be found in [71, 74, 78]. The general fuzzy unions from (i) to (vi) are shown in figure 2.3. Out of the whole class of fuzzy unions, the drastic union is the point wise largest t-conorm and standard union is the weakest t-conorm possible [71, 74].

The standard fuzzy intersection, min operator, when applied to a fuzzy set produces the largest membership value of all the t-norms, and the standard fuzzy union, max operator, when applied to a fuzzy set produces the smallest membership value of all the t-conorms. These features of the standard fuzzy intersection and union are significant because they both prevent the compounding of errors in the operands [74]. Most of the alternative norms lack this significance.
Figure 2.3: Graphs of fuzzy unions
2.2.4 Residuum: Let $\odot$ be a continuous t-norm. Then there is a unique binary operation $a \Theta b$ satisfying for all $a, b, x \in [0,1]$, the condition $a \odot x \leq b$ if and only if $x \leq (a \Theta b)$ namely $a \Theta b = \sup \{ x | a \odot x \leq b \}$ called as the residuum of the t-norm.

In the standard semantics of t-norm based fuzzy logics, where conjunction is interpreted by a t-norm, the residuum plays the role of implication (often called R-implication). The residuum operators associated with some prominent t-norms are given as:

(i). Lukasiewicz: $a \odot b = \max (a + b - 1, 0)$, $a \Theta b = \min (1 - a + b, 1)$

(ii). Gödel: $a \odot b = \min (a, b)$, $a \Theta b = \begin{cases} b & \text{if } a > b \\ 1 & \text{otherwise} \end{cases}$

(iii). Goguen: $a \odot b = a \cdot b$, $a \Theta b = \begin{cases} b / a & \text{if } a > b \\ 1 & \text{otherwise} \end{cases}$

(iv). Nilpotent minimum:

$$a \odot b = \begin{cases} \min(a, b) & \text{if } a + b > 1 \\ 0 & \text{otherwise} \end{cases}, \quad a \Theta b = \begin{cases} \max(1 - a, b) & \text{if } a > b \\ 1 & \text{otherwise} \end{cases}$$

The graphical representation of these four residuum operators is shown in figure 2.4.

Theorem 2.2.1. Let $\odot$ be a left-continuous t-norm then for all $a, b, x \in [0,1]$, the associated residuum operator defined as $\Theta_r = \sup \{ x | a \odot x \leq b \}$ satisfies the following axioms:

(i). $a \leq b \Rightarrow a \Theta_c c \geq b \Theta_c c$

(ii). $a \leq b \Rightarrow a \Theta b = 1$

(iii). $1 \Theta a = a$
(iv). \( b \leq a \Theta b \)

(v). \( a \Theta_i (b \Theta_i (a \circ b)) = 1 \)

(vi). \( (a \circ b) \Theta_i c \leq a \Theta_i (b \Theta_i c) \)

(vii). \( a \Theta_i b \leq (a \circ c) \Theta_i (b \circ c) \)

(viii). \( (a \circ b) \Theta_i b \Rightarrow a \Theta_i c \geq b \Theta_i c \)

(ix). \( a \circ (a \Theta_i b) \leq b \)

Figure 2.4: Graphs of residua operators
2.2.5 Aggregation operations

Aggregation operations on fuzzy sets are operations by which several fuzzy sets are combined in a desirable way to produce a single fuzzy set. An aggregation operation is an $n$-ary ($n \geq 2$) function $h: [0,1]^n \rightarrow [0,1]$ satisfying the following requirements:

(i) $h(0,0,\ldots,0) = 0$ and $h(1,1,\ldots,1) = 1$,

(ii) For any pair $\langle a_1,a_2,\ldots,a_n \rangle$ and $\langle b_1,b_2,\ldots,b_n \rangle$ of $n$-tuples, such that $a_i, b_i \in [0,1]$, if $a_i \leq b_i, \forall i = 1,2,\ldots,n$, then $h(a_1,a_2,\ldots,a_n) \leq h(b_1,b_2,\ldots,b_n),$

(iii) $h$ is a continuous function.

Besides these essential requirements, aggregation operations on fuzzy sets are usually expected to satisfy two additional requirements:

(iv) $h$ is symmetric in all its arguments, that is, $h(a_1,a_2,\ldots,a_n) = h(a_{p(1)},a_{p(2)},\ldots,a_{p(n)})$ for any permutation $p$,

(v) $h$ is an idempotent function; that is, $h(a,a,\ldots,a) = a, \forall a \in [0,1]$. Idempotent aggregation operators are also called as averaging operations.

The fuzzy intersections and fuzzy unions are also the aggregation operators by definition. In fact, the whole class of aggregation operators is bounded above by the weakest union and bounded below by largest intersection. The classification of the family of fuzzy aggregation operators is shown in figure 2.5.
2.3 Fuzzy numbers

In most of cases in our life, the information obtained for decision making is approximately known [104,105]. Fuzzy numbers allow us to make the mathematical model of quantities in fuzzy environment. Basically, a fuzzy number is a quantity whose value is imprecise, rather than exact ordinary" (single-valued) numbers such as “close to 5”, “approximately 3”, “several” etc. Mathematically, they can be defined as fuzzy subsets of real line defined by membership function \( A: \mathbb{R} \rightarrow [0,1] \) that satisfy the condition of convexity and normality.
2.3.1 Arithmetic of fuzzy numbers

The arithmetic operations on fuzzy numbers are basic content in fuzzy arithmetic. The arithmetic of fuzzy numbers can be explained from two different but equivalent approaches. The first approach is to use the interval arithmetic on the \( \alpha \)-cuts of given fuzzy numbers with the application of the resolution principle based on decomposition of fuzzy set \( A \) in terms of special fuzzy sets \( _{\alpha}A \) defined as \( _{\alpha}A(x) = \alpha \cdot A(x), \, \alpha \in (0,1) \) . The second approach is based on the Zadeh’s extension principle that results in a new fuzzy number as the outcome.

**Approach based on \( \alpha \)-cuts:** Let \( A \) and \( B \) be two fuzzy numbers and \( _{\alpha}A = [a_{\alpha L}, a_{\alpha U}], \, _{\alpha}B = [b_{\alpha L}, b_{\alpha U}] \), be the \( \alpha \)-cuts, \( \alpha \in [0,1] \), of \( A \) and \( B \) respectively. Let \( \ast \) denote any of the arithmetic operations \(+, -, \cdot, /\) on fuzzy numbers. Then,

\[
_{\alpha}(A \ast B) = _{\alpha}A \ast _{\alpha}B, \, \alpha \in (0,1] \quad \text{and} \quad A \ast B = \bigcup_{\alpha \in [0,1]} _{\alpha}(A \ast B).
\]

Since, \( _{\alpha}(A \ast B) \) is a closed interval for each \( \alpha \in (0,1] \) and \( A, \ B \) are fuzzy numbers, \( A \ast B \) is also a fuzzy number.

**Approach based on the Zadeh’s extension principle:** Let \( A \) and \( B \) be two fuzzy numbers and \( \ast \) be any of the arithmetic operations \(+, -, \cdot, /\). Then by using the Zadeh’s extension principle, the fuzzy number \( A \ast B \) is defined as:

\[
(A \ast B)(z) = \sup_{z=x+y} \min (A(x), B(y)), \text{ for all } z \in \mathbb{R}.
\]

2.3.2 Linguistic variables

The concept of linguistic variables (or fuzzy variable) is the key notion in fuzzy set theory and fuzzy logic. A linguistic variable is a variable whose values are words or
sentences in natural or artificial language that enables to model linguistic characteristics and notions in nature (as introduced originally by Zadeh [208]). For instance, a numerical variable can assume only numeric (precise) values but a fuzzy variable say ‘Age’ assume values such as young, old, very old, not very old etc. Figure 2.6 shows an example of a linguistic variable.

A linguistic variable is characterized by a quintuple denoted by \((v, T, X, g, m)\) in which \(v\) is the name of the variable, \(T\) is the term set of linguistic values of \(v\) that refer to a base variable whose values range over a universal set \(X\), \(g\) is generally grammar for generating linguistic terms, \(m\) is a semantic rule for associating with each label its meaning \(m(t)\), which is a fuzzy set on \(X\), i.e., \(m:T \rightarrow \mathcal{S}(X)\) [208].

In this work, fuzzy sets are defined in terms of mappings into residuated lattices. The abstraction from \([0,1]\) to residuated lattices enables to formulate the properties for a broad class of useful structures of truth values. The following sections describe the concepts that are required to establish the algebraic framework required in the work.
2.4 Algebraic system

An algebraic system \( \langle A, F \rangle \), is a system consisting of a non-empty set \( A \) equipped with one or more \( n \)-ary operations \( F \) defined on it.

**Semigroup:** Let \( A \) be a non-empty set and \( * \) be a binary operation on \( A \). The algebraic system \( \langle A, * \rangle \) is called a semigroup if the operation \( * \) is associative i.e. for all \( a, b, c \in A \), \( (a * b) * c = a * (b * c) \).

**Monoid:** A semigroup \( \langle A, * \rangle \) with an identity element with respect to operation \( * \) is called a monoid i.e. for any \( a, b, c \in A \), \( (a * b) * c = a * (b * c) \) and there exists an element \( e \in A \) such that for any \( a \in A \), \( e * a = a * e = a \). For example, t-norms are binary operations on the closed unit interval \([0,1]\) such that \( ([0,1], \leq, \odot) \) is an abelian, totally ordered semigroup with identity 1, hence are also considered as a monoidal operation on \([0,1]\).

**Lattice:** Let \( L \) be a non-empty partially ordered set. \( L \) is called a lattice if \( a \lor b \) and \( a \land b \) exists for all \( a, b \in L \), where \( \lor \) denotes the join or sum and \( \land \) denotes the meet or product of \( a \) and \( b \) respectively. \( L \) is complete if each of its nonempty subsets has a least upper bound and a greatest lower bound. The least and greatest elements of a lattice, if they exist, are called the bounds of the lattice and are denoted by 0 and 1 respectively. A lattice which has both elements 0 and 1 is called a bounded lattice. In a bounded lattice \( \langle L, \lor, \land, 0, 1 \rangle \), an element \( b \in L \) is called a complement of an element \( a \in L \) if \( a \land b = 0 \) and \( a \lor b = 1 \). A lattice is said to be a complemented lattice if every element of \( L \) has at least one complement in it. A lattice \( \langle L, \lor, \land, 0, 1 \rangle \) is called a distributive lattice if for any \( a, b, c \in L \), \( a \land (b \lor c) = (a \land b) \lor (a \land c) \) and \( a \lor (b \land c) = (a \lor b) \land (a \lor c) \).

**Brouwerian lattice:** A Brouwerian lattice is a lattice \( L \) in which for any given elements \( a \) and \( b \), the set of all \( x \in L \) such that \( a \land x \leq b \) contains the greatest element, denoted
\( a \circ b \) (called the relative pseudocomplement of \( a \) in \( b \)). In a dually Brouwerian lattice for any given elements \( a \) and \( b \), the set of all \( x \in L \) such that \( a \vee x \geq b \) contains a least element, denoted \( a \circ b \) (dually relative pseudocomplement of \( a \) in \( b \)). A Brouwerian lattice is always distributive, so is a dually Brouwerian lattice. A Brouwerian lattice \( L \), is also called an implicative lattice as it always has the greatest element 1 since \( \{ x \in L \mid a \wedge x \leq a \} = L \).

**Boolean algebra:** A Boolean algebra is a complemented, distributive lattice generally denoted by \( \langle L, \wedge, \vee, ', 0, 1 \rangle \) in which \( \langle L, \wedge, \vee \rangle \) is a lattice with two binary operations \( \wedge \) and \( \vee \) called meet and join respectively and 0 and 1 be the two bounds and ‘’’ is the complementation operation.

**Lattice as an algebraic system:** A lattice is an algebraic system \( \langle L, \wedge, \vee \rangle \) with two binary operations \( \wedge \) and \( \vee \) on \( L \), which are both (i) commutative, (ii) associative, and (iii) satisfy the absorption laws. Let \( \langle L, \wedge, \vee \rangle \) be a lattice and let \( A \subseteq L \). The algebra \( \langle A, \wedge, \vee \rangle \) is a sublattice of \( \langle L, \wedge, \vee \rangle \) iff \( A \) is closed under both operations \( \wedge \) and \( \vee \).

**Residuated lattice:** A complete residuated lattice is an algebra \( \langle L, \wedge, \vee, \circ, \Theta, 0, 1 \rangle \) satisfying the following conditions:

(i). \( \langle L, \wedge, \vee, 0, 1 \rangle \) is a complete bounded lattice.

(ii). \( \langle L, \circ, 1 \rangle \) is a commutative monoid.

(iii). \( \circ \) and \( \Theta \), form an adjoint couple i.e. \( x \leq (a \Theta b) \) if and only if \( a \circ x \leq b \) holds for all \( a, b, x \in L \).

A residuated lattice \( \langle L, \wedge, \vee, \circ, \Theta, 0, 1 \rangle \) is called complete if \( \langle L, \wedge, \vee \rangle \) is complete as a lattice. In our study, we assume that \( L \) is to be complete. Most important examples of
complete residuated lattices \( L = \langle L, \land, \lor, \circ, \Theta, 0,1 \rangle \) are those with the universe \( L \) being a real unit interval \([0,1]\).

In our study we consider fuzzy set \( A \) as a scale \( L \) of truth degrees, and a rule that associates to each object \( x \) from \( X \) a truth degree \( a \) from \( L \) that is thought of as the truth degree to which \( x \) belongs to \( A \). We assume that the structure of truth degrees forms a complete residuated lattice. In the following study \( L \) denotes an arbitrary complete residuated lattice (with \( L \) being the universe set of \( L \)).

**Theorem 2.4.1.** Each complete residuated lattice satisfies the following properties:

(i) \( a \circ (a \Theta, b) \leq b, \ b \leq a \Theta, (a \circ b), \ a \leq (a \Theta, b) \Theta, b, \)

(ii) \( a \leq b \) iff \( a \Theta, b = 1, \)

(iii) \( a \Theta, a = 1, \ a \Theta, 1 = 1, \ 0 \Theta, a = 1, \)

(iv) \( 1 \Theta, a = a, \)

(v) \( a \circ 0 = 0, \)

(vi) \( a \circ b \leq a, \ a \leq b \Theta, a, \)

(vii) \( a \circ b \leq a \land b, \)

(viii) \( (a \circ b) \Theta, c = a \Theta, (b \Theta, c) = b \Theta, (a \Theta, c), \)

(ix) \( (a \Theta, b) \circ (b \Theta, c) \leq a \Theta, c, \)

(x) \( a_1 \leq a_2 \) and \( b_1 \leq b_2 \) implies \( a_1 \circ b_1 \leq a_2 \circ b_2, \)

(xi) \( b_1 \leq b_2 \) implies \( a \Theta, b_1 \leq a \Theta, b_2, \)

(xii) \( a_1 \leq a_2 \) implies \( a_2 \Theta, b \leq a_1 \Theta, b, \)

(xiii) \( (a \Theta, b) \circ (c \Theta, d) \leq (a \circ c) \Theta, (b \circ d), \)

(xiv) \( (a \Theta, b)'' \leq a'' \Theta, b'', \)

(xv) \( a \Theta, b \) is the greatest element of \( \{ x | a \circ x \leq b \}, \)

(xvi) \( a \circ b \) is the least element of \( \{ x | a \leq b \Theta, x \}. \)
**BL Algebra:** A residuated lattice \( L = \langle L, \wedge, \vee, \odot, \Theta, 0, 1 \rangle \) is a BL-algebra that satisfies the following additional conditions:

(i) \( a \wedge b = (a \odot (a \Theta, b)) \),

(ii) \( (a \Theta, b) \vee (b \Theta, a) = 1 \).

The basic fuzzy logic and BL-algebras were introduced by Hájek [45], to formalize a part of the reasoning in fuzzy logic. The operation \( \odot \) belongs to conjunction and \( \Theta \), belongs to implication (residuum) and the pair \( \langle \odot, \Theta \rangle \) forms an adjoint couple. For a given \( \odot \), the operation \( \Theta \), satisfying adjointness property is unique (if \( \Theta^* \) is another one, we get \( x \leq a \Theta, b \) iff \( a \odot x \leq b \) iff \( x \leq a \Theta^*, b \) for any \( x \), from which it clearly follows \( a \Theta, b = a \Theta^*, b \)). Similarly, \( \Theta \) uniquely determines \( \odot \).

The most applied complete residuated lattices are those with \( L = [0,1] \) with operations forming adjoint couple as follows:

(i). Lukasiewicz: \( a \odot b = \max (a + b - 1, 0) \), \( a \Theta, b = \min (1 - a + b, 1) \)

(ii). Gödel: \( a \odot b = \min (a,b) \), \( a \Theta, b = \begin{cases} b & \text{if } a > b \\ 1 & \text{otherwise} \end{cases} \)

(iii) Goguen: \( a \odot b = a \cdot b \), \( a \Theta, b = \begin{cases} b/a & \text{if } a > b \\ 1 & \text{otherwise} \end{cases} \)

The corresponding algebras \( L \) are called standard Lukasiewicz algebra, standard Gödel algebra, and standard Goguen (product) algebra on \([0,1]\).

**Theorem 2.4.2.** In each BL-algebra, the following conditions hold in each \( a, b, c \in [0,1] \)

(i) \( (a \vee b) \odot c = (a \odot c) \vee (b \odot c) \)

(ii) \( a \odot b \leq a \)
(iii) \(a \cap b \leq a \land c\)

(iv) \(a \lor b = 1\) implies \(a \circ b = a \land b\)

(v) \(a \land b = a \circ (a \Theta, b)\)

(vi) \(a \circ (b \lor c) = (a \circ b) \lor (a \circ c)\)

(vii) \(a \circ (b \land c) = (a \circ b) \land (a \circ c)\)

(viii) \((a \lor b) = ((a \Theta, b) \Theta, b) \land ((b \Theta, a) \Theta, a)\)

(ix) \((a \Theta, (b \Theta, c)) = ((a \circ b) \Theta, c)\)

2.5 Fuzzy relations--compositions and properties

The concept of fuzzy relation was first conceived by Zadeh [206] in 1965 and developed by Kaufmann [65]. The power of the fuzzy sets lies in their capability to model linguistic expressions in natural language. The situation with fuzzy relations is similar. Classical relations are structures that just represent the presence or absence of correlation or interaction among various objects, while fuzzy relations are based on the philosophy that everything is related to some extent. Basically, fuzzy relations capture and represent qualitative relationships between sets of real-world objects.

Fuzzy relations are also the basic tool for modeling the fuzzy systems. Zadeh and Desoer [205] also showed that, in the general study of systems the relationship between input and output parameters can be modeled by fuzzy relation between input and output spaces. Fuzzy relations allow the relationship between elements of two or more sets to take on an infinite number of degrees of relationship between the extremes of “completely related” and “not related,” as possible in crisp relations.

2.5.1 Cartesian product of fuzzy sets: Let \(A_1, A_2, \ldots, A_n\) be \(n\) fuzzy sets defined on universes \(X_1, X_2, \ldots, X_n\) respectively then the Cartesian product of \(A_1, A_2, \ldots, A_n\) \(A_1 \times A_2 \times \ldots \times A_n\) is given as:

\[
\mu_{A_1 \times A_2 \times \ldots \times A_n}(x_1, x_2, \ldots, x_n) = (\mu_{A_1}(x_1) \circ \mu_{A_2}(x_2) \ldots \circ \mu_{A_n}(x_n))
\]
where $\odot$ is any t-norm and $x_i \in X_i$, $i = 1, 2, \ldots, n$. Figure 2.7 shows the Cartesian product of two classical sets $[1,3]$ and $[2,4]$.

![Figure 2.7: Classical Cartesian product of sets [1,3] and [2,4]](image)

Let there be two fuzzy sets $A(x) = e^{-\frac{(x-5)^2}{2}}$ and $B(y) = e^{-\frac{(y-5)^2}{2}}$ on universe $X = [0,10]$ then their Cartesian product based on two t-norms is shown in figure 2.8.

![Figure 2.8: Cartesian product of two fuzzy sets A and B](image)

$$\mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$$

$$\mu_{A \times B}(x, y) = (\mu_A(x) \cdot \mu_B(y))$$

2.5.2 Binary fuzzy relations

Let $X$ and $Y$ be two fuzzy sets then any fuzzy relation $R$ between $X$ and $Y$ is a fuzzy set $R \subseteq X \times Y$ given by the membership function $\mu_R : X \times Y \rightarrow [0,1]$ such that
μ̂_R(x, y) ≤ (μ_X(x) ∩ μ_Y(y)) and denoted as \( R(X, Y) \) where \( μ_R(x, y) \) represents the strength of relation between the elements \( x \) and \( y \) with \( \odot \) be any t-norm.

A fuzzy binary relation \( R(X, Y) \) defined on finite universes can be conveniently represented by the two dimensional array \( R = [r_{ij}]_{nm} \) of membership values \( r_{ij} \in [0,1] \) as:

\[
R = \begin{bmatrix}
    r_{11} & r_{12} & \cdots & r_{1j} & \cdots & r_{1m} \\
    r_{21} & r_{22} & \cdots & r_{2j} & \cdots & r_{2m} \\
    \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
    r_{ij} & r_{ij} & \cdots & r_{ij} & \cdots & r_{im} \\
    \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
    r_{ni} & r_{ni} & \cdots & r_{nj} & \cdots & r_{nm}
\end{bmatrix}
\]

where \( i = \{1, 2, \ldots, n\} \), \( j = \{1, 2, \ldots, m\} \) and \( r_{ij} = μ_R(x_i, y_j) \) denotes the strength of relation present between the \( i^{th} \) element of \( X \) and \( j^{th} \) element of \( Y \).

From here on for a fuzzy relation \( R \) from \( X \) to \( Y \) we simply write \( R \subseteq X \times Y \).

### 2.5.3 Operations on binary fuzzy relations

Fuzzy relations are again the fuzzy sets defined on multidimensional universe, so all the set and algebraic operations defined for fuzzy sets hold for the fuzzy relations as well.

**Definition 2.5.1.** For fuzzy relations \( R \subseteq X \times Y \) and \( S \subseteq X \times Y \),

(i) The fuzzy union of \( R \) and \( S \) is a fuzzy relation \( R \cup S \subseteq X \times Y \) with the degree of membership \( μ_{R∪S}(x, y) = (μ_R(x, y) \oplus μ_S(x, y)), \forall (x, y) \in X \times Y \),

(ii) The fuzzy intersection of \( R \) and \( S \) is a fuzzy relation \( R \cap S \subseteq X \times Y \) with the degree of membership \( μ_{R∩S}(x, y) = (μ_R(x, y) \odot μ_S(x, y)), \forall (x, y) \in X \times Y \).
(iii) The standard fuzzy complement of \( R \) is a fuzzy relation defined as
\[
\mu'_R(x, y) = 1 - \mu_R(x, y), \quad \forall (x, y) \in X \times Y.
\]

2.5.4 Projection and cylindrical extension

Fuzzy relations are fundamentally fuzzy sets defined over product spaces. Projection is used to reduce the dimension of a fuzzy relation. This generally results in loss of data. On the other hand, the opposite of projection is cylindrical extension in which we go from a fuzzy relation defined over a lower dimensional space to a fuzzy relation on a higher dimensional space.

**Definition 2.5.2.** (Projection) We can project a fuzzy relation \( R \subseteq X \times Y \) with respect to \( X \) on \( Y \) as in the following manner:

\[
R_x(x) = \text{Proj}_X \mu_R(x, y) = \max_y \mu_R(x, y), \quad \forall x \in X, \ y \in Y
\]

\[
R_y(y) = \text{Proj}_Y \mu_R(x, y) = \max_x \mu_R(x, y), \quad \forall x \in X, \ y \in Y
\]

For example, let \( R(x, y) = e^{-\frac{(x-y)^2}{2}} \) be a fuzzy relation on \( X = Y = [0, 10] \) then, the projected relation of \( R \) to \( X \) is denoted by \( R_x \), and to \( Y \) is given by \( R_y \). Figure 2.9 shows the corresponding projections \( R_x \) and \( R_y \).

**Definition 2.5.3** (Projection in \( n \) dimension) Projection in 2-dimensions relation can be extended to projection in \( n \)-dimensions. Assume relation \( R \) is defined in the space of \( X_1 \times X_2 \times \ldots \times X_n \). Projecting this relation to subspace of \( X_{i1} \times X_{i2} \times \ldots \times X_{ik} \), gives a projected relation \( R_{X_{i1} \times X_{i2} \times \ldots \times X_{ik}} \), where

\[
\mu_{R_{X_{i1} \times X_{i2} \times \ldots \times X_{ik}}}(x_{i1}, x_{i2}, \ldots, x_{ik}) = \max_{x_{j1}, x_{j2}, \ldots, x_{jm}} \mu_R(x_1, x_2, \ldots, x_n).
\]
Here, \( X_{j_1}, X_{j_2}, \ldots, X_{j_m} \) represent the omitted dimensions, and \( X_{j_1}, X_{j_2}, \ldots, X_{j_k} \) the remained dimensions, and thus \( \{ X_1, X_2, \ldots, X_n \} = \{ X_{j_1}, X_{j_2}, \ldots, X_{j_k} \} \cup \{ X_{j_1}, X_{j_2}, \ldots, X_{j_m} \} \).

**Definition 2.5.4.** (Cylindrical extension) Let fuzzy relation \( R \) is defined over \( X \times Y \), this relation can be extended to \( X \times Y \times Z \) and we can obtain a new fuzzy set. This fuzzy set is written as \( C(R) \), where,

\[
\mu_{C(R)}(x, y, z) = \mu_R(x, y), \ x \in X, \ y \in Y, \ z \in Z.
\]

The projection and cylindrical extension are often used to make domains for more than one fuzzy set. Figure 2.9 shows the cylindrical extension operation on the fuzzy relation \( R_X \).

2.5.5 Binary fuzzy relations on a single set

A binary fuzzy relation \( R : X \times X \rightarrow [0,1] \) on a single set \( X \) is a fuzzy set \( R \) on the Cartesian product \( X \times X \). Then, the properties of a binary fuzzy relation are as follows:

(i) Reflexive: if \( \mu_R(x, x) = 1, \forall x \in X \),

\[41\]
(ii) Irreflexive: if $\exists x \in X$, s.t. $\mu_R(x, x) \neq 1$,

(iii) Antireflexive: if $\mu_R(x, x) \neq 1$, $\forall x \in X$,

(iv) $\varepsilon$-reflexive: if $\mu_R(x, x) \geq \varepsilon$, where $0 < \varepsilon < 1$,

(v) Symmetric: if $\mu_R(x, y) = \mu_R(y, x)$, $\forall x, y \in X$,

(vi) Asymmetric: if $\exists x, y \in X$, $\mu_R(x, y) \neq \mu_R(y, x)$,

(vii) Antisymmetric: if $\mu_R(x, y) > 0$ and $\mu_R(y, x) > 0$ implies that $x = y$, $\forall x, y \in X$,

(viii) Transitive: $R$ is max-$\odot$ transitive if $\mu_R(x, z) \geq \max \{\mu_R(x, y) \odot \mu_R(y, z)\}$,

$\forall (x, z) \in X^2$, $\odot$ being a t-norm. A relation failing to satisfy this inequality for some members of $X$ is called nontransitive,

(ix) Antitransitive: if $\mu_R(x, z) < \max \{\mu_R(x, y) \odot \mu_R(y, z)\}$, $\forall (x, z) \in X^2$.

**Definition 2.5.5.** A fuzzy binary relation $R \subseteq X \times X$ that is reflexive, symmetric, and transitive is known as fuzzy equivalence relation or similarity relation. Basically, equivalence relations generalize the idea of equality to similarity.

**Definition 2.5.6.** A fuzzy binary relation $R \subseteq X \times X$ that is reflexive and symmetric is known as fuzzy compatibility relation or tolerance relation.

**Definition 2.5.7.** A fuzzy binary relation $R \subseteq X \times X$ that is reflexive, antisymmetric, and transitive under some form of fuzzy transitivity is known as fuzzy ordering relation.

**2.5.6 Sup-$\odot$ composition of fuzzy relations**

Composition of fuzzy relations combines fuzzy relations on different product spaces to yield a new fuzzy relation. Let $P \subseteq X \times Y$ and $Q \subseteq Y \times Z$, be two fuzzy relations then the sup-$\odot$ composition of $P$ and $Q$ is a fuzzy relation $P \circ Q$ on $X \times Z$ defined by

$$(P \circ Q)(x, z) = \sup_{y \in Y} \odot \{P(x, y), Q(y, z)\}, \forall x \in X, z \in Z$$
where $\odot$ is a particular t-norm. When $\odot$ is chosen as the standard intersection the composition is said to be standard composition i.e.

$$(P \circ Q)(x, z) = \sup_{y \in Y} \min[P(x, y), Q(y, z)], \forall x \in X, z \in Z$$

It is clear from the definition that the result of the composition operation depends entirely upon the underlying composition operator used and the choice of the composition operator depends on the context or application.

Max–min and max–product methods of composition of fuzzy relations are the two most commonly used techniques. Many other techniques are mentioned in the literature [74]. Each method of composition of fuzzy relations reflects a special inference machine and has its own significance and applications. The max–min method is the one used by Zadeh in his original paper on approximate reasoning using natural language if–then rules. Many have claimed, since Zadeh’s introduction that this method of composition effectively expresses the approximate and interpolative reasoning used by humans when they employ linguistic propositions for deductive reasoning [148].

The compositions based on different compositional operators are used in numerous applications such as data clustering, medical diagnosis, approximate reasoning, fuzzy control, engineering design problems [22, 23, 74, 148, 202, 218].

2.6 Fuzzy relation equations

Fuzzy relational equations can be regarded as a generalization of Boolean equations or lattice equations [151], as well as an analogous version of linear algebraic equations in a lattice framework. According to the types of composite operations, fuzzy relational equations can be roughly characterized into two basic categories, namely, sup-$\odot$ / inf-$\Theta$, equations.
2.6.1 Sup-$\circ$ fuzzy relation equations

Let $P \subseteq X \times Y$, $Q \subseteq Y \times Z$, $R \subseteq X \times Z$ be three binary fuzzy relations. Let $P = [p_{ij}]_{n \times m}$, $Q = [q_{jk}]_{m \times s}$ and $R = [r_{ik}]_{n \times s}$ are their matrix representations respectively. If these three relations constrain each other with one of the relations to be unknown the problem is categorized as fuzzy relation equation given as:

$$P \circ Q = R,$$

(2.1)

where $\circ$ denotes the sup-$\circ$ composition of $P$ and $Q$, $\circ$ being a t-norm. Let $I = \{1, 2, \ldots, n\}$, $J = \{1, 2, \ldots, m\}$ and $K = \{1, 2, \ldots, s\}$ be the index sets. This means that,

$$r_{ik} = \sup_{j \in J} \circ [p_{ij}, q_{jk}]
$$  

(2.2)

for all $i \in I, k \in K$. Equation (2.1) is the typical form of a fuzzy relation equation for which the following problems arise:

(i) Given $R$ and $P$ the resolution of (2.1) for $Q$, (identification problem)
(ii) Given $Q$ and $R$ the resolution of (2.1) for $P$, (inverse problem).

Figure 2.10 shows a presentation of a fuzzy relational system.

![Fuzzy relational system](image)

Figure 2.10: A fuzzy relational system
It is remarked that if we have a solution for solving the first problem, using the same method for the equation $Q^{-1} \circ P^{-1} = R^{-1}$ that employs transposed matrices, the second problem could be solved. Thus, without loss of generality, we can consider only the first problem.

2.7 Fuzzy logic and reasoning

Fuzzy logic was introduced by Zadeh [206] to emulate the human reasoning and inferring capability of generating decision under uncertainty even with incomplete and approximate information. Fuzzy logic admits truth values that are fuzzy sets of the unit interval. Truth values may be regarded as a linguistic characterization of numerical truth values. Thus, fuzzy logic concerns the principles of approximate reasoning. Theoretical foundations of fuzzy logic can be viewed in [10,146,152].

A fuzzy rule generally assumes the form “$R : A$ then $Y$ is $B$” where $A$ and $B$ are linguistic variables defined by fuzzy sets on the universes $X$ then $Y$, respectively. The rule is also called as fuzzy conditional statement abbreviated as $R : A \rightarrow B$. In essence, the expression describes a relation between two variables $x$ and $y$. This suggests that a fuzzy rule can be defined as a binary relation $R$ on the product space $X \times Y$.

Based on the interpretations of the Cartesian product and various t-norm and t-conorm operators, a number of qualified methods can be formulated to calculate the fuzzy relation $R : A \rightarrow B$. When we model a knowledge system, it is often represented by the form of “fuzzy rule base” consisting of a collection of fuzzy if-then rules. The fuzzy reasoning on the fuzzy rule base is based on inference rules.

**Generalized inference rules:** Fuzzy inference rules serve as the basis for approximate reasoning. The generalization of the classical inference rules modus ponens, modus tollens and hypothetical syllogism takes place using the idea of fuzzy relations and compositional rule of inferences. Let $R \subseteq X \times Y$, $R_1 \subseteq Y \times Z$ and $R_2 \subseteq X \times Z$ be fuzzy
relations and \( A, A' \in \mathfrak{I}(X), B, B' \in \mathfrak{I}(Y) \) and \( C \in \mathfrak{I}(Z) \) be fuzzy sets, where \( \mathfrak{I}(X), \mathfrak{I}(Y) \) and \( \mathfrak{I}(Z) \) denote the fuzzy power sets of \( X, Y \) and \( Z \) respectively. Generalized rules of inference and compositional fuzzy relation equations are given as:

(i) Generalized modus ponens: \( A \to B, A' \Rightarrow B' \) or equivalently \( B' = A' \circ R \),

(ii) Generalized modus tollens: \( A \to B, B' \Rightarrow A' \) or equivalently \( A' = B' \circ R \).

(iii) Generalized hypothetical syllogism:
\[
A \to B, \quad B \to C \Rightarrow A \to C \quad \text{or equivalently} \quad R_2 = R \circ R_1.
\]

2.8 Fuzzy inference

In many cases, the fuzzy reasoning on the fuzzy rule base is based on one level forward data-driven inference (generalized modus ponens). Therefore in real systems, the inference is determined by two factors: the “implication operator” and “composition operator”. Based on the different interpretations of these two operators various results are obtained.

2.8.1 Fuzzy expert system

An expert system is a program which contains human expert’s knowledge and gives answers to the user’s query by using an inference method. The knowledge is often stored in the form of rule base. In our real world, a human expert has his knowledge in the form of linguistic terms. Therefore it is natural to represent the knowledge by fuzzy rules and thus to use fuzzy inference methods. The kernel of a fuzzy expert system basically comprises of three parts:

1. Knowledge base (A set of if-then production rules)
2. Database
3. Inference engine
The basic configuration of fuzzy expert system is shown in figure 2.11.

![Diagram](image)

Figure 2.11: Basic configuration of a fuzzy expert system

### 2.8.2 Fuzzy logic controller

The most successful application of fuzzy relations is the fuzzy logic controller. In general, fuzzy controllers are the special expert systems that utilize knowledge elicited from human operators and are used to solve control problems. Fuzzy control action is considered as the aggregated result derived from individual control rules. The basic configuration of a fuzzy logic controller (FLC) is shown in Figure 2.12.

![Diagram](image)

Figure 2.12: Basic configuration of a fuzzy logic controller

A fuzzy logic controller consists of three main operations: Fuzzification, Inference Engine and Defuzzification. The input (crisp or numerical) data are fed into fuzzy rule
based system where physical quantities are represented into linguistic variables with appropriate membership functions. These linguistic variables are then used in the antecedents (IF-Part) of a set of fuzzy “IF-THEN” rules within an inference engine to result in a new set of fuzzy linguistic variables or consequent (THEN-Part). The output data is obtained in form of fuzzified form that after being processed into defuzzifier gives the desired crisp form of result.

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